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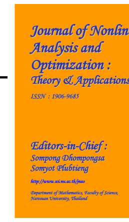

ON A GENERAL CLASS OF MULTI-VALUED STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, a general class of multi-valued strictly pseudocontractive mappings, which properly includes the class of multi-valued $k-$strictly pseudocontractive mappings, is introduced. Furthermore, it is proved that if T belongs to this class of mappings and the set of fixed points of T is nonempty, a Krasnoselskii-type sequence is constructed and proved to be an approximate fixed point sequence of T. Finally, convergence of the sequence to a fixed point of T is proved under appropriate additional conditions.

KEYWORDS : Iterative Approximation, Multi-valued Maps, Strictly Pseudocontractive Mappings.
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1. INTRODUCTION

Let E be a metric space endowed with a metric d and K be a nonempty closed, convex subset of E. Let $CB(K)$ denote the collection of closed and bounded subsets of K and $T : K \to CB(K)$ be a multi-valued mapping.
In recent years, the study of fixed point theory for multi-valued nonlinear mappings T has attracted, and continues to attract, the interest of several well known mathematicians. Interest in this type of mappings is, perhaps, due to its many real world applications, for example, in Game Theory and Market Economy and in Non-Smooth Differential Equations (see e.g., Chidume *et al.* [7] for details).

The applications of fixed point theory for multi-valued mappings on the problem of differential equations (DEs) with discontinuous right-hand sides gave birth to the existence theory of differential inclusions (DIs). Most recent results for game theory showed that equilibrium points of games correspond to fixed points of some multi-valued mappings, under appropriate conditions. A model example of such an application, the *Nash equilibrium theorem*, (see e.g Nash [21, 22]) showed that

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the existence of equilibria for non-cooperative static games is a direct consequence of fixed point theorems of Brouwer [2] or Kakutani[15].

From the point of view of social recognition, game theory is, perhaps, the most successful area of application of fixed point theory of multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a nonequilibrium point and convergent to an equilibrium solution. This is part of the problem that is being addressed by *iterative methods* for fixed point of multi-valued mappings. It is worth mentioning, at this juncture also, that iterative methods for approximating fixed points of nonexpansive mappings constitute the central tools used in signal processing and image reconstruction (see, e.g., Byrne[5]).

For early results involving fixed points of multi-valued mapping, (see, for example, Brouwer [2], Kakutani [15], Nash [21, 22], Geanakoplos [14], Downing and Kirk [10]). For details on the applications of this type of mappings in Nonsmooth Differential Equations, one may consult, for example, Chang [6], Chidume [7], Deimling [9], Erbe and Krawcewicz [11], Frigon [12], Nadler [20], Ofoedu and Zegeye [23], Reich *et al.* [25, 26, 27] and the references therein.

Fixed point problems involving a multi-valued mapping T can be reformulated as a zero problem for a multi-valued mapping A, namely;

$$\text{Find } 0 \in Ax, \qquad \text{where } A = I - T$$

and I is the identity mapping of K.

Many problems in applications can be modeled in the form of $0 \in Ax$, where, for example. $A : H \to 2^H$ is a monotone operator, that is $\langle u - v, x - y\rangle \geq 0$ for all $u \in Ax, v \in Ay, x, y \in H$. Typical examples include the equilibrum state of evolution equations and critical points of some functionals defined on Hilbert spaces. For example, let $f : H \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. It is known (see e.g., Rockafellar [29], Minty [19]) that the multi-valued mapping $T := \partial f$, *the subdifferential* of f, is maximal monotone, where for each $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w\rangle, \quad \forall y \in H, \\ &\Leftrightarrow x \in Argmin(f - \langle ., w\rangle). \end{aligned}$$

In this case, a solution of the inclusion $0 \in \partial f(x)$, if any, is a critical point of f, which is precisely a minimizer of f.

The *proximal point algorithm* introduced by Martinet [18], and studied extensively by Rockafeller [28], which has also been studied by a host of other authors, is connected with the iterative algorithm for solutions of $0 \in Ax$ where A is a maximal monotone operator on a Hilbert space.

In studying the equation $Au = 0$, Browder [4], introduced an operator T given by $T := I - A$, where I is the identity mapping on H. He called such an operator a *pseudocontractive mapping*. It is easily seen that the solutions of $Ax = 0$ when A is monotone are precisely the fixed points of pseudocontractive mapping T. Every nonexpansive mapping is pseudocontractive and continuous but a pseudocontractive mapping is not neccesarily continuous. Thus the study of iterative methods

for fixed points of pseudcontractive mappings requires some continuity conditions (e.g., Lipschitz condition).
While pseudocontractive mappings are generally not continuous, a subclass of pseudocontractive mappings, the *strictly pseudocontractive mappings*, inherits Lipschitz property from their definitions. The study of fixed point theory for strictly pseudocontractive mappings helps in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudocontractive mappings. Consequently, the study by several authors of iterative methods for fixed point of multi-valued strictly pseudocontractive mappings has motivated our study of a more *general class* of multi-valued strictly pseudocontractive mappings which certainly includes the important class of multi-valued nonexpansive maps.

In this paper, we extend the notion of single-valued strictly pseudo-contractive mappings, defined by Browder and Petryshyn [3] on Hilbert spaces, and the notion of multi-valued stricly pseudocontractive mappings (see e.g., [7], [23]), to a *general class* of multi-valued strictly pseudocontractive mappings. This class is shown to properly contain the class of multi-valued strictly pseudocontractive mappings introduced by Chidume *et al* [7] which itself contains the important class of multi-valued nonexpansive mappings and consequently properly contains the class of single-valued strictly pseudo-contractive maps introduced by Browder and Petryshyn [3].

Part of the novelty of this paper is that for the general class of maps considered here, convergence theorems for Krasnoselskii-type sequence, which is known to be superior to the Mann-type and Ishikawa-type sequences, are still applicable. In particular, the Krasnoselskii-type iteration sequence $\{x_n\}$ constructed is proved to be an approximate fixed point sequence of T and then, under appropriate additional conditions, convergence to a fixed point is established.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, we will adopt the following:

(i) $x_n \to x$: $\{x_n\}$ converges strongly to x.
(ii) H: a Hilbert Space with an induced norm $\|.\|$.
(iii) $F(T) := \{x \in K : x \in Tx\}$.
(iv) 2^H is the power set of H.
(v) $CB(K)$, is the family of nonempty, closed and bounded subsets of K

We recall some definitions and facts that are needed in our study.

Definition 2.1. Let $(X; d)$ be a metric space and K a nonempty subset of X. The *Hausdorf metric* on $CB(X)$ is given by

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A, B \in CB(X)$.

To simplify notation, we shall denote $(D(A,B))^2$ by $D^2(A,B)$ for all $A, B \in CB(X)$.

Browder and Petryshin [3] introduced this class of single-valued mappings.

Definition 2.2. Let K be a nonempty subset of a Hilbert space H. A map $T : K \to H$ is called *strictly pseudo-contractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in K. \tag{2.1}$$

Definition 2.3. (Chidume *et al.* [7]) Let H be a real Hilbert space and let D be a nonempty, open and convex subset of H. Let $T : \overline{D} \to CB(\overline{D})$ be a mapping. Then, T is called *a multi-valued $k-$strictly pseudocontractive mapping* if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{2.2}$$

for all $u \in Tx, \ \ v \in Ty$.

Remark 2.4. If $k = 1$, in definition (2.3), the mapping T is called *pseudocontractive* and if $k = 0$, it is called *nonexpansive*. Notice also that the mapping becomes nonexpansive if $x \in Tx \ \ \forall x \in \overline{D}$.

Definition (2.3) is an extension of the definition of single-valued pseudo-contractive mappings to multi-valued maps.

Definition 2.5. A multi-valued mapping $T : K \subseteq H \to CB(H)$ is called Lipschitzian if there exists $L > 0$ such that

$$D(Tx, Ty) \leq L\|x - y\|, \tag{2.3}$$

for each $x, y \in K$. If $L < 1$ in inequality (2.3), the mapping T is called a *contraction* and if $L = 1$, it is *nonexpansive*.

Several authors have studied the problem of approximating fixed points of *multi-valued* nonexpansive mappings (see, e.g., [1], [8], [16, 17], [24], [30], [32], and the references therein), and of their generalizations (see, e.g., [8], [13]).

Definition 2.6. A map $T : K \to CB(K)$ is said to be *hemicompact* if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Definition 2.7. Let H be a real Hilbert space and let T be a multi-valued mapping. The multi-valued mapping $I - T$ is said to be *strongly demiclosed* at 0 (see, e.g., [13]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $x_n \to p$ and $d(x_n, Tx_n)$ converges strongly to 0, then $d(p, Tp) = 0$.

The projection mapping P_K onto a nonempty, closed and convex subset of H has the following characterization:

Lemma 2.8. *Let H be a Hilbert space, $K \subset H$ be nonempty, closed and convex, $z \in H$ and $x \in K$. Then $x = P_K z$ if and only if*

$$\langle z - x, w - x\rangle \leq 0 \quad \forall w \in K.$$

The following lemma will also be used in the sequel.

Lemma 2.9. *([34]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n, \ \ n \geq 0,$$

such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.

3. MAIN RESULTS

We first prove the following preliminary results.

Lemma 3.1. *Let E be a normed linear space , $B_1, B_2 \in CB(E)$ and $x_0, y_0 \in E$ arbitrary. The following hold;*

(a) $D(B_1, B_2) = D(x_0 + B_1, x_0 + B_2)$.
(b) $D(B_1, B_2) = D(-B_1, -B_2)$.
(c) $D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2)$.
(d) $D(\{x_0\}, B_1) = \sup_{b_1 \in B_1} \|x_0 - b_1\|$.
(e) $D(\{x_0\}, B_1) = D(0, x_0 - B_1)$.

Proof. (a) By definition, we have

$$\begin{aligned} D(x_0 + B_1, x_0 + B_2) &= \max\Big\{ \sup_{b_1 \in B_1} d(x_0 + b_1, x_0 + B_2); \sup_{b_2 \in B_2} d(x_0 + b_2, x_0 + B_2)\Big\} \\ &= \max\{ \sup_{b_1 \in B_1} d(b_1, B_2); \sup_{b_2 \in B_2} d(b_2; B_1)\} \\ &= D(B_1, B_2). \end{aligned}$$

(b) We have

$$\begin{aligned} D(-B_1, -B_2) &= \max\{ \sup_{-b_1 \in -B_1} d(-b_1; -B_2); \sup_{-b_2 \in -B_2} d(-b_2; -B_1)\} \\ &= \max\{ \sup_{b_1 \in B_1} d(b_1; B_2); \sup_{b_2 \in B_2} d(b_2; B_1)\} \\ &= D(B_1, B_2). \end{aligned}$$

(c) It is known that for any set $B \subseteq E$, $x, y \in E$ arbitrary, the inequality

$$d(x, B) \leq \|x - y\| + d(y, B)$$

holds. Using this inequality we have

$$\begin{aligned} d(x_0 + b_1, y_0 + B_2) &\leq \|(x_0 + b_1) - (y_0 + b_1\| + d(y_0 + b_1, y_0 + B_2) \\ &= \|x_0 - y_0\| + d(b_1, B_2), \end{aligned}$$

and similarly

$$d(y_0 + b_2, x_0 + B_1) \leq \|x_0 - y_0\| + d(b_2, B_1).$$

Therefore, taking sup over B_1 and B_2 respectively, we have

$$\sup_{b_1 \in B_1} d(x_0 + b_1, y_0 + B_2) \leq \|x_0 - y_0\| + \sup_{b_1 \in B_1} d(b_1, B_2),$$

and

$$\sup_{b_2 \in B_2} d(y_0 + b_2, x_0 + B_1) \leq \|x_0 - y_0\| + \sup_{b_2 \in B_2} d(b_2, B_1).$$

Thus $D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2)$.

(d) It is obvious that $d(x_0; B_1) = \sup_{x_0 \in \{x_0\}} d(x_0, B_1)$. On the otherhand, for any $b_1 \in B_1$, we have

$$d(b_1; \{x_0\}) = \|b_1 - x_0\| \geq d(x_0; B_1).$$

Taking sup over B_1 we have

$$\sup_{b_1 \in B_1} d(b_1, \{x_0\}) \geq d(x_0, B_1),$$

and therefore

$$D(\{x_0\}, B_1) := \max\{\sup_{b_1 \in B_1} d(b_1; \{x_0\}); \sup_{x_0 \in \{x_0\}} d(x_0; B_1)\} = \sup_{b_1 \in B_1} d(b_1, \{x_0\}).$$

(e)

$$\begin{aligned} D(\{x_0\}, B_1) &:= \max\{\sup_{b_1 \in B_1} d(b_1, \{x_0\}), d(x_0, B_1)\} \\ &= \max\{\sup_{b_1 \in B_1} \|x_0 - b_1\|), \inf_{b_1 \in B_1} \|x_0 - b_1\|\} \\ &= \max\{\sup_{b_1 \in B_1} d(0, x_0 - B_1), d(0, x_0 - B_1)\} \\ &= D(\{0\}, x_0 - B_1). \end{aligned}$$

□

We now introduce the following class of *generalized $k-$ strictly pseudocontractive multi-valued mappings*.

Definition 3.2. Let H be a real Hilbert space and let K be a nonempty subset of H. Let $T : K \to CB(K)$ be a multi-valued mapping. Then T is called **generalized $k-$strictly pseudocontractive multi-valued mapping** if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + kD^2(Ax, Ay), \quad A := I - T, \tag{3.1}$$

and I is the identity operator on K.

Remark 3.3. Definition (3.2) seems to be a more natural generalization of the single-valued definition (2.2) given by Browder and Petryshin [3] than the definition (2.3) given by Chidume *et al.* [7].

Proposition 3.4. *Let $T : K \to CB(K)$ be a multi-valued $k-$strictly pseudocontractive mapping, then T is a generalized $k-$strictly pseudocontractive multi-valued mapping.*

Proof. Given that T is a multi-valued $k-$strictly pseudocontractive mapping, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k \inf_{(u,v) \in (Tx, Ty)} \|(x - u) - (y - v)\|^2. \tag{3.2}$$

We now show that inequality (3.2) implies inequality (3.1).

$$\begin{aligned} D(x - Tx, y - Ty) :&= \max\left\{\sup_{u \in Tx} d(x - u; y - Ty); \sup_{v \in Ty} d(y - v; x - Tx)\right\} \\ &\geq \sup_{u \in Tx} d(x - u; y - Ty) \\ &\geq d(x - u_0; y - Ty), \qquad u_0 \in Tx. \end{aligned}$$

Now, given $\epsilon > 0$, there exist $v_\epsilon \in Ty$ such that

$$\begin{aligned} d(x-u_0; y-Ty) &\geq \|(x-u_0)-(y-v_\epsilon)\| - \epsilon \\ &\geq \inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| - \epsilon. \end{aligned}$$

Thus, for arbitrary $\epsilon > 0$, we have

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty) + \epsilon,$$

and therefore, since $\epsilon > 0$ is arbitrary, we have:

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty). \tag{3.3}$$

We therefore obtain from (3.2) and (3.3) that:

$$D^2(Tx,Ty) \leq \|x-y\|^2 + kD^2(x-Tx, y-Ty).$$

□

Thus, every multi-valued $k-$strictly pseudocontractive mapping is also a *generalized* $k-$strictly pseudocontractive multi-valued mapping.

We now give an example to show that this inclusion is proper.

For the example, we shall need the following lemma which is trivially proved.

Lemma 3.5. *Let* a, b *be real numbers such that* $0 \leq a \leq 4b$. *Then,*

$$(a-b)^2 \leq b^2 + \frac{1}{2}a^2. \tag{3.4}$$

Example 3.6. Let H be a real Hilbert space. Define a mapping

$$T : H \to CB(H) \quad \text{by}$$

$$Tx := \begin{cases} \overline{B}(-x, \|x\|), & \|x\| > 0 \\ 0, & x = 0, \end{cases}$$

where

$$\overline{B}(-x, \|x\|) = \{u \in H : \|u + x\| \leq \|x\|\}.$$

Then, for distinct nonzero x and y, we have the following identities which follow from the definition of T:

$$\begin{aligned} x - Tx &= \overline{B}(2x, \|x\|), \\ y - Ty &= \overline{B}(2y, \|y\|), \\ Tx &= \{w \in H : \|w + x\| \leq \|x\|\}, \\ Ty \setminus Tx &= \{z \in H : \|z + y\| \leq \|y\|, \|z + x\| > \|x\|\}. \end{aligned}$$

We now establish the following equation:

$$D(Tx,Ty) = \|x-y\| + \Big|\|y\| - \|x\|\Big|. \tag{3.5}$$

First, we consider the case $\|y\| \geq \|x\|$. We proceed as follows:

Claim 1: $\forall z \in Ty \setminus Tx$, $d(z, Tx) = \|z - P_{(Tx)}z\|$, where

$$P_{(Tx)}z := -x + \frac{\|x\|}{\|x+z\|}\Big(z + x\Big). \tag{3.6}$$

Proof of Claim 1. Let $w \in Tx$. Then, $\|w + x\| \leq \|x\|$. Furthermore,

$$\begin{aligned}
\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle &= \Big\langle z + x - \frac{\|x\|}{\|x+z\|}(z+x), w + x - \frac{\|x\|}{\|x+z\|}(z+x)\Big\rangle \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \frac{\|x\|}{\|x+z\|}\Big\langle z+x, z+x\Big\rangle\Big) \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \|x\|\|z+x\|\Big) \\
&\leq \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\|z+x\|\|w+x\| - \|x\|\|z+x\|\Big) \\
&\leq (\|x+z\| - \|x\|)\Big(\|x\| - \|x\|\Big) \\
&= 0.
\end{aligned}$$

Thus, it follows that,

$$\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle \leq 0,$$

and applying Lemma(2.8), the claim is proved. Also note that $P_{(Tx)}z$ is unique for each z since H is a real Hilbert space. Now, set

$$z_0 := -x + \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)(x - y). \tag{3.7}$$

Clearly $z_0 \in Ty \setminus Tx$ since

$$\|z_0 + y\| = \Big\|\frac{\|y\|}{\|x-y\|}(x-y)\Big\| = \|y\|$$

and,

$$\begin{aligned}
\|z_0 + x\| = \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)\|x - y\| &= \|x-y\| + \|y\| \\
&\geq \|x-y\| + \|x\| \\
&> \|x\|.
\end{aligned}$$

Moreover, from equation (3.6),

$$\begin{aligned}
P_{(Tx)}z_0 &= -x + \frac{\|x\|}{\|x+z_0\|}\Big(z_0 + x\Big) \\
&= -x + \frac{\|x\|}{\|x-y\| + \|y\|}\Big[\frac{\|x-y\| + \|y\|}{\|x-y\|}(x-y)\Big] \\
&= -x + \frac{\|x\|}{\|x-y\|}\Big(x - y\Big).
\end{aligned} \tag{3.8}$$

Therefore , using (3.8) and (3.7) we obtain that

$$d(z_0, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x-y\| + \|y\| - \|x\|, \tag{3.9}$$

establishing Claim 1.

Claim 2: $d(z_0, Tx) = \sup_{v \in Ty} d(v, Tx)$.

Proof of Claim 2. Let $z \in Ty \setminus Tx$ be arbitrary. We have,

$$\|z+y\| \le \|y\|, \ \ \|z+x\| > \|x\|,$$

and so, using equation (3.9),

$$\begin{aligned}\|z - P_{(Tx)}z\| &= \Big\|z + x - \frac{\|x\|}{\|x+z\|}\Big(z+x\Big)\Big\| \\ &= \|z+x\|\Big(1 - \frac{\|x\|}{\|x+z\|}\Big) \\ &= \|x+z\| - \|x\| \\ &\le \|x-y\| + \|z+y\| - \|x\| \\ &\le \|x-y\| + \|y\| - \|x\| \\ &= \|z_0 - P_{(Tx)}z_0\|.\end{aligned}$$

For $z \in (Ty \cap Tx)$, we have $d(z, Tx) = 0$. Thus, we obtain that,

$$d(z, Tx) \le \|z_0 - P_{(Tx)}z_0\| \quad \forall z \in Ty.$$

Using the fact that $z_0 \in Ty$, we obtain

$$\sup_{v \in Ty} d(v, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x-y\| + \|y\| - \|x\|.$$

Thus, Claim 2 is established.

We now consider the case $\|x\| \ge \|y\|$.
For $\|x\| \ge \|y\|$, we have by interchanging the roles of x and y,

$$\sup_{u \in Tx} d(u, Ty) = \|x-y\| + \|x\| - \|y\|.$$

Therefore,

$$\max\Big\{\sup_{y \in Ty} d(y, Tx), \sup_{x \in Tx} d(x, Ty)\Big\} = \|x-y\| + \Big|\|y\| - \|x\|\Big|. \tag{3.10}$$

For $x = y$, $Tx = Ty$, and $D(Tx, Ty) = 0$. Moreover, for $x = 0, \ y \neq 0$, a straightforward computation gives

$$D(0, Ty) = 2\|y\| = \|0 - y\| + \Big|0 - \|y\|\Big|.$$

Thus, the identity (3.5) is fully satisfied for arbitrary $x, y \in H$.

Following a similar procedure, we obtain

$$D(x - Tx, y - Ty) = 2\|x-y\| + \Big|\|y\| - \|x\|\Big| \quad \forall x, y \in H. \tag{3.11}$$

We set

$$\begin{aligned} a &:= D(x - Tx, y - Ty). \\ b &:= \|x - y\|. \end{aligned}$$

Then, using equations (3.11) and (3.5), we obtain that,

$$a - b = D(Tx, Ty).$$

Clearly, by equation (3.11),

$$a = 2\|x-y\| + \Big|\|y\| - \|x\|\Big| \le 4\|x-y\| = 4b.$$

Therefore, by Lemma (3.5),

$$D^2(Tx,Ty) \leq \|x-y\|^2 + \frac{1}{2}\Big(D(x-Tx,y-Ty)\Big)^2 \qquad \forall x,y \in H.$$

Therefore, T is a generalized $k-$strictly pseudocontractive multi-valued mapping with $k=\frac{1}{2}$.

We now show that T is not a multi-valued $k-$strictly pseudocontractive mapping in the sense of definition (2.3).

We establish this by contradiction. So, assume that there exists $k \in [0,1)$ such that inequality (2.2) holds. Choose $x \in H \setminus \{0\}$. Set $y=2x$, $u=v=0 \in (Tx \cap Ty)$. Then,

$$||x-y|| = \|x\|,$$

$$D(Tx,Ty) = \|x-y\| + \Big|\|y\| - \|x\|\Big| = 2\|x\|,$$

and

$$\|(x-u)-(y-v)\| = \|x-y\| = \|x\|.$$

Thus,

$$4\|x\|^2 = D^2(Tx,Ty) \leq \|x-y\|^2 + k\|(x-u)-(y-v)\|^2 \leq 2\|x\|^2.$$

This is a contradiction to $x \in H \setminus \{0\}$. Therefore, T is not a multi-valued $k-$strictly pseudocontractive mapping for any $k \in (0,1)$.

To prove our main theorem, we first prove the following important propositions.

Proposition 3.7. *Let K be a nonempty subset of a real Hilbert space H and $T: K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then T is Lipschitzian.*

Proof. Let $x,y \in D(T)$. Then,

$$\begin{aligned} D^2(Tx,Ty) &\leq \|x-y\|^2 + kD^2(x-Tx,y-Ty) \\ &\leq \|x-y\|^2 + k\Big(\|x-y\| + D(Tx,Ty)\Big)^2, \text{by Lemma (3.1), (c), (b).} \\ &\leq \Big(\|x-y\| + \sqrt{k}\|x-y\| + \sqrt{k}D(Tx,Ty)\Big)^2. \end{aligned}$$

Thus,

$$D(Tx,Ty) \leq (1+\sqrt{k})\|x-y\| + \sqrt{k}D(Tx,Ty),$$

and hence,

$$D(Tx,Ty) \leq \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x-y\Big\|,$$

as proposed. ∎

Remark 3.8. Propostion (3.7) is an improvement of Proposition 8 of [7] because it does not assume that Tx *is weakly closed for each* $x \in K$.

Proposition 3.9. *Let K be a nonempty and closed subset of a real Hilbert space H and let $T: K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then, $(I-T)$ is strongly demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in K such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$. For each $n \in \mathbb{N}$, take $y_n \in Tx_n$ such that $\|x_n - y_n\| \leq d(x_n, Tx_n) + \frac{1}{n}$.
Then,

$$\begin{aligned} d(x,Tx) &\leq \|x - x_n\| + \|x_n - y_n\| + d(y_n, Tx) \\ &\leq \|x - x_n\| + d(x_n, Tx_n) + \frac{1}{n} + D(Tx_n, Tx) \\ &\leq \|x - x_n\| + d(x_n, Tx_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_n - x\Big\|. \end{aligned}$$

Thus, taking limits on both sides as $n \to \infty$, we have $d(x, Tx) = 0$. Since Tx is closed, $x \in Tx$. □

Now, using lemma (3.1)(d), we have that

$$D(\{x_n\}, Tx_n) = \sup_{y_n \in Tx_n} \|x_n - y_n\|.$$

Thus, for any given sequence $\{x_n\} \subseteq K$, the set

$$U^n := \Big\{ y_n \in Tx_n : D^2(\{x_n\}, Tx_n) \leq \|x_n - y_n\|^2 + \frac{1}{n^2} \Big\},$$

is nonempty.

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty, closed, convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping such that $F(T) \neq \emptyset$. Assume $Tp = \{p\}$ $\forall p \in F(T)$. Define a sequence $\{x_n\}$ by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n \tag{3.12}$$

where $y_n \in U^n$ and $\lambda \in (0, 1-k)$. Then, $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Proof. Let $p \in F(T)$. Then, using Lemma(3.1), (d) and (e), we have

$$\begin{aligned} &\|x_{n+1} - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda D^2(Tx_n, Tp) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big(\|x_n - p\|^2 + kD^2(x_n - Tx_n, 0)\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k D^2(\{x_n\}, Tx_n) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k\Big(\|x_n - y_n\|^2 + \frac{1}{n^2}\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2. \\ &= \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2. \end{aligned}$$

Thus,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2. \tag{3.13}$$

By Lemma (2.9), the sequence $\Big\{\|x_n - p\|\Big\}$ has a limit and therefore, $\{x_n\}$ is bounded. Moreover, we have from inequality (3.13) that

$$\lambda(1-\lambda-k)\|x_n - y_n\|^2 \le \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \|x_{n+1} - p\|^2,$$

and then,

$$\begin{aligned}\lambda(1-\lambda-k)\sum_{n=1}^{m}\|x_n - y_n\|^2 &\le \sum_{n=1}^{m}\|x_n - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2} - \sum_{n=1}^{m}\|x_{n+1} - p\|^2 \\ &= \|x_1 - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2} - \|x_{m+1} - p\|^2 \\ &\le \|x_1 - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2.}\end{aligned}$$

This implies that

$$\lambda(1-\lambda-k)\sum_{n=1}^{\infty}\|x_n - y_n\|^2 \le \|x_1 - p\|^2 + \sum_{n=1}^{\infty}\lambda k\frac{1}{n^2} < \infty,$$

and therefore $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $d(x_n, Tx_n) \le \|x_n - y_n\|$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Corollary 3.10. *Let K be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Suppose that T is hemicompact. Then, the sequence $\{x_n\}$ defined by equation (3.12) converges strongly to a fixed point of T.*

Proof. By Theorem (3.1), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemicompact, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \to q$ as $n \to \infty$ and let $y_{n_k} \in Tx_{n_k}$ such that
$\|x_{n_k} - y_{n_k}\| \le d(x_{n_k}, Tx_{n_k}) + \frac{1}{k}$.. Then

$$\begin{aligned}d(q, Tq) &\le \|q - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| + d(y_{n_k}, Tq) \\ &\le \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + D(Tx_{n_k}, Tq) \\ &\le \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_{n_k} - q\Big\|.\end{aligned}$$

Thus, taking limits on the righthand side as $k \to \infty$, we have $d(q, Tq) = 0$. Since Tq is closed, $q \in Tq$. Moreover, $x_{n_k} \to q$ as $n \to \infty$ gives $\|x_{n_k} - q\| \to 0$ as $n \to \infty$. Thus, using inequality (3.13) and Lemma (2.9), $\lim_{n\to\infty}\|x_n - q\| = 0$. Therefore $\{x_n\}$ converges strongly to a fixed point q of T as claimed. □

Remark 3.11. Observe that we did not assume that Tx is proximinal for each $x \in K$ neither did we require any continuity assumption on T nor any compactness assumption on K. Consequenctly, Corollary(3.10) is a significant improvement on Corollary13 of [7]

Corollary 3.12. *Let K be a nonmepty, compact and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Then, the sequence $\{x_n\}$ defined by equation (3.12) converges strongly to a fixed point of T.*

Proof. Since K is compact, every map $T : K \to CB(K)$ is hemicompact. Thus, by Corollary 3.10, we have that $\{x_n\}$ converges strongly to some $p \in F(T)$.

□

Remark 3.13. Our theorem and corollaries improve convergence theorems for multi-valued nonexpansive mappings in [1], [7], [16, 17],[24, 30, 32], in the following sense:

- (i) The class of mappings considered in this paper contains the class of multi-valued $k-$ strictly pseudocontractive mappings as special case, which itself properly contain the class of multi-valued nonexpansive maps.
- (ii) The algorithm here is Krasnoselkii type, which is known to have a geometric order of convergence, and the theorem is proved for the much larger class of generalized multi-valued strict pseudocontractive mappings.
- (iii) Inequality (3.1) of definition (3.2) is a more natural generalisation of the single-valued psudo-contractive mappings as given by inequality(2.1).
- (iv) The condition that Tx be weakly closed for each $x \in K$ imposed in [7] is dispensed with here.

We conclude, by saying that the condition $T(p) = \{p\}$ for all $p \in F(P)$, which is imposed in our theorem and corollaries is not crucial. Certainly our example (3.6) satisfies the condition since $T0 = \{0\}$ is the unique fixed point of T. However, some work in the litrature shows that this condition can be replaced with another condition which does not assume that the multi-valued mapping is single-valued on the nonempty fixed point set. Deatails of this can be found, for example, in [31], [33].

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