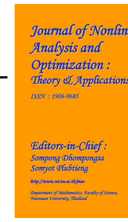
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 1-6
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



A NOTE ON QUASI SPLIT NULL-POINT FEASIBILITY PROBLEMS

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ABSTRACT. Inspired by the very recent work by M.-A. Noor and Kh.-I Noor [9] and given a closed convex set-valued mapping C, we propose a split algorithm for solving the problem of finding an element x^* in $C(x^*)$ such that its image, Ax^*, under a linear operator, A, is a zero of a given maximal monotone operator T in Hilbert spaces setting. Then, we present a strong convergence result and state some examples as applications.

KEYWORDS : Fixed-point; monotone operator; quasi split feasibility problem.
AMS Subject Classification: Primary, 49J53, 65K10; Secondary, 49M37, 90C25.

1. INTRODUCTION AND PRELIMINARIES

Throughout, H is a Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the inner product and $\| \cdot \|$ stands for the corresponding norm. The split feasibility problem (SFP) has received much attention due to its applications in image denoising, signal processing and image reconstruction, with particular progress in intensity-modulated therapy. For a complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography see, for example [1] and for split convex feasibility problem see, for instance, the excellent paper [5] and the references therein. Inspired by the idea developed in [9], our interest in this paper is on the study of the convergence of an algorithm for solving a Quasi Split Null-point Feasibility Problem, i.e., the case where the constrained set, instead of being fixed, is a set-valued mapping. Besides being a more general case, it also has many applications, see for example [2]) and is an extension of the problem introduced in [9]. At this stage, we would like to emphasize that the result in [9] is not correct. Indeed, clearly the constant θ defined by relation (16) in [9] is greater than 1. As a consequence the application F defined by relation (12) in [9] is not a contraction and thus the Banach fixed-point principle is not applicable. Actually, since the operator $A^*(I - P_C)A$ is firmly nonexpansive, it is easily seen

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Article history : Received September 04, 2014, Accepted July 21, 2013.

then that the best value of θ is in fact $\theta = 1 + \mu$ which is still greater than 1. Hence, we cannot apply again the Banach fixed-point principle. To overcome this difficulty, one way (maybe the only one) is to consider the more general problem (2.1) and assume that both the linear operator A and the set-valued mapping T are strongly monotone operators. Note that by taking $A = I$, the identity mapping and $T = \partial\phi$, the subdifferential of a strongly convex proper lower semi-continuous function, problem (2.1) reduces to minimizing the strongly convex function ϕ with respect to the implicit convex set $C(x)$. To be in a position to apply the fixed-point Banach principle and by observing that the fixed-point reformulation of the problem considered in [9] involves the projection operator over convex sets and that the techniques are strongly based on its properties which do not depend on any parameter in contrast to the resolvent and proximal mappings, we will consider a quasi split null-point feasibility problem, propose a strong convergence result and provide some applications. This will be done by taking advantage of the resolvent techniques which depend on parameters that allow more flexibility. Moreover, an appropriate choice amounts to weakened the assumptions on the data.

To begin with, let us recall that the split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Assuming that the (SFP) is consistent (i.e (1.1) has a solution), it is no hard to see that $x \in C$ solves (1.1) if and only if it solves to fixed-point equation

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x,\ x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.

To solve the (1.2), Byrne [4] proposed his CQ algorithm which generates a sequence (x_k) by

$$x_{k+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_k,\ k \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

2. MAIN RESULTS

In the sequel, we will focus our attention on the following implicit null-point feasibility problem

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* \in T^{-1}(0), \tag{2.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $T : H_1 \to H_1$ a maximal monotone operator and $C : H_1 \to 2^{H_1}$ be a set-valed map with closed convex values.

It is easy to see, using the normal cone to $C(x^*)$, that (2.1) is equivalent to the following fixed-point formulation $x^* = P_{C(x^*)}(x^* - \gamma A^*(I - J^T_{\lambda_k})Ax^*)$. To solve (2.1), the latter suggests the use of the following algorithm:

Algorithm (QSFA): *Initialization*: Let $\lambda_0 > 0$ and $x_0 \in H_1$ be arbitrary.
Iterative step:

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - J^T_{\lambda_k})Ax_k),\ k \in \mathbb{N}, \tag{2.2}$$

where $\gamma \in]0, \frac{2\nu_k\mu^2}{L_A^2}[$ with L_A being the spectral radius of the operator A^*A, ν_k and μ will be defined in the sequel.

Lemma 2.1. *If T is strongly monotone with constant α, then (see for example [10])*

$$\|J_\lambda^T(x) - J_\lambda^T(y)\| \leq \frac{1}{1+\alpha\lambda}\|x-y\|, \quad \forall x, y.$$

A simple computation shows that its complement $I - J_\lambda^T$ (which is firmly nonexpansive) is strongly monotone. More precisely, we have

$$\langle (I - J_\lambda^T)x - (I - J_\lambda^T)y, x - y\rangle \geq \frac{\alpha\lambda}{1+\alpha\lambda}\|x-y\|^2, \quad \forall x, y.$$

We are now in a position to prove our convergence result.

Theorem 2.1. *Given a bounded linear μ-strongly positive operator $A : H_1 \to H_2$, H_1, H_2 two Hilbert spaces, $T : H_2 \to 2^{H_2}$ is a α-strongly monotone set-valued operator and $C : H_1 \to 2^{H_1}$ is a set-valued mapping with closed convex values and assume that for every x, y, z we have*

$$\|P_{C(x)}z - P_{C(y)}z\| \leq \beta\|x-y\|. \tag{2.3}$$

Then any sequence (x_k) generated by the algorithm (2.2) strongly converges to the unique solution of (2.1), provided that

$$\beta \in]0,1[, L_A\sqrt{\beta(2-\beta)} < \nu_k < L_A \text{ and } \gamma \in]0, \frac{2\nu_k\mu^2}{L_A^2}[\text{ with } \nu_k := \frac{\alpha\lambda_k}{1+\alpha\lambda_k}. \tag{2.4}$$

Proof. Let x^* be the solution to (2.2), then $P_{C(x^*)}(x^*) = x^*$, $J_{\lambda_k}^T(Ax^*) = Ax^*$. By Lemma 2.1, we know that $I - J_{\lambda_k}^T$ is nonexpansive and strongly monotone with constant ν_k. Therefore, we successively have

$$\begin{aligned}
& \langle A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*, x_k - x^*\rangle \\
= \ & \langle (I - J_{\lambda_k}^T)Ax_k - (I - J_{\lambda_k}^T)Ax^*, Ax_k - Ax^*\rangle \\
\geq \ & \nu_k\|Ax_k - Ax^*\|^2 \geq \nu_k\mu^2\|x_k - x^*\|^2.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
& \|(I - \gamma A^*(I - J_{\lambda_k}^T)A)x_k - x^*\|^2 \\
= \ & \|(x_k - x^*) - \gamma(A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*)\|^2 \\
= \ & \|x_k - x^*\|^2 - 2\gamma\langle A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*, x - x^*\rangle \\
+ \ & \gamma^2\|A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*\|^2 \\
\leq \ & (1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2)\|x_k - x^*\|^2.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|P_{C(x_k)}(x_k - \gamma A^*(I - J_{\lambda_k}^T)Ax_k) - P_{C(x_k)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*)\| \\
&+ \|P_{C(x_k)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*) - P_{C(x^*)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*)\| \\
&\leq \beta\|x_k - x^*\| + \|(I - \gamma A^*(I - S)A)x_k - x^*\| \\
&\leq (\beta + \sqrt{1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2})\|x_k - x^*\|,
\end{aligned}$$

λ_k and γ_k were chosen judiciously such that $\theta := \beta + \sqrt{1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2} \in]0,1[$. The latter assures the strong convergence of (x_k) to x^* the unique solution of (2.1).

Remark 2.2. It is worth mentioning that we can develop the same analysis for the following quasi fixed-point feasibility problem

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* \in FixP, \tag{2.5}$$

where $P : H_2 \to H_2$ is a κ-contraction, by considering the following algorithm

Algorithm:
Initialization: Let $x_0 \in H_1$ be arbitrary.
Iterative step:

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - P)Ax_k),\ k \in I\!N. \quad (2.6)$$

Following the same lines of the proof of the above Theorem, we obtain

Proposition 2.3. *Given a bounded linear μ-strongly positive operator $A : H_1 \to H_2$, H_1, H_2 are two Hilbert spaces, $P : H_2 \to 2^{H_2}$ a κ-contraction and $C : H_1 \to 2^{H_1}$ a set-valued mapping with closed convex values and assume that for every x, y, z we have*

$$\|P_{C(x)}z - P_{C(y)}z\| \leq \beta\|x - y\|.$$

Then any sequence (x_k) generated by the algorithm (2.6) strongly converges to the unique solution of (2.5), provided that

$$\nu := 1-\kappa, \beta \in]0, 1[, L_A\sqrt{\beta(2 - \beta)} < \nu\mu^2 < L_A \text{ and } |\gamma-\nu\mu^2| < \sqrt{\nu^2\mu^4 - \beta(2 - \beta)L_A^2}.$$

□

3. SPECIAL CASES

Now, let us consider the following special cases:

(i) **Quasi Split minimization problem:** Let $\phi : H_1 \to I\!R$ be a lower semicontinuous convex function by setting $T = \partial\phi$ in (2.1), we obtain the following Quasi Split Minimization Problem (QSMP):

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* = argmim\phi \quad (3.1)$$

and (2.2) reduces to

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - prox_{\lambda_k\phi})Ax_k),\ k \in I\!N, \quad (3.2)$$

where $prox_{\lambda_k\phi}(x) := argmin_y\{\phi(y) + \frac{1}{2\lambda}\|x - y\|^2\}$ is the proximal mapping of ϕ. The assumption of strong monotonicity of $\partial\phi$ is equivalent to the strong convexity of ϕ.

(ii) **Split Saddle-point problem:** Let X, Y be two Hilbert spaces, a function $L : X \times Y \to I\!R \cup \{-\infty, +\infty\}$ is convex-concave if it is convex in the variable x and concave in the variable y. To such a function, Rockafellar associated the operator T_L, defined by

$$T_L = \partial_1 L \times \partial_2(-L),$$

where ∂_1 (resp. ∂_2) stands for the subdifferential of L with respect to the first (resp. the second) variable.
T_L is a maximal monotone operator if and only if L is closed and proper in Rockafellar sense (see, [10]). Moreover, it is well known that (x^*, y^*) is a saddle-point of L, namely $L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*),\ \forall(x, y) \in X \times Y$ if and only if the following monotone variational inclusion holds true $(0, 0) \in T_L(x^*, y^*)$.
Now, if in the (2.1) we set $H_1 = X_1 \times Y_1$, $H_2 = X_2 \times Y_2$, $T = T_L$ with L be a proper closed convex-concave function, then we obtain the following Quasi Split Minimax Problem (QSMMP):

$$\text{find } (x^*, y^*) \in C(x^*, y^*);\ A(x^*, y^*) = argminmax_{(x,y)\in H_1} L(x, y), \quad (3.3)$$

and (2.2) reduces to

$$(x_{k+1}, y_{k+1}) = P_{C(x_k,y_k)}((x_k, y_k) - \gamma A^*(I - prox_{\lambda_k L})A(x_k, y_k)),\ k \in \mathbb{N}, \quad (3.4)$$

where $prox_{\lambda_k L}(x, y) := argminmax_{(u,v)}\{L(u, v)) + \frac{1}{2\lambda}\|x-u\|^2 - \frac{1}{2\lambda}\|y-v\|^2\}$. The assumption of strong monotonicity of T_L is equivalent to the strong convexity of L with respect to the first variable and its strong concavity with respect to the second one.

(iii) **Quasi Split equilibrium problem:** Having in mind the connection between monotone operators and equilibrium functions, we may consider the following problem

$$A_F(x) \ni 0, \quad (3.5)$$

with A_F defined as follows $v \in A_F(x) \Leftrightarrow F(x, y) + \langle v, x - y\rangle \geq 0,\ \forall y \in D$, D is a closed convex set and $F : D \times D \to \mathbb{R}$ belongs in the class of bifunctions F verifying the following usual conditions:

(A1) $F(x, x) = 0$ for all $x, y \in D$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in D$;
(A3) $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$ for any $x, y, z \in D$;
(A4) for each $x \in D, y \longrightarrow F(x, y)$ is convex and lower-semicontinuous.

It is well-known; see [7], that A_F is maximal monotone and that the associated resolvent operator $T_\lambda : H \longrightarrow D$ is defined by

$$T^F_\lambda(x) = \{z \in D :\ F(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,\ \forall y \in D\}.$$

If in the (2.1) we take $T = A_F$ a monotone bifunction, then we obtain the following Quasi Split Equilibrium Problem (QSEP):

$$\text{find } x^* \in C(x^*) \text{ such that } F(Ax^*, x) \geq 0 \text{ for all } x \in C(x^*), \quad (3.6)$$

and (2.2) is nothing but

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - T^F_{\lambda_k})Ax_k),\ k \in \mathbb{N}. \quad (3.7)$$

It is well known that in this case, the strong monotonicity of A_F is equivalent to

$$F(x, y) + F(y, x) \leq \alpha\|x - y\|^2 \text{ for all } x, y \in D.$$

(iv) **A special form of the implicit set:**
In many applications (see for example [2]) the set-valued mapping has the form $C(x) = K + \psi(x)$, where K is a fixed closed subset in H_1 and $\psi : H_1 \to H_1$ is a single-valued mapping. In this case, assumption on C is satisfied provided the mapping ψ is Lipschitz continuous. Indeed, it is not hard (using the relation below) to show that, if ψ is κ-Lipschitz then assumption (2.3) satisfies with $\beta = 2\kappa$. Using the well known relation

$$x = P_{K+v}(u) \Leftrightarrow x - v = P_K(u - v),$$

Algorithm can be rewritten in the simpler form

$$x_{k+1} = \psi(x_k) + P_K(x_k - \psi(x_k) - \gamma_k A^*(I - J^T_{\lambda_k})Ax_k),\ k \in \mathbb{N}. \quad (3.8)$$

Conclusion: Only the existence of solutions to a quasi split feasibility problem has been considered in [9] and the result is more than questionable. Also, only some algorithms are mentioned! To the best of our knowledge, nothing has been done concerning the construction of solutions in this case. Inspired by this work and to overcome the difficulties that arise in applying the Banach principle, we proposed a quasi feasibility null-point problem and study the convergence of a related algorithm. Applications to some applied nonlinear analysis problems are also provided. The techniques used in solving our problem are strongly based on the resolvent mapping which depends on a parameter. The latter allows more flexibility and an appropriate choice amounts to assume mild assumptions on the data (see Theorem 2.1).

Acknowledgment: I would like also to thank the anonymous referees for their careful reading which permitted me to improve the first version of this paper.

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