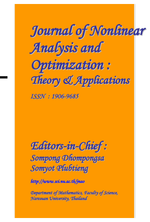


Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 21-35
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

NON-STANDARD LAGRANGIANS WITH DIFFERENTIALS OPERATORS

RAMI AHMAD EL-NABULSI*,[1] AND HAMIDREZA REZAZADEH[2]

[1]College of Mathematics and Information Science, Neijiang Normal University, Neijiang, Sichuan 641112, China.

[2]School of Mathematics, Iran University of Science and Technology, Narmak, 16844 Tehran,Iran

ABSTRACT. Nonlinear dynamics from fractional variational approach characterized by non-standard Lagrangians holding singular Weinstein and higher-order extended Euler-Poisson derivative operators are presented in this work. Many results that can enrich the theory of analytical mechanics are obtained and discussed.

KEYWORDS : fractional variational approach, non-singular Weinstein derivative operator, extended Euler-Poisson derivative operator, non-standard Lagrangians

1. INTRODUCTION

Fractional calculus of variations (FCV) is a new branch of applied mathematics which has many successful applications in physics and engineering [1] - [22]. Different formulations of FCV were introduced in literature depending on the nature of the problem under study. In fact, the FCV starts with the work of Riewe in 1996 where he formulated the problem of the calculus of variations by replacing the standard derivatives operators by a fractional one and derived the respective fractional Euler-Lagrange equations [23, 24]. The main result of Riewe's fractional approach is that non-conservative forces can be computed directly from potentials represented by fractional derivatives. It is noteworthy that the method developed by Riewe was applied to several systems by Dreisigmeyer and Young [25, 26] and by Rabei, Alhalholy and Taani [27] who also showed the limitations of the method caused by the complexity of fractional calculus. More importantly these authors also demonstrated that the resulting equations are casual and that the procedure to convert these equations into casual ones is not well-defined. In other words, past and future become linked through the fractional Euler-Lagrange equation which results on a broken causality, i.e. violation of the causality principle. Some advances in this approach are done in [28], nevertheless much work is required to settle the problem. Despite the fact that fractional causal dynamical equations may be derived from the least action principle, some of the resulting fractional functions are unclear, their physical meaning is still obscure and their physical validity is not obvious [29]. In all cases, the fractional approach derived by Riewe may represent a powerful tool to understand many hidden properties of complex dynamical systems, like

* *Corresponding author.*
Email address : nabulsiahmadrami@yahoo.fr(R.A. EL-NABULSI) and hr_rezazadeh@MathDep.iust.ac.ir(H. Rezazadeh).
Article history : Received November 12, 2012 Accepted June 26, 2013.

Hamiltonian chaos for example [30, 31]. The literature dealing with the FCV is immense; nevertheless much remains to be done. For a good introduction of the topic, we refer the reader to [32, 33].
One more successful form is known by the Fractional ActionLike Variational Approach (FALVA) introduced by the author in 2005 [8, 9]. Although in general, FCV and FALVA are under strong research studies, much remains to be done at both classical and quantum levels. The main purpose of this work is to extend FALVA for the case of non-standard Lagrangians (NSL) with singular derivatives operators (SDO) introduced by Weinstein, Euler and Poisson [34]. In fact, some basic equations of mathematical physics are obtained from NSL with SDO for dissipative dynamical systems and besides, it was observed that they can be applied successfully to a broad range of classical and quantum problems [35]-[39]. These NLS may increase the number of the initial data required to fix the classical trajectory and generate dynamical equations that go beyond the standard CityplaceNewton's law. In a more recent work [40], we derived a large numbers of NSL which were disregarded in the literature and which result on a number of familiar dynamical equations of motion. One chief observation is that the same equation of motion of a certain dynamical equation may be derived from dissimilar Lagrangians functionals. In reality, the significance of NSL was recognized in dissipative dynamical systems [36, 37]. Some nice examples are the Lienard type nonlinear oscillator [41, 42] and the 2^{nd} order Riccati equation [43]. Some related interesting results are found in [44]-[47].
The paper is organized as follows: the basic formalism is constructed in Sec. 2 and examples are illustrated in Sec. 3; conclusions and perspectives are given in Sec. 4.

2. BASIC FORMALISM

We start by introducing the new extended FALVA formalism with NSL and SDO:
Definition 2.1: *Consider a smooth manifold* M *and let* L *be an admissible smooth Lagrangian function* $L:\mathbb{R}\times\mathrm{TM}\longrightarrow\mathbb{R}$ *on* $\mathbb{R}\times\mathbb{R}^d\times C^d, d\geq 1$ *defined on the tangent bundle* TM. *The extended FALVA with NSL and SDO on the set of paths* $q(\tau)$,$0\leq\tau\leq t$ *between two given points* $A=\mathrm{Q}(a)$ *and* $B=\mathrm{Q}(b)$, *is defined as for any piecewise smooth differentiable path* $\mathrm{Q}:[a,b]\longrightarrow\mathrm{M}$,

$$S[\mathrm{Q}]=\frac{1}{\Gamma(\alpha)}\int_a^b L\left(\tau,\mathrm{Q}(q(\tau)),\dot{\mathrm{Q}}(q(\tau),\dot{q}(\tau)),\ddot{\mathrm{Q}}(q(\tau),\dot{q}(\tau),\ddot{q}(\tau))\right)(t-\tau)^{\alpha-1}\,d\tau, \tag{2.1}$$

with $\mathrm{Q}(q(\tau))=q(\tau)+\beta\int\frac{q(\tau)}{\tau-t}d\tau$*and*

$$\dot{\mathrm{Q}}(q(\tau),\dot{q}(\tau))=\dot{q}(\tau)+\frac{\beta}{\tau-t}q(\tau), \tag{2.2}$$

$$\ddot{\mathrm{Q}}(q(\tau),\dot{q}(\tau),\ddot{q}(\tau))=\ddot{q}(\tau)+\frac{\beta}{\tau-t}\dot{q}(\tau)-\frac{\beta}{(\tau-t)^2}q(\tau), \tag{2.3}$$

being respectively the Weinstein and the extended Euler-Poisson [**34**] *singular derivative operator*[1]. *Here* $\dot{q}(\tau)=dq(\tau)/d\tau$,$\ddot{q}(\tau)=d^2q(\tau)/d\tau^2$, α*is a real or a complex number and* $\beta\in\mathbb{R}$; τ*is the intrinsic time,* t *is the observer time with* $t\neq\tau$.

Theorem 2.1: *If* $q(\cdot)$ *are solutions of equation* (2.1), *then* $\mathrm{Q}(\cdot)$*satisfies the following Euler-Lagrange equation:*

$$\left\{1+\ln(\tau-t)^{\beta}\right\}\frac{\partial L}{\partial\mathrm{Q}}-\left\{\frac{1}{(t-\tau)^{\alpha-1}}+\frac{\alpha-1}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\mathrm{Q}}}\right)+\left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}}+\frac{(\alpha-1)^2}{(t-\tau)^2}\right\}\frac{\partial L}{\partial\dot{\mathrm{Q}}}$$

[1] The Euler-Poisson singular derivative operator is in reality of the form

$$\ddot{\mathrm{Q}}(\dot{q}(\tau),\ddot{q}(\tau))=\ddot{q}(\tau)+(\beta/(\tau-t))\dot{q}(\tau)$$

and it is known as the Bessel's operator nevertheless it appeared in the work of Euler and Poisson [34]. In our work, we called equation (3) the extended Euler-Poisson operator.

$$+\left\{\frac{\beta}{(t-\tau)^{\alpha}}-\frac{2(\alpha-1)}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)+\left\{\frac{(\alpha-1)(\alpha-2)}{(t-\tau)^{2}}-\frac{\beta(\alpha-1)}{(t-\tau)^{\alpha+1}}\right\}\frac{\partial L}{\partial \ddot{\mathrm{Q}}}+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)=0. \tag{2.4}$$

Proof: This problem is the same as extremizing the fractional action with Lagrangian $L(\tau, \dot{\mathrm{Q}}, \dot{\mathrm{Y}})$ subject to the constraints $\mathrm{Y}=\dot{\mathrm{Q}}$. The action of the theory is now given by:

$$S[\mathrm{Q}]=\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left\{L\left(\tau,\mathrm{Q},\dot{\mathrm{Q}},\dot{y}\right)-\lambda\left(\mathrm{Y}-\dot{\mathrm{Q}}\right)\right\}(t-\tau)^{\alpha-1}\,d\tau, \tag{2.5}$$

where λ is the Lagrange multiplier. It is easy to prove after elementary calculations that the following relation hold:

$$\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1}-\frac{d}{d\tau}\left(\left(\frac{\partial L}{\partial \dot{q}}\right)(t-\tau)^{\alpha-1}+\lambda\frac{\partial \dot{\mathrm{Q}}}{\partial \dot{q}}(t-\tau)^{\alpha-1}\right)=0, \tag{2.6}$$

and $\mathrm{Y}=\dot{\mathrm{Q}}$.[2] Making use of the chain rules:

$$\frac{\partial L}{\partial q}=\frac{\partial L}{\partial \mathrm{Q}}\frac{\partial \mathrm{Q}}{\partial q}+\frac{\partial L}{\partial \dot{\mathrm{Q}}}\frac{\partial \dot{\mathrm{Q}}}{\partial q}+\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\frac{\partial \ddot{\mathrm{Q}}}{\partial q}, \tag{2.7}$$

$$\frac{\partial L}{\partial \dot{q}}=\frac{\partial L}{\partial \dot{\mathrm{Q}}}\frac{\partial \dot{\mathrm{Q}}}{\partial \dot{q}}+\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\frac{\partial \ddot{\mathrm{Q}}}{\partial \dot{q}}, \tag{2.8}$$

we obtain after simple algebraic manipulation:

$$\left(\frac{\partial L}{\partial \mathrm{Q}}\frac{\partial \mathrm{Q}}{\partial q}+\frac{\partial L}{\partial \dot{\mathrm{Q}}}\frac{\partial \dot{\mathrm{Q}}}{\partial q}+\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\frac{\partial \ddot{\mathrm{Q}}}{\partial q}\right)(t-\tau)^{\alpha-1}$$

$$-\frac{d}{d\tau}\left(\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\frac{\partial \dot{\mathrm{Q}}}{\partial \dot{q}}+\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\frac{\partial \ddot{\mathrm{Q}}}{\partial \dot{q}}\right)(t-\tau)^{\alpha-1}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\frac{\partial \ddot{\mathrm{Q}}}{\partial \dot{q}}(t-\tau)^{\alpha-1}\right)\frac{\partial \dot{\mathrm{Q}}}{\partial \dot{q}}(t-\tau)^{\alpha-1}\right)=0. \tag{2.9}$$

Using equations (2.2) and (2.3) and the fact that

$$\frac{\partial \mathrm{Q}}{\partial q}=1+\beta\frac{\partial}{\partial q}\int\frac{q(\tau)}{\tau-t}d\tau=1+\beta\int\frac{\partial}{\partial q}\frac{q(\tau)}{\tau-t}d\tau=1+\ln(\tau-t)^{\beta}, \tag{2.10}$$

we find effortlessly:

$$\left\{1+\ln(\tau-t)^{\beta}\right\}\frac{\partial L}{\partial \mathrm{Q}}-\left\{\frac{1}{(t-\tau)^{\alpha-1}}+\frac{\alpha-1}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right)+\left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}}+\frac{(\alpha-1)^{2}}{(t-\tau)^{2}}\right\}\frac{\partial L}{\partial \dot{\mathrm{Q}}}$$

$$+\left\{\frac{\beta}{(t-\tau)^{\alpha}}-\frac{2(\alpha-1)}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)+\left\{\frac{(\alpha-1)(\alpha-2)}{(t-\tau)^{2}}-\frac{\beta(\alpha-1)}{(t-\tau)^{\alpha+1}}\right\}\frac{\partial L}{\partial \ddot{\mathrm{Q}}}+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)=0. \tag{2.11}$$

Remark 2.1: *When* $\beta=0$*and* $\alpha=1$, *equation* (2.4) *is reduced to:*

$$\frac{\partial L}{\partial q}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}}\right)+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial \ddot{q}}\right)=0. \tag{2.12}$$

For $\beta\neq 0$ *and* $\alpha=1$, *equation* (2.4) *is reduced to:*

$$\left\{1+\ln(\tau-t)^{\beta}\right\}\frac{\partial L}{\partial \mathrm{Q}}+\frac{\beta}{\tau-t}\left\{\frac{\partial L}{\partial \dot{\mathrm{Q}}}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\}-\frac{d}{d\tau}\left\{\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right)-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\}=0. \tag{2.13}$$

Illustrations and exemplifications are done in the subsequent section.

[2] *Equation (6) is the fractional Euler-Lagrange equation obtained in FALVA approach*

3. ILLUSTRATIONS AND APPLICATIONS

Before illustrating our approach, it is noteworthy we do not claim at this stage the superiority of the FALVA method over the standard FCV method originally developed by Riewe [23, 24]. Both approaches are different nevertheless in our opinion, FALVA approach is simpler. In Riewe's approach, fractional derivatives are used to study nonconservative dynamical systems (NDS) and in that context, the generalization of Lagrangians taken into account the dissipative effects was performed. Whereas in FALVA approach we used fractional integrals to study NDS and not fractional derivatives. In general, dealing with fractional derivatives is much complicated than dealing with fractional derivatives, e.g. the fractional integration by parts require some hard mathematical restrictions on the functions involved. The form of the fractional equation of motion may be comparable with the classical one but the physical meaning of the classical derivatives is loosing the meaning in the fractional case [48]. The replacement of the classical derivative in equation (2.1) by fractional derivative renders the mathematical manipulation of the theory very hard and consequently, numerical techniques are required in order to compare FALVA approach to Riewe's approach. From another direction, by replacing the standard Lagrangians in Riewe's approach by NSL augmented by fractional SDO renders the mathematical manipulations much hard. To appreciate FALVA and the importance of NSL, let us start by discussing the following simplest example:

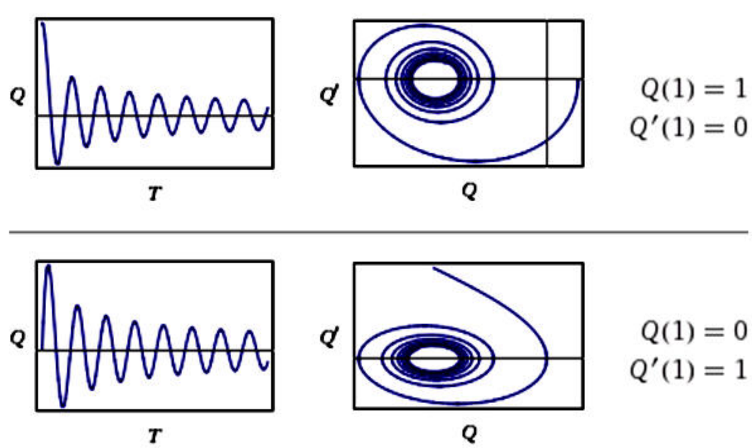
Example 1-We consider at the beginning the NSL $L = -\frac{1}{2}\{1+\ln(\tau-t)^{\beta}\}^{-1}\mathrm{Q}^2 + \frac{1}{2}\dot{\mathrm{Q}}^2 + \ddot{\mathrm{Q}}$ for $\alpha = 1$ where equation (2.13) holds accordingly. Equation (2.13) results into the following dynamical equation:

$$\ddot{\mathrm{Q}} - \frac{\beta}{\tau - t}\dot{\mathrm{Q}} + \mathrm{Q} = 0. \tag{3.1}$$

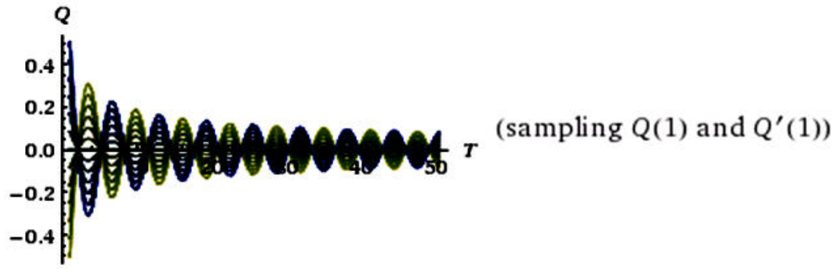
The solution is given by:

$$\mathrm{Q}(\tau) = c_1(\tau-t)^{\frac{\beta+1}{2}} J_{\frac{\beta+1}{2}}(\tau-t) + c_2(\tau-t)^{\frac{\beta+1}{2}} Y_{\frac{\beta+1}{2}}(\tau-t), \tag{3.2}$$

where c_1,c_2 are integration constants and $J_n(z)$ and $Y_n(z)$ are respectively the Bessel functions of 1^{st} and 2^{nd} kind. For $\beta = -1$, we plot in graphs 1 and 2 sample individual solutions and sample family solution of equation (3.1) respectively ($T = \tau - t$):



Graph 1: sample individual solutions of equation (3.2)



Graph 2: sample family solution of equation (3.2)

These graphs show decaying oscillations due to the presence of a positive friction term. From equation (2.2) we find:

$$q(\tau)=c_1 2^{-\frac{\beta+3}{2}}(\tau-t)^{\beta+2}\Gamma(\beta+1)\,{}_1\tilde{F}_2\left(\beta+1;\frac{\beta+3}{2};\beta+2,-\frac{(\tau-t)^2}{4}\right)$$

$$+\left\{(\tau-t)^{\beta+1}\cos\left(\frac{\pi(\beta+1)}{2}\right)\Gamma(\beta+1)\,{}_1\tilde{F}_2\left(\beta+1;\frac{\beta+3}{2};\beta+2;-\frac{(\tau-t)^2}{4}\right)\right.$$

$$\left.-\,2^{\beta+1}\Gamma\left(\frac{\beta+1}{2}\right){}_1\tilde{F}_2\left(\frac{\beta+1}{2};\frac{1-\beta}{2};\frac{\beta+3}{2};-\frac{(\tau-t)^2}{4}\right)\right\}c_2 2^{-\frac{\beta+3}{2}}(\tau-t)\sec\left(\frac{\pi\beta}{2}\right)+c_3, \tag{3.3}$$

where c_3 is another integration constant and ${}_p\tilde{F}_q\left(a_1,...,a_p;b_1,...,b_q;z\right)$ is the generalized regularized hypergeometric function. If for instance $\beta=-1$, then $\mathrm{Q}(\tau)=c_1J_0(\tau-t)+c_2Y_0(\tau-t)$and

$$q(\tau)=-c_1\frac{1}{2}(\tau-t)^2\,{}_1F_2\left(\frac{1}{2};\frac{3}{2};2;-\frac{(\tau-t)^2}{4}\right)-c_2\frac{1}{2}(\tau-t)G^{2,1}_{2,4}\left(\frac{\tau-t}{2},\frac{1}{2}\middle|\begin{array}{c}1,-1\\-\frac{1}{2},\frac{1}{2},-1,0\end{array}\right). \tag{3.4}$$

${}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)$ is the hypergeometric function and $G^{m,n}_{p,q}\left(z\middle|\begin{array}{c}a_1,....,a_p\\b_1,...,b_q\end{array}\right)$ is the Meijer G-function [48]:

$$G^{m,n}_{p,q}\left(z\middle|\begin{array}{c}a_1,....,a_p\\b_1,...,b_q\end{array}\right)=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{j=m+1}^{q}\Gamma(1-b_j+s)}x^s ds. \tag{3.5}$$

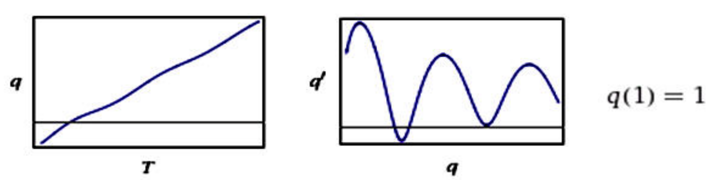
Assuming the initial conditions $\mathrm{Q}(2.1)=1$and $\mathrm{Q}'(2.1)=0$, then the solution of equation (3.1) which is illustrated in graph 1 is:

$$\mathrm{Q}(\tau)=\frac{J_1(1)Y_0(T)-Y_1(1)J_0(T)}{J_1(1)Y_0(1)-Y_1(1)J_0(1)}, \tag{3.6}$$

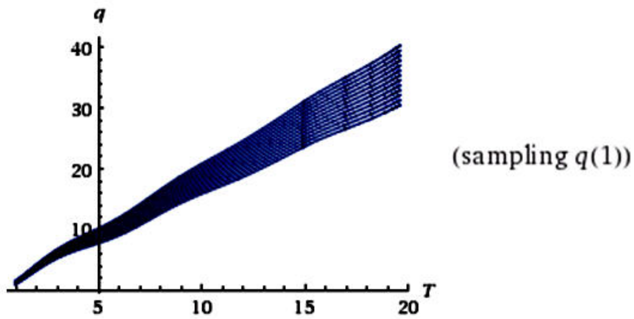
and accordingly$q(\tau)$, is equal to:

$$\frac{(\tau-t)Y_1(1)G^{2,0}_{1,3}\left(\left(\frac{\tau-t}{2}\right)^2\middle|\begin{array}{c}1\\0,0,0\end{array}\right)}{2J_1(1)Y_0(1)-2J_0(1)Y_1(1)}-\frac{2J_1(1)G^{3,0}_{2,4}\left(\frac{\tau-t}{2},\frac{1}{2}\middle|\begin{array}{c}0,\frac{3}{2}\\\frac{1}{2},\frac{1}{2},\frac{1}{2},0\end{array}\right)}{2J_1(1)Y_0(1)-2J_0(1)Y_1(1)}+c_3(\tau-t). \tag{3.7}$$

We plot in graphs 3 and 4 sample individual solutions and sample family solution of equation (3.7) respectively:



Graph 3: sample individual solutions of equation (3.7)



Graph 4: sample family solution of equation (3.7)

It is very hard to describe analytically the dynamical problem by replacing the ordinary derivative operator by a fractional and numerical techniques are required afterward. It is noteworthy that that the dynamical equation (3.1) may be derived in general from the standard time-dependent Lagrangian $L(\tau, \mathrm{Q}, \dot{\mathrm{Q}}) = \frac{1}{2}(\tau - t)(\dot{\mathrm{Q}}^2 - \mathrm{Q}^2)$ and using the standard Euler-Lagrange equation

$$\frac{\partial L}{\partial \mathrm{Q}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right) = 0. \tag{3.8}$$

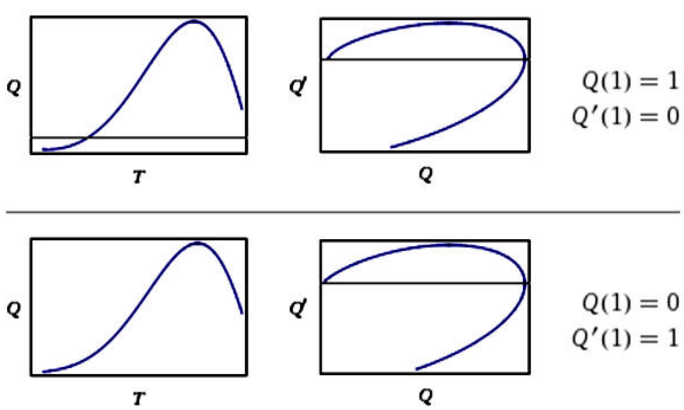
Whereas in our approach, equation (2.11) gives for $L(\tau, \mathrm{Q}, \dot{\mathrm{Q}}) = \frac{1}{2}\tau(\dot{\mathrm{Q}}^2 - \mathrm{Q}^2)$

$$\left\{\frac{1}{(t-\tau)^{\alpha-2}} + 1 - \alpha\right\}\ddot{\mathrm{Q}} + \left\{\frac{\alpha(\alpha-1)}{t-\tau} + \frac{\alpha-\beta}{(t-\tau)^{\alpha-1}}\right\}\dot{\mathrm{Q}} + \left\{1 + \ln(\tau - t)^{\beta}\right\}(\tau - t)\mathrm{Q} = 0. \tag{3.9}$$

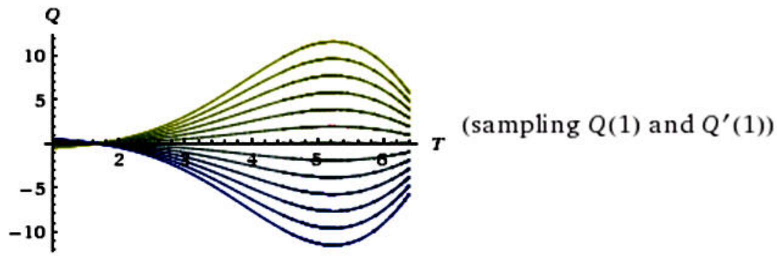
Mainly for $\alpha = 1$, we get:

$$(\tau - t)\ddot{\mathrm{Q}} + (\beta - 1)\dot{\mathrm{Q}} - \left\{1 + \ln(\tau - t)^{\beta}\right\}(\tau - t)\mathrm{Q} = 0. \tag{3.10}$$

Obviously, for $\beta = -1$ the dynamics which result from equation (3.10) is totally different from the one obtained from equation (3.1) as shown in graphs 5 and 6:



Graph 5: sample individual solutions of equation (3.10)



Graph 6: sample family solution of equation (3.10)

Comparing graphs 1 and 5, we observe that for some critical values of the parameter β, the dynamics are totally dissimilar.

Example2-We consider now the NSL $L = -\frac{1}{2}\frac{(\tau-t)^m}{1+\ln(\tau-t)^\beta}\mathrm{Q}^2 + \dot{\mathrm{Q}} + \ddot{\mathrm{Q}}$ with $\alpha - 1 = \beta$and $m \in \mathbb{R}$. Hence we get:

$$\mathrm{Q}(\tau) = \frac{(\alpha-1)(2\alpha-3)}{(t-\tau)^{2+m}} + \frac{(\alpha-1)^2}{(t-\tau)^{\alpha+m+1}}, \tag{3.11}$$

and accordingly:

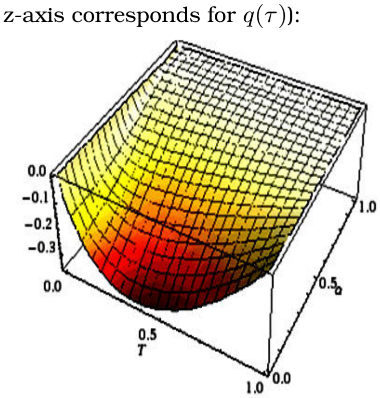
$$\dot{q} + \frac{\alpha-1}{\tau-t}q = (2+m)(\alpha-1)(2\alpha-3)\,(t-\tau)^{-3-m}$$

$$+\,(\alpha+m+1)(\alpha-1)^2\,(t-\tau)^{-\alpha-m-2}\,. \tag{3.12}$$

One particular class of solution is obtained for $m = -2$ where the solution is given by:

$$q(\tau) = (1-\alpha)^3(\tau-t)^{1-\alpha}\ln|\tau-t| + c_4. \tag{3.13}$$

c_4 is one more integration constant. It is noteworthy that the dynamics of Qis complexified for $0 < \alpha < 1$ and $T = \tau - t > 0$ whereas the dynamics of $q(\tau)$ is real. We plot in Graph 7 the variations of equations (3.13) and in Graphs 8 and 9 the variations of equation (3.11) (the z-axis corresponds for $q(\tau)$):



Graph 7: Plot of equation (3.13)

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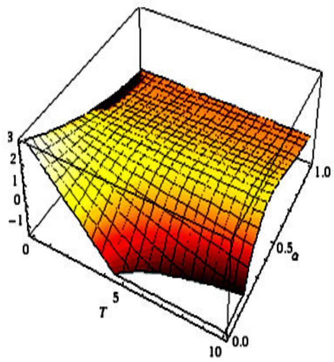

Graph 8: Plot of equation (3.13): *real part*

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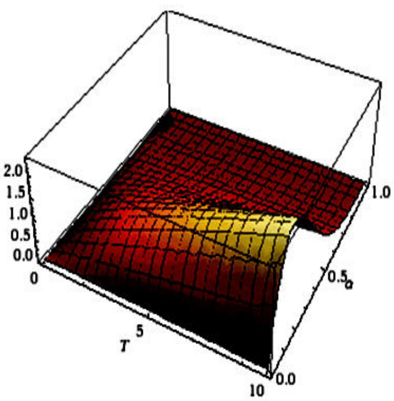

Graph 9: Plot of equation (3.13): *imaginary part*

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We may conclude that equations (2.4) and (2.13) are motivating if we want to describe Lagrangian systems holding a NSL and starting from FALVA.

Example 3-We could at present consider the following NSL $L = (\dot{Q} + f(\tau,t)Q)^{-1}$considered in [36] to model dissipations. Here $f(\tau,t)$is an arbitrary function of time. Equation (2.11) gives now:

$$\left\{\frac{1}{(t-\tau)^{\alpha-1}} + \frac{\alpha-1}{t-\tau}\right\}\left((\dot{Q} + f(\tau,t)Q)^{-1}\left(\ddot{Q} + \frac{df}{d\tau}Q + f\dot{Q}\right)\right) - \left\{1 + \ln(\tau-t)^{\beta}\right\}f(\tau,t) + \left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}} + \frac{(\alpha-1)^2}{(t-\tau)^2}\right\} = 0. \tag{3.14}$$

For simplicity, we choose $f(\tau,t) = 1$, then equation (3.14) is simplified to:

$$\left\{\frac{1}{(t-\tau)^{\alpha-1}} + \frac{\alpha-1}{t-\tau}\right\}\left(\ddot{Q} + \dot{Q}\right) + \left(\left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}} + \frac{(\alpha-1)^2}{(t-\tau)^2}\right\} - \left\{1 + \ln(\tau-t)^{\beta}\right\}\right)(\dot{Q}+Q) = 0. \tag{3.15}$$

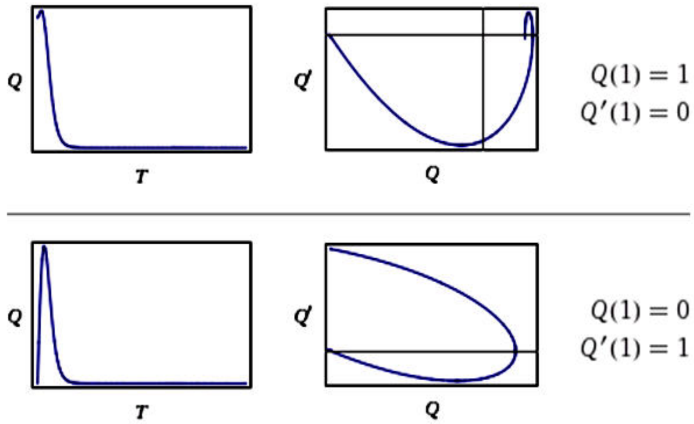
To illustrate, we choose $\beta << 1$and $\alpha = 2$; then equation (3.15) is simplified to

$$-\frac{2}{\tau-t}\ddot{Q} + \left\{-\frac{2}{\tau-t} + \frac{2}{(\tau-t)^2} - 1\right\}\dot{Q} + \left\{\frac{2}{(\tau-t)^2} - 1\right\}Q = 0. \tag{3.16}$$

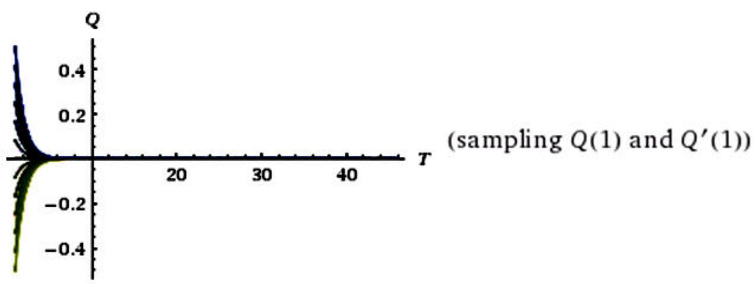
and the solution is given by:

$$Q(\tau) = c_5 e^{-\frac{1}{4}(\tau-t)((\tau-t+4))}\left(\sqrt{\pi}e^{\frac{1}{4}(\tau-t)^2+1}\operatorname{erf}\left(\frac{\tau-t-2}{2}\right) - e^{\tau-t}\right) + c_6 e^{-(\tau-t)}, \tag{3.17}$$

where $\operatorname{erf}(x)$ is error function. We plot the resulting solutions in Graphs 10 and 11

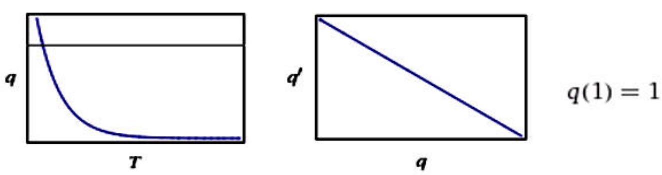


Graph 10: sample individual solutions of equation (3.17)

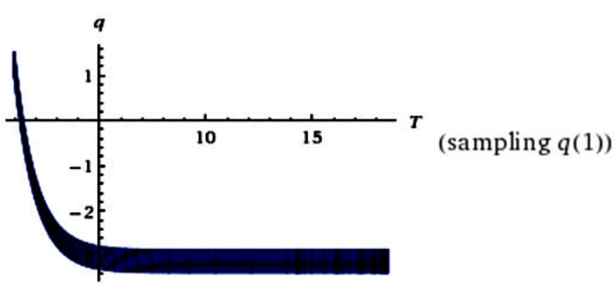


Graph 11: sample family solution of equation (3.17)

For very large time, we plot in Graphs 12 and 13 the solutions which correspond for $q(\tau)$ as deduced from equation (2.2):

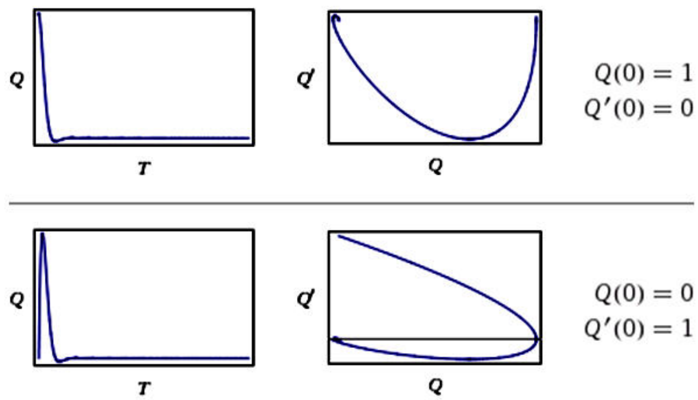


Graph 12: sample individual solutions for $q(\tau)$

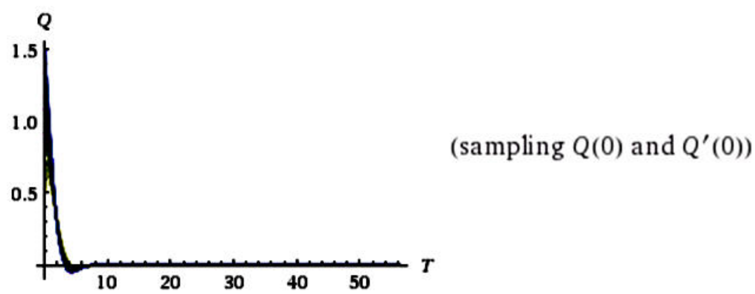


Graph 13: sample family solution of for $q(\tau)$

In [36], the resulting equation of motion which results from the NSL $L = (\dot{\mathrm{Q}}+f(\tau,t)\mathrm{Q})^{-1}$with $f(\tau,t) = 1$is: $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$and the solutions are plotted in Graphs 14 and 15:

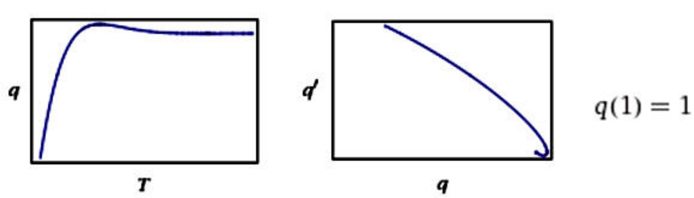


Graph 14: sample individual solutions of equation $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$

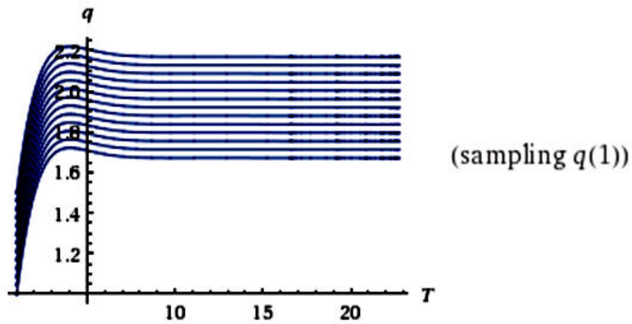


Graph 15: sample family solution of equation $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$

Graphs 10 and 14 are similar however in [36], we have $\beta = 0$whereas in our approach, for the particular case $\beta << 1$, e.g. $\beta = 10^{-3}$, the following graphical solutions (Graphs 16 and 17) hold for $q(\tau)$:



Graph 16: sample individual solutions of $q(\tau)$



Graph 17: sample family solution of $q(\tau)$

Remark 3.1: *If the Lagrangian is independent of* Q, *then equation* (2.13) *is reduced to*:

$$\frac{\beta}{\tau - t}\left\{\frac{\partial L}{\partial \dot{\mathrm{Q}}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\} - \frac{d}{d\tau}\left\{\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right) - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\} = 0. \quad (3.18)$$

It is easy to check after straightforward algebra that:

$$\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right) - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right) = K(\tau - t)^{\beta}, \quad (3.19)$$

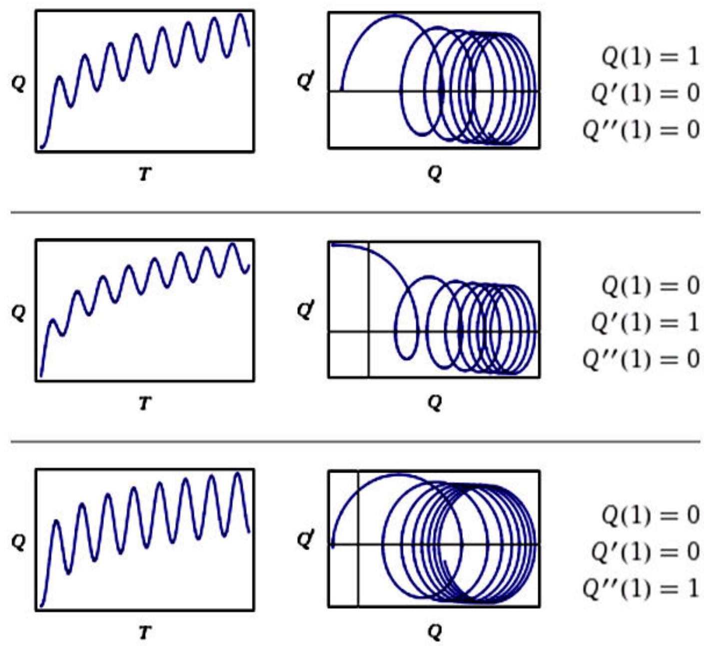
where K is an integration constant. This equation is similar to the Euler-Lagrange equation with a time-dependent dissipative term S in particular when $\dot{\mathrm{Q}} \longrightarrow \mathrm{Q}$*and* $S = K(\tau - t)^{\beta}$. *Here* Q*is a new generalized coordinate.*
To illustrate we discuss the following examples:

Example 4-We consider the Lagrangian $L = \frac{1}{2}\dot{\mathrm{Q}}^2 - \frac{1}{2}\ddot{\mathrm{Q}}^2$and we choose $\beta = -1$ Equation (3.18) gives:

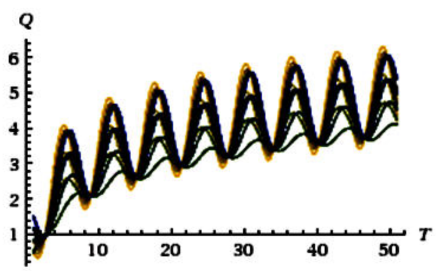
$$\mathrm{Q}(\tau) = c_5 \sin(\tau - t) + c_6 \cos(\tau - t) + c_7$$

$$+ \mathrm{Ci}(\tau - t)(-\cos(\tau - t)) - S\mathrm{i}(\tau - t)(\sin(\tau - t)) + \log(\tau - t), \quad (3.20)$$

where $\mathrm{Ci}(\tau - t)$ and $\mathrm{Si}(\tau - t)$are respectively the cosine and the sine integrals [49]. We plot in Graph 18 sample individual solutions and in Graph 19 a sample family solution:



Graph 18: sample individual solutions of equation (3.20)



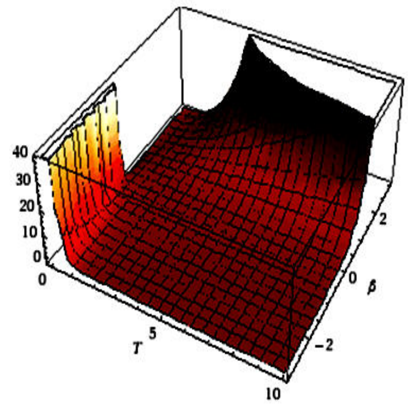
(sampling $Q(1)$, $Q'(1)$ and $Q''(1)$)

Graph 19: sample family solution of equation (3.20)

Example 5-We consider now the Lagrangian $L = \frac{1}{2}m\dot{Q}^2 - U(\ddot{Q})$where $U(\ddot{Q})$ is a function of $\ddot{Q}$and $m$is the mass of the particle. Then the equation of motion is:

$$p - \frac{d}{d\tau}\left(\frac{\partial U}{\partial \ddot{Q}}\right) = K(\tau - t)^{\beta}, \tag{3.21}$$

where $p = m\dot{Q}$is the momentum. In the absence of the function $U(\ddot{Q})$, we get $p = K(\tau - t)^{\beta}$and hence the momentum is time-dependent. It increases with time for $\beta > 0$and decreases for $\beta < 0$. We plot in Graph 20 the variation of equation $p = K(\tau - t)^{\beta}$ (z axis corresponds for p):



Graph 20: Plot of equation $p = K(\tau - t)^{\beta}$for $K = 1$and $-3 < \beta < 3$

4. CONCLUSIONS AND PERSPECTIVES

In conclusion, we have explored nonlinear dynamics from both fractional and non-fractional FALVA holding non-standard Lagrangians. The new formalism is characterized by non-standard Lagrangians holding singular Weinstein and higher-order extended Euler-Poisson derivative operators. The examples discussed in this paper are simple, however they prove that non-standard Lagrangians are important and deserves attention. The results obtained here are very relevant to those presented by Riewe. It is notable that in 1931, Bauer proved that for a given dissipative dynamical systems with constant coefficients, the resulting dynamical equations of motion are not derived from a variational principle [50]. In Bauer' approach all the derivatives are of integer order. This is one of the main reasons why Riewe use fractional derivatives operators to model dissipative dynamical systems. Here we proved that FALVA augmented by NSL and Weinstein differential operator could be used to model nonconservative and dissipative systems without implementing fractional derivatives. In other words, the Lagrangian formulation is constructed for different kinds of dissipative systems without using the notion of fractional derivative operator. Further investigations should be carried to elucidate and understand this approach.

In literature there exist many approaches to deal with dynamical systems [51, 52, 53]; nevertheless the equations of motion are derived from the standard action principle. However, the presented approach is in fact widespread and permits obtaining miscellaneous forms of non-standard Lagrangians and equations of motion. We argue that this new formalism has important implications in many physical systems. The main advantage of the new action explored in this work is that it offers a new view of the dynamical system under study. We expect that the new arguments proposed in this work can be applied successfully to a broad range of physical problems ranging from dissipative mechanics to field theories [54]-[58]. It will be as well interesting to apply our approach to Hamiltonian and chaotic dynamics where time takes on a fractal structure [30]. Work in these directions is under progress.

References

1. R. Almeida, D. F. M. Torres, Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives, *Commun. Nonlinear Sci. Numer. Simulat.* 16 (2011), No. 3, 1490-1500.
2. R. Almeida, placeS. Pooseh, D. F. M. Torres, Fractional variational problems depending on indefinite integrals, *Nonlinear Anal.* 75 (2012), No. 3, 1009-1025.
3. O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, *J. Math. Anal. Appl.* 272 (2002) 368-379.
4. R. A. El-Nabulsi, Fractional variational problems from extended exponentially fractional integral, *Appl. Math. Comp.* 217 (2011), No. 22, 9492-9496.
5. R. A. El-Nabulsi, A periodic functional approach to the calculus of variations and the problem of time dependent damped harmonic oscillators, *Appl. Math. Lett.* 24, (2011) No. 10, 1647-1653.

6. R. A. El-Nabulsi, D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann- Liouville derivatives of order (alfa,beta), *Math. Methods Appl. Sci.* 30, (2007) No 15, 1931-1939.
7. R. A. El-Nabulsi, D. F. M. Torres, Fractional actionlike variational problems, *J. Math. Phys.* 49 (2008), No 5, 053521-053528.
8. R. A. El-Nabulsi, A fractional approach to nonconservative Lagrangian dynamical systems, *Fiz.* A14, No. 4, (2005) 289-298.
9. R. A. El-Nabulsi, A fractional action-like variational approach of some classical, quantum and geometrical dynamics, *Int. J. Appl. Math* 17, (2005), No. 3, 299-317.
10. R. A. El-Nabulsi, The fractional calculus of variations from extended Erdelyi-Kober operator, *Int. J. Mod. Phys.* B23, (2009) No. 16, 3349-3361.
11. R. A. El-Nabulsi, Universal fractional Euler-Lagrange equation from a generalized fractional derivate operator, Central Europ. J. Phys. 9, (2010), No.1, 250-256.
12. A. B. Malinowska, M. R. S. Ammi, D. F. M. Torres, Composition Functionals in Fractional Calculus of Variations, *Commun. Frac. Calc.* 1 (2010) 32-40.
13. A. B. Malinowska, D. F. M. Torres, Fractional calculus of variations for a combined Caputo derivative, *Fract. Calc. Appl. Anal.* 14 (2011), No. 4, 523-537.
14. A. B. Malinowska, D. F. M. Torres, Multiobjective fractional variational calculus in terms of a combined Caputo derivative, *Appl. Math. Comput.* 218 (2012), No. 9, 5099-5111.
15. T. M. Atanackovi, S. Konjik, S. Pilipovi, placeS. Simi, Variational problems with fractional derivatives: Invariance conditions and Noether's theorem, *Nonlinear Anal. Theory, Methods Appl.* 71 (2009) 1504-1517.
16. D. Baleanu, S. I. Muslih, Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives. *J. Math. Anal. Appl.* 304 (2005) 599-606.
17. T. M. Atanackovic, S. Konjik, placeS. Pilipovic, Variational problems with fractional derivatives: Euler-Lagrange equations, *J. Phys.* A41(2.9) (2008) 095201-095213.
18. T. M. Atanackovic, S. Konjik, Lj. Oparnica, S. Pilipovic, Generalized Hamilton's principle with fractional derivatives, *J. Phys.* A43 (2010), No. 25, 255203-255215.
19. M. Klimek, Fractional sequential mechanics-models with symmetric fractional derivative, *Czechoslovak J. Phys.*, 51(2.12) (2001) 1348–1354.
20. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst.* 29 (2011), No. 2, 417–437.
21. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Discrete-time fractional variational problems, *Signal Process.* 91 (2011), No. 3, 513–524.
22. G.-C. Wu, Ji-H. He. Fractional Calculus of Variations in Fractal Spacetime *Nonlinear Sci. Lett. A*1.3 (2010), 281-287.
23. F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev.* E53 (1996), No. 2, 1890-1899.
24. F. Riewe, Mechanics with fractional derivatives. *Phys. Rev.* E(2.3) 55 (1997), No 3, part B, 3581–3592.
25. D. W. Dreisigmeyer, P. M. Young, Nonconservative Lagrangian mechanics: a generalized function approach, *J. Phys.* A36 (2003) 8297–8310.
26. D. W. Dreisigmeyer, P. M. Young, Extending Bauer's corollary to fractional derivatives, *J. Phys. A: Math. Gen.* 37(2.11) (2004), L117-121.
27. E. M. Rabei, T. S. Alhalholy, A. A. Taani, On Hamiltonian Formulation of Non-Conservative Systems, *Turk. J. Phys.* 28 (2004), 213-221.
28. J. Cresson, P. Inizan, Irreversibility, least action principle and causality, arXiv: 0812.3529.
29. P. Inizan, Compatibility between fractional Hamiltonian systems, *Int. J. Ecol. Econ. Stat.* 9, (2007) No. F07, 83-91.
30. A. A. Stanislavsky, Hamiltonian formalism of fractional systems, *Eur. Phys. J. B*49 (2006), 93-101.
31. G.M. Zaslavsky, Hamiltonian Chaos & Fractional Dynamics, Oxford University Press, Oxford (2005).
32. M. D. Ortigueira, Fractional calculus for scientists and engineers, Lecture Notes in Electrical Engineering, 84, Springer, Dordrecht (2011).
33. A. B. Malinowska and D. F. M. Torres, Introduction to the fractional calculus of variations, Imperial College Press, London & World Scientific Publishing, Singapore, (2012).
34. A. Weinstein, On a singular differential operator, Ann. Mat. Pura Appl. 49 (1960), 359–365.
35. F. Bagarello, Nonstandard Variational Calculus with applications to Classical Mechanics. 1. An Existence Criterion, *Int. J. Theor. Phys.* 38 (1999), No. 5, 1569-1592.

36. Z. E. Musielak, Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, *J. Phys. A: Math. Theor.* 41 (2008), 055205-055222.
37. Z. E. Musielak, General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems, *Chaos Solitons and Fractals* 42 (2009), No. 15, 2645-2652.
38. V. Neves, Nonstandard calculus of variations, *J. Math. Scie.* 120 (2004), No. 1, 940-954.
39. R. A. El-Nabulsi, Quantum field theory from an exponential action functional, *Indian J. Phys.* 87 (2013), 379-383.
40. R. A. El-Nabulsi, Modified Proca equation and modified dispersion relation from a power-law Lagrangian functional, *Indian J. Phys.* 87 (2013), 465-470; Erratum *Indian J. Phys.* 87 (2013), p. 1059.
41. V. K. Chandrasekar, S. N. Pandey, M. Senthilvelan, M. Lakshmanan, A simple and unified approach to identify integrable nonlinear oscillators and systems, *J. Math. Phys.* 47, (2006) 023508-023545.
42. V. K. Chandrasekar, M. Senthilvelan, M. Lakshmanan M., On the Lagrangian and Hamiltonian description of the damped linear harmonic oscillator, *Phys. Rev.* E72, (2005) 066203-066211.
43. J. G. Carinena, M. F. Ranada, M. Santander, Lagrangian formalism for nonlinear second-order Riccati Systems: one-dimensional integrability and two-dimensional superintegrability, *J. Math. Phys.* 46, (2005) 062703-062721.
44. V. I. Arnold, Mathematical Methods of Classical Mechanics, placeStateNew York: Springer, (1978).
45. D. D. Dimitrijevic, M. Milosevic, About non-standard Lagrangians in cosmology, PROCEEDINGS OF THE PHYSICS CONFERENCE TIM - 11. AIP Conference Proceedings, Volume 1472, (2012) pp. 41-46.
46. N. A. Tyurin, Nonstandard Lagrangian tori and pseudotoric structures, *Teoret. Mat. Fiz.* 171, (2012) 321-325.
47. D. Lukkassem, Reiterated homogenization of non-standard Lagrangians, *C. R. Acad. Sci. Paris*, t. 332, Srie I, (2001) p. 999–1004.
48. D. Baleanu, S. I. Muslih, A. K. Golmankhaneh, Fractional mechanics on the extended phase space, Proceedings of the ASME 2009 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference, IDETC/CIE 2009, August 30-Sptember 2, San Diego, California USA.
49. Abramowitz M., Stegun I. A.: eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, NBS Applied Mathematics Series 55, National Bureau of Standards, CityplaceWashington, StateDC (1964).
50. P. S. Bauer. Dissipative dynamical systems I, *Proc. Nat. Acad. Scie.* 17 (1931), 311–314.
51. R.A. El-Nabulsi, M. Jamil, G.-c. Wu, Complexified Lie algebroids from generalized Stieltjes action approach to the calculus of variations, Sarajevo J. Math. 8, 20, (2012) 1-16.
52. R. A. El-Nabulsi, Nonlinear dynamics with non-standard Lagrangians, *Qual. Theo. Dyn. Syst.* 12 (2013), 273-291.
53. R. A. El-Nabulsi, Calculus of variations with hyperdifferential operators from Tabasaki–Takebe–Toda lattice arguments, *RACSAM* 107 (2013) 419-436.
54. D. Baleanu, S. I. Muslih. Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives. *Phys. Scrip.* 72(2-3) (2005) 119-121.
55. V. E. Tarasov, Fractional vector calculus and fractional Maxwell's equations, *Ann. Physics* 323, (2008) 2756–2778 (2008)
56. R. Herrmann, Gauge invariance in fractional field theories, *Phys. Lett.* A372, (2008) 5515–5522.
57. M. J. Lazo, Gauge invariant fractional electromagnetic fields, *Phys. Lett.* A375, (2011) 3541–3546.
58. R. A. El-Nabulsi, G.-c. Wu, Fractional Complexified Field Theory from Saxena-Kumbhat Fractional Integral, Fractional Derivative of Order (alpha, beta) and Dynamical Fractional Integral Exponent, *Afr. Diaspora. J. Math.*13, (2012) No. 1, 45-61.