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A NONLINEAR INTEGER PROGRAMMING FORMULATION FOR THE AIRLIFT LOADING PROBLEM WITH INSUFFICIENT AIRCRAFT

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ABSTRACT. The Airlift Loading Problem with Insufficient Aircraft (ALPIA) is frequently faced by members of the United States Department of Defense when conducting airlift missions. The ALPIA is a combination of knapsack, assignment, and packing problems; items are *selected* for shipment based on a utility measure then *assigned* to pallets which will be loaded into an aircraft in a specific pallet position. These pallets are then *packed* in a manner to optimize both the pallet and aircraft characteristics, such as item utility, aircraft and pallet utilization, pallet center of gravity, aircraft center of balance, etc. Since not all items have the same destination, it is necessary to perform the packing in an intelligent fashion to ensure ease of unpacking at a destination. This paper formulates the ALPIA as an integer programming problem which allows items to be stably packed onto pallets with any specified orientation (i.e. accounting for “this side up” constraints). Rather than addressing the knapsack, assignment and packing problems separately in a hierarchical manner, this formulation simultaneously accounts for each of these problems.

KEYWORDS : Nonlinear Programming, Integer Programming, Knapsack, Bin-Packing.
AMS Subject Classification : 90C30

1. INTRODUCTION AND LITERATURE REVIEW

The airlift loading problem (ALP) was first defined by Chocolaad [2] as a knapsack problem and redefined by Roesener, et al. [8] as a bin-packing problem. This airlift process involves: (1) *packing* cargo items onto pallets, (2) *partitioning* the set of packed pallets into aircraft loads, (3) *selecting* a set of aircraft from a pool of aircraft, and (4) *placing* the cargo in the best available positions within the aircraft. There are very strict differences between the ALP and other packing problems. In addition to the normal spatial packing constraints, factors such as weight, center of balance, and temporal restrictions on cargo loading availability and cargo delivery requirements must be considered while solving the ALP [8].

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The two common problems used to model the ALP are knapsack and bin-packing problems. The single knapsack problem was proven to be a combinatorial NP-Hard problem by Karp [5]. It involves the selection of items from an available set of n items each with weight w_i and utility u_i, i =1,2,. . . ,n, to be packed in a container with a total weight capacity of b (i.e. a single constraint). The overall goal is to maximize the combined utility of the items placed into the container. The multidimensional knapsack problem extends the single dimensional knapsack problem by allowing more than one constraint in the problem.

The bin packing problem has also been proven to be a combinatorial NP-Hard problem [5]. It is defined as the placement of objects into a given number of bins of limited capacity in a way that minimizes the number of bins required. There are different types of bin packing problems named according to descriptions of the items to be packed. In this research effort, only two-dimensional (2D) and three-dimensional (3D) bin packing problems of orthogonal items are considered due to their relationship to the ALP. Numerous exact, heuristic, and meta-heuristic solution methods for 2D bin packing problems exist [3, 6]. In 2006, Harwig et al. [4] used tabu search to solve 2D bin packing problems using two dimensional packing, achieving excellent results on a well-known problem set. Nance, et al. [7] and Roesener, et al. [8] defined and solved special cases of aircraft loading problems as 2D bin packing problems. The 3D bin packing problem is an extension of the 2D bin packing problem in which an additional dimension is added to the problem. Pallet packing and container loading are common applications of 3D bin packing problems for which solution methods exist.

2. PROBLEM DESCRIPTION

This research focuses on a problem called the Aircraft Loading Problem with Insufficient Aircraft (ALPIA). In general, the ALPIA includes *selecting* cargo items to be transported, *packing* the items onto pallets, *partitioning* the pallets into aircraft sized loads, and *assigning* the pallets to specific positions within the available aircraft. These sub-problems are described in more detail below.

1. *Selecting Cargo Items*: For regularly scheduled missions, the cargo items to be transported could surpass the amount of available space within the airlift aircraft. Thus, some of the items will remain at the aerial port of embarkation (APOE). In order to carry the maximum amount of cargo items and ensure that the most important items are transported, a special evaluation or utility for each cargo item is needed. The goal of this sub-problem involves maximization of *both* the number of items and the utility associated with those items.
2. *Packing Items onto Pallets*: For some airlift missions (including deployments), the deploying command packs the pallets prior to the aircraft arrival. In other airlift missions, members of an aerial-port squadron are responsible for pallet packing. Unless the pallet is properly balanced (i.e. the pallet center of gravity (CG) is approximately in the geometric center of the pallet), the safety of the ground handlers and aircrew could be in jeopardy. This sub-problem considers how to best pack selected items onto a pallet while ensuring safety requirements (proper CG position, heavier items on lower levels, etc.).
3. *Partitioning Packed Pallets*: For multi-aircraft/multi-destination missions, pallets that have the same destinations should be partitioned and assigned

to the same aircraft. This sub-problem does not occur in single aircraft operations.

4. *Assigning Pallets to Aircraft*: This sub-problem involves assigning the pallets to specific pallet stations inside the aircraft while reducing the number of aircraft required. For this sub-problem, the aircraft's center of balance (CB), or the point within the aircraft where the cargo load is balanced, and any regional constraints associated with a specific pallet position in an aircraft (i.e. height and weight of packed pallet) must still be satisfied.

Additionally, each of the ALPIA sub-problems is restricted to the same aircraft constraints which affect the ALP. These constraints were defined by Roesener, et al. [9]; they are modified for this problem formulation and presented in subsequent sections.

1. *Aircraft CB*
2. *Operational Allowable Cabin Load (ACL)*
3. *Pallet Position Restrictions*
4. *Available Space for Loading*
5. *Route of Flight and Item Destination*
6. *Pallet CG*

Although the sub-problems of the ALPIA can be viewed separately, they are not independent. Rather, changes in one sub-problem can have a dramatic impact on the feasibility and/or optimality of another sub-problem. With this in mind, it is best to formulate and solve the ALPIA in its entirety, rather than sequentially solving the sub-problems.

3. ALPIA FORMULATION

To enhance understanding, initially only constraints for a single pallet will be presented. These constraints with an associated objective function were previously detailed by Roesener, et al. [9]. After the single pallet constraints are detailed, multi-pallet, multi-destination and multi-aircraft packing formulations will be presented, respectively. Lastly, the objective function will be explained.

3.1. **ALPIA Set Notation and Variable Description.** Before a valid formulation can be presented, the set notation used in the formulation must be adequately explained. Additionally, the variables associated with these sets must also be defined.

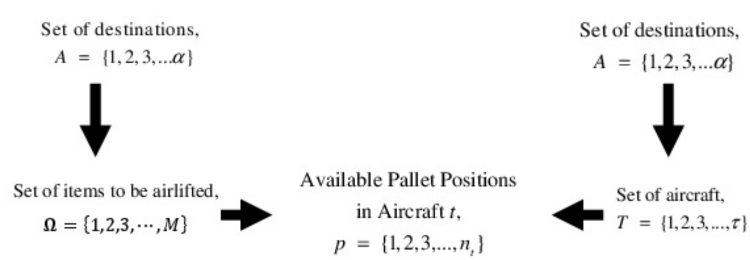

FIGURE 1. ALPIA Sets and their Relationships

There are three major data sets associated with the ALPIA: *Destinations*, *Cargo Items*, and *Available Aircraft*. The aircraft set has a subset, *Available Pallet Positions*, which varies among aircraft of different types. The overall goal of this research is to efficiently and feasibly place the items on the available pallets. The sets and their relationships are shown in Figure 1.

The constants and decision variables used in this formulation will be explained in the context of these sets. Some of the input data for the ALPIA are values that vary as a function of the decision variables. Although these values are not constant throughout the formulation, their values are not allowed to vary arbitrarily. Thus, they will be explained in the same section as the constants (i.e., maximum, minimum and optimal CB values). The constants and functions of decision variables are:

a. *Destination Set* (A)*:* This is the set of destinations for cargo items and aircraft.
   - Item ($a \in A$): This index refers to a destination, where a is a positive integer value (i.e., $a \in \{1,2,\ldots,\alpha = |A|\}$).

b. *Item Set* (Ω)*:* This is the set of cargo items that may be loaded. The following defines the parameters (i.e., utility, weight, and dimensions) associated with each item. Figure 2 presents a visual illustration of an element of this set.

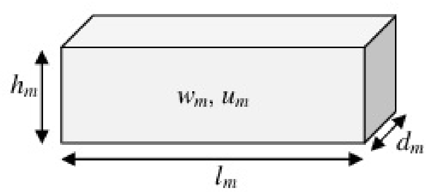

FIGURE 2. Physical Characteristics of Item m ($m \in \Omega$)

   - Item ($m \in \Omega$): This is an index which refers to the items to be transported, where m is a positive integer value (i.e., $m \in \{1,2,\ldots,M = |\Omega|\}$).
   - Destination of the item (I_{ma}, $m \in \Omega$, $a \in A$): This is a binary constant; $I_{ma} = 1$ if item $m \in \Omega$ has destination $a \in A$. Otherwise, $I_{ma} = 0$.
   - Utility of the item (u_m): This is a positive, integer-valued constant. It is the assigned utility of item m. This value accounts for the priority of the item and the usefulness that can be currently derived from transporting the item.
   - Weight (w_m): This is a real-valued constant. It is the actual weight of item m (in pounds).
   - Dimensions, (d_m, l_m, h_m): In this context, d_m is the depth, l_m is the length, and h_m is the height of item m (in inches). They are positive, real-valued constants.

   The CG of item $m \in \Omega$ (not of a packed pallet) is assumed to be in the geometric center of the item.

c. *Aircraft Set* (T)*:* This is the set of aircraft that are available for loading.
   - Available Aircraft ($t \in T$): This designates the index for an available aircraft; it is a positive integer value.

- Available Aircraft Routes ($R_t = \{a_1, \ldots, a_\alpha\}$, $t \in T, a \in A$): This is the set of ordered destinations that gives the route of aircraft t.
- Cargo Load (Ψ_t): This is the total weight of the packed items that can be placed on aircraft t.
- CB Limits for aircraft t ($CB_{t(max)}$, $CB_{t(min)}$, $CB_{t(ideal)}$): These are the maximum, minimum, and ideal values of the CB for aircraft t, respectively. These values are predetermined for each aircraft type, and they depend upon the total cargo weight assigned to the aircraft. The maximum and minimum CB values are constraints which cannot be violated without the aircraft departing from safe flight. The ideal CB, however, denotes the target CB value; it is the CB location (for a given cargo load) at which the aircraft exhibits the best fuel consumption rate. Each of these CB limits is a real-valued constant that is a function of the decision variables.
- Number of Pallet Positions, (n_t, $t \in T$): This is a positive, integer valued constant. It is the total number of the pallet positions inside the t^{th} aircraft.

d. *Pallet Positions Set* (P_t, $t \in T$)*:* This is the set of available pallet positions within aircraft t.

- Pallet Positions ($p \in P_t$, $t \in T$)*:* This is the index for the pallet positions for each aircraft in the aircraft set. Note that $|P_t| = n_t$, and that this is directly dependent upon the type and route of aircraft t.
- Assigned Arrival Point (I_{pa}, $p \in P_t, t \in T, a \in A$): This is a binary constant. A value of 1 indicates that the pallet p of aircraft t is assigned to the destination a.
- Fuselage station (b_p, $p \in P_t, t \in T$): This is a real-valued constant that refers to the distance from the aircraft's reference datum point to the center of pallet position p in aircraft t.
- Pallet Position Restrictions (W_p, H_p, D_p , L_p, $p \in P_t, t \in T$): These are constants that represent the weight, height, depth and length (respectively) restrictions for a packed pallet located in pallet position p in aircraft t.

In addition to the parameters, several decision variables which require detailed explanation are used in the formulation. Pixel based packing (i.e., the packing bins and items are partitioned into uniform unit pixels) was first used in a nonlinear 3D bin packing formulation by Ballew [1]; however, the advantages for implementing this type of approach was not adequately addressed by Ballew. In this research, ALPIA is formulated as a multi-constraint bipartite maximal matching problem. The objective is matching the maximum number of item unit pixels to the pallet pixels, which is equivalent to occupying the maximum amount of available space on the pallet.

As seen in Figure 3, items are placed on the pallets on specific grids. A similar idea was proposed by Ballew [1].

a. Coordinates inside the pallet: (i, j, k) denote the coordinates of a cubic grid made up of small "pixels" or grid cubes of unit volume. The volume of each grid cube depends upon the units of the cargo items to be packed. For example, if all items are measured in inches, then a 1 in^3 pixel would logically be used as a grid cube; if the items are measured in centimeters, then a 1 cm^3 pixel would be used. A smaller grid cube volume allows for higher fidelity in the packing procedure, but requires more computation

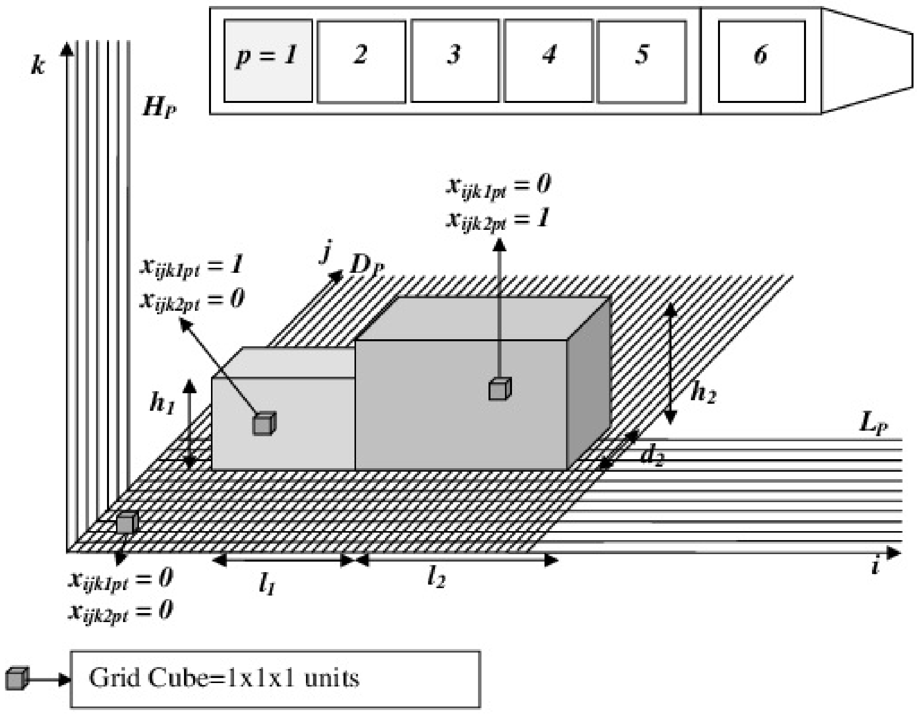

FIGURE 3. Required variables for packing an item m ($m \in \Omega$) on a pallet p ($p \in P_t$) based on grids or pixels

time. For this research, all items are measured in inches; thus, a grid cube volume of 1 in^3 is used.

b. Occupation of pixels (x_{ijkmpt}): This is a binary decision variable. A value of 1 indicates that the grid cube on the (*i*,*j*,*k*) coordinates of the pallet occupying the p^{th} pallet position in aircraft t is occupied by item m.
c. Item-Pallet Relation (X_{mpt}): This is a binary decision variable. A value of 1 indicates that item m is packed on the pallet occupying pallet position p in aircraft t. When considering a given pallet p in aircraft t, the decision variable is denoted by X_{mpt}.
d. Item-Pallet Orientation (y_{mptsz}): This is a binary decision variable. A value of 1 indicates that the z^{th} (z = 1,2,3) dimension of item m is parallel to the s^{th}(s = 1,2,3) dimension of the pallet occupying the p^{th} pallet position in aircraft t. It will be used to determine the orientation of a packed item with reference to the pallet. This variable allows for different orientations of items as well as ensuring any item with a "This side up" constraint is properly packed. Figure 4 provides a visual description of these decision variables.

Now that the sets, constants and variables have been adequately explained, the actual formulation for the ALPIA can be presented. First, the formulation of a single pallet, which was previously detailed by Roesener, et al. [9], will be explained. The single pallet formulation will then be expanded to encompass multiple pallets with a single destination on a single aircraft. This formulation will be further expanded

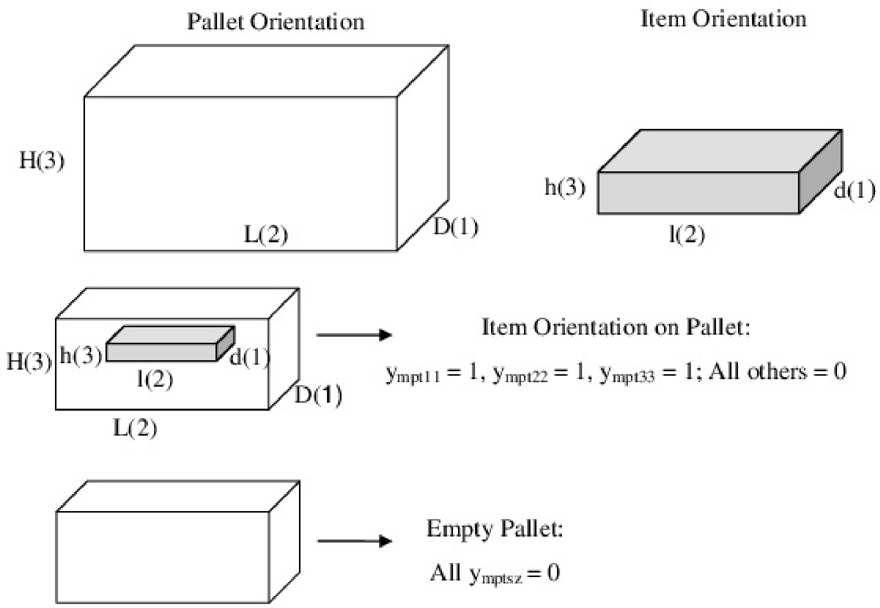

FIGURE 4. Pallet Orientation Variables

to allow for multiple destinations for items and pallets that are on a single aircraft. Finally, the formulation of a problem with multiple aircraft which have multiple destinations will be presented.

3.2. **Formulation of the Single Aircraft, Single Destination, Single Pallet Packing Problem.** For a single pallet formulation, the indices for the aircraft t and the pallet occupying the p^{th} pallet position will have a constant value. These subscripts will therefore be omitted in the mathematical presentation. The constraints necessary for this formulation are:

a. Overlap Constraint [1]: Multiple items cannot simultaneously occupy the same grid cube.

$$\sum_{m=1}^{M} x_{ijkm} \leq 1 \;\; (\forall\, i, j, k) \tag{3.1}$$

b. Stability Constraint [1]: Each occupied grid cube requires support; in other words, it must be placed upon another *occupied* grid cube or on the surface of the pallet.

$$\sum_{m=1}^{M} x_{ij(k+1)m} - \sum_{m=1}^{M} x_{ijkm} \leq 0 \;\; (\forall\, i, j, k) \tag{3.2}$$

c. Dimensional Constraints: The total number of grid cubes occupied by items along each depth, length and height dimension cannot exceed the pallet's limitation for depth (D), length (L) and height (H), respectively.

$$\sum_{i=1}^{D}\sum_{m=1}^{M} x_{ijkm} \leq D \;\; (\forall\, j,k) \tag{3.3}$$

$$\sum_{j=1}^{L}\sum_{m=1}^{M} x_{ijkm} \leq L \;\; (\forall\, i,k) \tag{3.4}$$

$$\sum_{k=1}^{H}\sum_{m=1}^{M} x_{ijkm} \leq H \;\; (\forall\, i,j) \tag{3.5}$$

d. Volume Constraint: The total volume of packed items on a pallet can be at most the allowable volume for that pallet, which is $D{\cdot}L{\cdot}H$. In other words, the total number of grid cubes occupied by all packed items cannot exceed the total number of available grid cubes.

$$\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}\sum_{m=1}^{M} x_{ijkm} \leq D \cdot L \cdot H \tag{3.6}$$

e. Weight Constraint: Total weight of packed items on a pallet cannot exceed the structural weight limitations (W) for the actual pallet or the pallet position within the aircraft (W represents the smallest of these weight limitations).

$$\sum_{m=1}^{M} w_m \cdot X_m \leq W \tag{3.7}$$

Figure 5 shows the 2-dimensional placement (i.e. the "footprint") of two different items on a theoretical pallet or dimensions ten units by eight units. Observe that for each column and each row the total occupied pixels are different from each other. This is one of the reasons for using a grid-based formulation.

f. CG Constraints: A set of constraints are required to ensure the Pallet CG in the lateral (CG_{length}) and longitudinal (CG_{depth}) direction from the center of the pallet does not exceed a given amount. The ideal center of balance is in the center of the pallet, which is determined by $L/2$ and $D/2$ for the lateral and longitudinal dimensions, respectively. These values along with the CG for the vertical dimension must be calculated as a "soft" constraint which will negatively impact the objective function if it is violated (but not cause the problem to become infeasible).

$$\left[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{D \cdot X_m}{2}\right)\cdot w_m \right. \\ \left. - CG_{depth}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.8}$$

$$\left[-\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{D \cdot X_m}{2}\right)\cdot w_m \right. \\ \left. + CG_{depth}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.9}$$

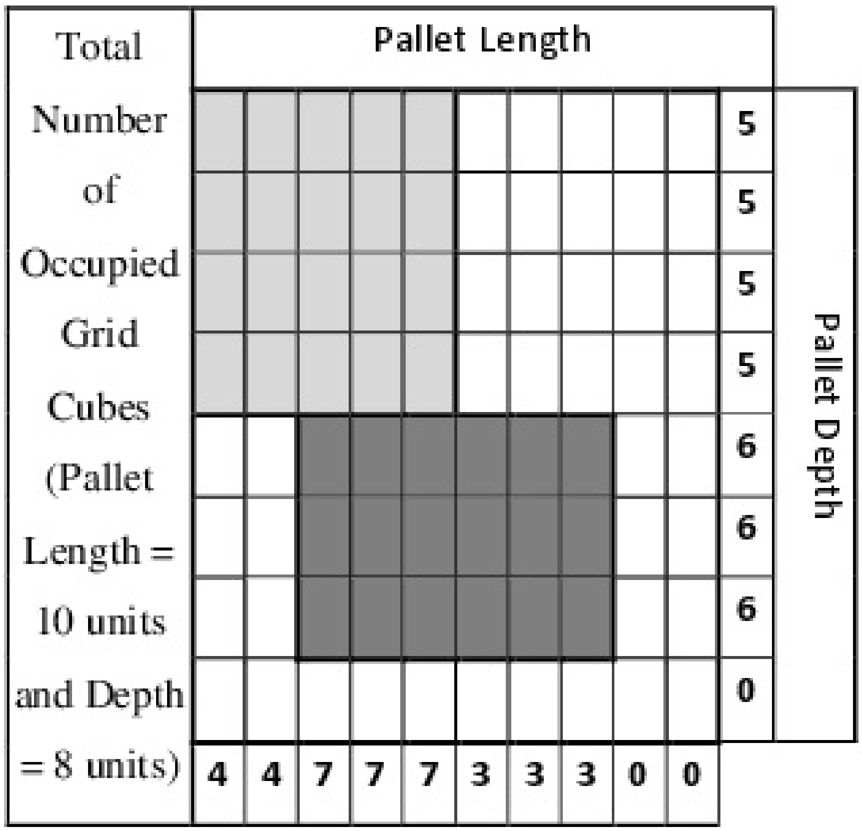

FIGURE 5. Example of different items ($m \in \Omega$) occupying grids on the pallet p ($p \in P_t$)

$$\left[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{L \cdot X_m}{2}\right) \cdot w_m - CG_{length}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.10}$$

$$\left[-\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{L \cdot X_m}{2}\right) \cdot w_m + CG_{length}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.11}$$

g. Vertical CG Constraint: The pallet CG in the vertical dimension should be in the lower half of the pallet to prevent tipping. This constraint ensures that heavier items are placed under lighter items without imposing unnecessary constraints on the problem. The smaller values imply the vertical CG is below half the height ($H/2$) of the pallet and the majority of the heavier items are closer to the pallet surface.

$$\begin{aligned}\Bigg[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(k\cdot x_{ijkm})}{d_m\cdot l_m\cdot h_m}-\frac{H\cdot X_m}{2}\right)\cdot w_m \\ -\sum_{m=1}^{M}w_m\cdot X_m\Bigg]\leq\frac{H}{2}\end{aligned} \tag{3.12}$$

h. Item Integrity: A set of constraints are necessary to ensure that all occupied cube grids associated with a single item are contiguous. In other words, cargo items cannot be divided into small pieces and placed on the pallet. These constraints also ensure that the item's dimensions and volume coincide with the actual values for the item while allowing the item to be packed in different orientations. This non-linear formulation ensures the item integrity while allowing packing in different orientations.

$$\begin{aligned}\Bigg[y_{m11}\left(d_m\sum_{i=1}^{D-d_m}\prod_{s=0}^{d_m-1}x_{(i+s)jkm}\right)+y_{m12}\left(l_m\sum_{i=1}^{D-l_m}\prod_{s=0}^{l_m-1}x_{(i+s)jkm}\right) \\ +y_{m13}\left(h_m\sum_{i=1}^{D-h_m}\prod_{s=0}^{h_m-1}x_{(i+s)jkm}\right)\Bigg]=\left(\sum_{i=1}^{D}x_{ijkm}\right)(\forall\ j,k,m)\end{aligned} \tag{3.13}$$

$$\begin{aligned}\Bigg[y_{m21}\left(d_m\sum_{j=1}^{L-d_m}\prod_{s=0}^{d_m-1}x_{i(j+s)km}\right)+y_{m22}\left(l_m\sum_{j=1}^{L-l_m}\prod_{s=0}^{l_m-1}x_{i(j+s)km}\right) \\ +y_{m23}\left(h_m\sum_{j=1}^{L-h_m}\prod_{s=0}^{h_m-1}x_{i(j+s)km}\right)\Bigg]=\left(\sum_{j=1}^{L}x_{ijkm}\right)(\forall\ i,k,m)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Bigg[y_{m31}\left(d_m\sum_{j=1}^{H-d_m}\prod_{s=0}^{d_m-1}x_{ij(k+s)m}\right)+y_{m32}\left(l_m\sum_{j=1}^{H-l_m}\prod_{s=0}^{l_m-1}x_{ij(k+s)m}\right) \\ +y_{m33}\left(h_m\sum_{j=1}^{H-h_m}\prod_{s=0}^{h_m-1}x_{ij(k+s)m}\right)\Bigg]=\left(\sum_{j=1}^{L}x_{ijkm}\right)(\forall\ i,j,m)\end{aligned} \tag{3.15}$$

i. Item Orientation Constraint: The last set of constraints only applies to items which have a "this side up" constraint. These constraints force the height dimension of the item to be used in the vertical orientation.

$$y_{m11}+y_{m21}+y_{m31}\leq 1\ \ (\forall\ m) \tag{3.16}$$

$$y_{m12}+y_{m22}+y_{m32}\leq 1\ \ (\forall\ m) \tag{3.17}$$

$$y_{m13}+y_{m23}+y_{m33}\leq 1\ \ (\forall\ m) \tag{3.18}$$

$$y_{m11}+y_{m12}+y_{m13}\leq 1\ \ (\forall\ m) \tag{3.19}$$

$$y_{m21} + y_{m22} + y_{m23} \leq 1 \ (\forall\ m) \tag{3.20}$$

$$y_{m31} + y_{m32} + y_{m33} \leq 1 \ (\forall\ m) \tag{3.21}$$

$$y_{m33} = X_m \tag{3.22}$$

3.3. **Formulation of the Single Aircraft, Single Destination, Multi-Pallet Packing Problem.** For a multi-pallet ALPIA formulation in a single aircraft with a single destination, the previously defined constraints are used with an additional index ($p \in P$) to account for multiple pallet positions. The aircraft index will still remain constant and will therefore be omitted. Additional constraints are necessary to ensure that none of the items are placed on multiple pallets and to distinguish between pallet positions.

$$\sum_{p=1}^{n} X_{mp} \leq 1 \ (\forall\ m) \tag{3.23}$$

Another important constraint for flight safety is the aircraft CB. A constraint that assures that the aircraft CB is within the acceptable range is given by:

$$CB_{min} \leq \left(\frac{\sum_{p=1}^{n} \left(b_p \cdot \sum_{m=1}^{M} (w_m \cdot X_{mp}) \right)}{\sum_{m=1}^{M} w_m \cdot X_{mp}} \right) \leq CB_{max} \tag{3.24}$$

The last constraint for aircraft is the total ACL. The total weight of the cargo load cannot exceed the allowable cabin load for the aircraft.

$$\sum_{p=1}^{n} \sum_{m=1}^{M} (w_m \cdot X_{mp}) \leq \psi \tag{3.25}$$

3.4. **Formulation of the Single Aircraft, Multi-Destination, Multi-Pallet Packing Problem.** In the multi-destination, single-aircraft instance of the ALPIA problem, an additional constraint is added to the problem. The aircraft subscript is still not required in this formulation. The additional constraint is:

$$I_{ma} \cdot I_{pa} = X_{mp} \ (\forall\ a, m, p) \tag{3.26}$$

This constraint ensures that none of the items are allowed to be packed on a pallet that has a different destination than the item.

3.5. **Formulation of Multi-Aircraft, Multi-Destination, Multi-Pallet Packing Problem with Insufficient Aircraft.** When the formulations used in the previous subproblems are augmented with the index accounting for multiple aircraft ($t \in T$) and combined, the following formulation is derived. The first constraints are for packing the pallet:

$$\sum_{m=1}^{M} x_{ijkmpt} \leq 1 \ (\forall\ i, j, k, p, t) \quad \rightarrow Avoid\ Item\ Overlap \tag{3.27}$$

$$\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \leq D_{pt} \cdot L_{pt} \cdot H_{pt} \, (\forall\ p, t) \quad \rightarrow Pallet\ Volume \tag{3.28}$$

$$\sum_{i=1}^{D_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \le D_{pt} \, (\forall\; j,k,p,t) \quad \rightarrow Pallet\ Depth \tag{3.29}$$

$$\sum_{j=1}^{L_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \le L_{pt} \, (\forall\; i,k,p,t) \quad \rightarrow Pallet\ Length \tag{3.30}$$

$$\sum_{k=1}^{H_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \le H_{pt} \, (\forall\; i,j,p,t) \quad \rightarrow Pallet\ Height \tag{3.31}$$

$$\sum_{m=1}^{M} X_{mpt} \cdot w_m \le W_{pt} \;\; (\forall\; p,t) \quad \rightarrow Pallet\ Weight \tag{3.32}$$

$$\sum_{m=1}^{M} x_{ij(k+1)mpt} - \sum_{m=1}^{M} x_{ijkmpt} \le 0 \quad (\forall\; i,j,k,p,t) \quad \rightarrow Packing\ Stability \tag{3.33}$$

$$\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} x_{ijkmpt} \le d_m \cdot l_m \cdot h_m \cdot X_{mpt} \, (\forall\; m,p,t) \quad \rightarrow Linking\ Constraint \tag{3.34}$$

$$I_{ma} \cdot I_{pa} = X_{mpt} \, (\forall\; a,m,p,t) \quad \rightarrow Destination\ Constraint \tag{3.35}$$

The Pallet CG constraints are given by (depth, length, and height, respectively):

$$\left| \sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (i \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{D_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right| \le CG_{depth,pt} \tag{3.36}$$

$$\left| \sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (j \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{L_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right| \le CG_{length,pt} \tag{3.37}$$

$$\left(\sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (k \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{H_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right) \le \frac{H_{pt}}{2} \tag{3.38}$$

As previously mentioned in the single pallet packing process, pixels must be kept contiguous by using another constraint set. This formulation requires the same type of constraints with the additional indices for multiple pallets and aircraft.

$$\left[y_{m11pt}\left(d_m\sum_{i=1}^{D-d_m}\prod_{s=0}^{d_m-1}x_{(i+s)jkmpt}\right)+y_{m12pt}\left(l_m\sum_{i=1}^{D-l_m}\prod_{s=0}^{l_m-1}x_{(i+s)jkmpt}\right)\right.$$
$$\left.+y_{m13pt}\left(h_m\sum_{i=1}^{D-h_m}\prod_{s=0}^{h_m-1}x_{(i+s)jkmpt}\right)\right]=\left(\sum_{i=1}^{D}x_{ijkmpt}\right)(\forall\, j,k,m,p,t) \quad (3.39)$$

$$\left[y_{m21pt}\left(d_m\sum_{j=1}^{L-d_m}\prod_{s=0}^{d_m-1}x_{i(j+s)kmpt}\right)+y_{m22pt}\left(l_m\sum_{j=1}^{L-l_m}\prod_{s=0}^{l_m-1}x_{i(j+s)kmpt}\right)\right.$$
$$\left.+y_{m23pt}\left(h_m\sum_{j=1}^{L-h_m}\prod_{s=0}^{h_m-1}x_{i(j+s)kmpt}\right)\right]=\left(\sum_{j=1}^{L}x_{ijkmpt}\right)(\forall\, i,k,m,p,t) \quad (3.40)$$

$$\left[y_{m31pt}\left(d_m\sum_{j=1}^{H-d_m}\prod_{s=0}^{d_m-1}x_{ij(k+s)mpt}\right)+y_{m32pt}\left(l_m\sum_{j=1}^{H-l_m}\prod_{s=0}^{l_m-1}x_{ij(k+s)mpt}\right)\right.$$
$$\left.+y_{m33pt}\left(h_m\sum_{j=1}^{H-h_m}\prod_{s=0}^{h_m-1}x_{ij(k+s)mpt}\right)\right]=\left(\sum_{j=1}^{L}x_{ijkmpt}\right)(\forall\, i,j,m,p,t) \quad (3.41)$$

These constraints make the problem distinctly non-linear. Furthermore, they are very detailed and (possibly) difficult to understand. The following constraints could replace the pixel contiguity constraints; however, both sets of constraints will result in a non-linear formulation of the problem.

$$max\left(\sum_{s=i}^{i+d_m}x_{(s)jkmpt}\right)-d_m y_{m11pt}=0 \qquad (\forall\, j,k,m,p,t) \quad (3.42)$$

$$max\left(\sum_{s=j}^{j+d_m}x_{i(s)kmpt}\right)-d_m y_{m21pt}=0 \qquad (\forall\, i,k,m,p,t) \quad (3.43)$$

$$max\left(\sum_{s=k}^{k+d_m}x_{ij(s)mpt}\right)-d_m y_{m31pt}=0 \qquad (\forall\, i,j,m,p,t) \quad (3.44)$$

$$max\left(\sum_{s=i}^{i+l_m}x_{(s)jkmpt}\right)-l_m y_{m11pt}=0 \qquad (\forall\, j,k,m,p,t) \quad (3.45)$$

$$max\left(\sum_{s=j}^{j+l_m}x_{i(s)kmpt}\right)-l_m y_{m21pt}=0 \qquad (\forall\, i,k,m,p,t) \quad (3.46)$$

$$max\left(\sum_{s=k}^{k+l_m}x_{ij(s)mpt}\right)-l_m y_{m31pt}=0 \qquad (\forall\, i,j,m,p,t) \quad (3.47)$$

$$max\left(\sum_{s=i}^{i+h_m}x_{(s)jkmpt}\right)-h_m y_{m11pt}=0 \qquad (\forall\, j,k,m,p,t) \quad (3.48)$$

$$max\left(\sum_{s=j}^{j+h_m} x_{i(s)kmpt}\right) - h_m y_{m21pt} = 0 \qquad (\forall\ i,k,m,p,t) \tag{3.49}$$

$$max\left(\sum_{s=k}^{k+h_m} x_{ij(s)mpt}\right) - h_m y_{m31pt} = 0 \qquad (\forall\ i,j,m,p,t) \tag{3.50}$$

To account for different orientations and still ensure feasible solutions, the following set of constraints is required.

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m11pt} + y_{m21pt} + y_{m31pt}) \leq 1 \qquad (\forall\ m) \tag{3.51}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m12pt} + y_{m22pt} + y_{m32pt}) \leq 1 \qquad (\forall\ m) \tag{3.52}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m13pt} + y_{m23pt} + y_{m33pt}) \leq 1 \qquad (\forall\ m) \tag{3.53}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m11pt} + y_{m12pt} + y_{m13pt}) \leq 1 \qquad (\forall\ m) \tag{3.54}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m21pt} + y_{m22pt} + y_{m23pt}) \leq 1 \qquad (\forall\ m) \tag{3.55}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m31pt} + y_{m32pt} + y_{m33pt}) \leq 1 \qquad (\forall\ m) \tag{3.56}$$

$$y_{m33pt} = X_{mpt} \tag{3.57}$$

Finally, the constraints related to the available aircraft are given by:

$$\sum_{p=1}^{n} X_{mpt} \leq 1 \ \ (\forall\ m,t) \tag{3.58}$$

$$CB_{min,t} \leq \left(\frac{\sum_{p=1}^{n}\left(b_{pt}\cdot\sum_{m=1}^{M}(w_m\cdot X_{mpt})\right)}{\sum_{m=1}^{M} w_m\cdot X_{mpt}}\right) \leq CB_{max,t} \quad (\forall\ t) \tag{3.59}$$

$$\sum_{p=1}^{n}\sum_{m=1}^{M}(w_m\cdot X_{mpt}) \leq \psi_t \quad (\forall\ t) \tag{3.60}$$

3.6. **Objective Function.** Unlike bin, pallet or container packing problems which have an objective of minimizing the number of bins required to pack all items, the ALPIA primary objective is to maximize the total utility of the packed items while also maximizing the total volume and weight of the transported items. Of course, leaving any available space on the pallet empty may cause the overall problem to require an additional bin (possibly more). Thus, without exceeding the weight or volume limitations of pallets, a goal of the ALPIA is to pack them efficiently; this aids the primary objective by ensuring pallet space is available for as many items as possible.

The ALPIA objective function is also similar to that of a knapsack problem. In both types of problems, the number of "bins" or "knapsacks" is limited. In knapsack problems, the objective involves maximizing the total utility of the selected items; however, for the ALPIA, simply maximizing the number of the packed, high priority items while only focusing on utility may lead to inefficient use of available capacity (weight and space).

In the following section, two different objective functions are introduced. The first function is a weighted sum of sub-objectives. The sub-objectives of the ALPIA are:

a. *Maximizing the utility of the packed items and reducing the number of unpacked high utility items*: Both packed and unpacked items are relevant in the ALPIA. Failing to pack items of high priority is as undesirable as neglecting to utilize the available aircraft capacity. Thus, the first objective function, f_1, is given by:

$$f_1 = \left[\frac{\left(\sum_{m=1}^{M} u_m\right) - \left(\sum_{m=1}^{M} u_m \cdot X_m\right)}{\left(\sum_{m=1}^{M} u_m\right)} \right] \cdot 100 \tag{3.61}$$

This objective function is a percentage of the utility of the packed items. Unfortunately, minimizing this value may lead to packing the highest priority items whenever there is sufficient space (regardless of whether some other item is better suited for the space).

b. *Maximum Aircraft Capacity Usage*: ALPIA involves packing the aircraft efficiently by placing properly packed pallets within the aircraft. Placing packed pallets with the available volume maximized may not result in the best possible solution; weight and volume maximization should also be included. Additional portions of the objective function which account for these considerations are labeled f_2 and f_3 and are given by:

$$f_2 = \left[\frac{min\left[\left(\sum_{t=1}^{\tau}\sum_{p=1}^{n_t} W_p\right), \left(\sum_{t=1}^{\tau} \psi_t\right)\right] - \left(\sum_{m=1}^{M} w_m \cdot X_m\right)}{min\left[\left(\sum_{t=1}^{\tau}\sum_{p=1}^{p_t} W_p\right), \left(\sum_{t=1}^{\tau} \psi_t\right)\right]} \right] \cdot 100 \tag{3.62}$$

$$f_3 = \left[\frac{\left(\sum_{t=1}^{\tau}\sum_{p=1}^{n_t} L_{pt} \cdot D_{pt} \cdot H_{pt}\right) - \left(\sum_{m=1}^{M} w_m \cdot X_m\right)}{\left(\sum_{t=1}^{\tau}\sum_{p=1}^{n_t} L_{pt} \cdot D_{pt} \cdot H_{pt}\right)} \right] \cdot 100 \tag{3.63}$$

The overall goal of these two portions of the objective function is to minimize the unused capacity by minimizing the unused weight and volume pallet capacity, respectively. These functions make the objective function non-linear. The weight limitation of the aircraft depends on the relationship between the ACL and the total weight capacity of the pallet positions on the same aircraft.

c. *Balanced Aircraft:* An unbalanced aircraft with pallets whose volume is maximized will be an infeasible solution to the ALPIA; packing the pallets with respect to each other is important. CB feasibility is assured with the constraints; thus this does not require inclusion in the objective function.

d. *Efficiently Packed Pallets*: Packing balanced and stable pallets is more important than completely maximizing their volume. CG feasibility and stable packing of the items are assured by the constraints. As a result, this does not require inclusion in the objective function.

After defining the sub-objectives, the overall objective is a weighted summation of the sub-objective functions and is given by:

$$min\left[(\lambda_1 \cdot f_1) + (\lambda_2 \cdot f_2) + (\lambda_3 \cdot f_3)\right] = min\sum_{i=1}^{3}(\lambda_i \cdot f_i) \tag{3.64}$$

where f_1, f_2, and f_3 are as previously defined, and λ_1, λ_2, and λ_3 are penalty weights.

Despite the similarities with knapsack and bin packing problems, the ALPIA has additional aspects which require consideration. These considerations may result in a non-convex solution space of ALPIA. In addition to the drawbacks of a non-linear function, possible non-convex portions of the solution set of ALPIA may not be obtained by minimizing convex combinations of the objectives.

Thus, another objective function for the ALPIA may be given as:

$$max\left[\sum_{m=1}^{M} u_m^{\lambda} \cdot l_m \cdot d_m \cdot h_m \cdot w_m \cdot X_m\right] \tag{3.65}$$

This objective function attempts to simultaneously maximize the total utility, total volume and total height without using penalty multipliers. The use of the superscript ensures the importance of the priority aspect of the items. Lower values may be used in routine missions for higher aircraft utilization; higher values may be used in deployment (less frequent) missions for value based aircraft utilization.

4. COMPLEXITY OF ALPIA

Clearly, ALPIA is an NP-Hard optimization problem since the 0-1 Knapsack problem is a special case of sub-problem 1 (i.e., *Selecting Cargo Items*). Although the polynomial transformation is not presented in this research, the 0-1 Knapsack problem can be reduced to the ALPIA in polynomial time. Karp [5] previously proved the 0-1 Knapsack problems to be NP-Hard.

5. SUMMARY

In this research effort, the Airlift Loading Problem with Insufficient Aircraft (ALPIA) was introduced and explained in detail. Similarities and differences between the ALPIA and knapsack, bin-packing and multi-constraint bipartite maximal matching problems were also presented. For the first time, a formulation considering all the constraints of "packing an aircraft" and an objective function that achieves the ALPIA objective is presented. Except for the contiguity constraint and the objective function, this is an integer-linear formulation.

The number of variables associated with the constraints presented in the ALPIA formulation is very large; there are

$$\left(\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(L_{pt}\cdot D_{pt}\cdot H_{pt})\right)+7 \qquad (t\in T, p\in P_T) \tag{5.1}$$

variables required for each item. Couple this with the fact that the ALPIA is an NP-Hard problem, and classical optimization methods are insufficient to solve the ALPIA in a reasonable amount of computational time and effort. Therefore, heuristics or other algorithmic techniques can be applied to this problem to provide a high quality solution in a reasonable amount of time.

References

1. B. Ballew, The Distributor's Three Dimensional Pallet-Packing Problem: A Mathematical Formulation and Heuristic Solution Approach. MS Thesis. Graduate School of Engineering, Air Force Institute of Technology, WPAFB, OH (2000).
2. C. Chocolaad, Solving Geometric Knapsack Problems Using Tabu Search Heuristics. MS Thesis. Graduate School of Engineering, Air Force Institute of Technology, WPAFB, OH (1998).
3. E. Coffman, G. Galambos, S. Martello, and D. Vigo, Bin Packing Approximation Algorithms: Combinatorial Analysis, in Handbook of Combinatorial Optimization, D. Du and P. Pardalos, (eds.), Kluwer, Amsterdam (1998).
4. J.W Harwig, J.W. Barnes, and J.T. Moore, An Adaptive Tabu Search Approach for 2-Dimensional Orthogonal Packing Problems. Military Oper. Res. J., 11 (2006), no. 2, 5-26.
5. R. Karp, Reducibility among combinatorial problems. Complexity of Computer Computations (R.E. Miller and J.M. Thatcher, eds.), Plenum Press, 85-103 (1972).
6. A. Lodi, S. Martello, and M. Monaci, Two-dimensional Packing Problems: A Survey, European J. Oper. Res. 141 (2002), 241-252.
7. R.L. Nance, A.G. Roesener, and J.T. Moore, An Advanced Tabu Search for the Mixed Payload Airlift Loading Problem, J. Oper. Res. Soc., 62 (2011), 337-347.
8. A.G. Roesener, J.W. Barnes, J.T. Moore, and D. Van Veldhuizen, An Advanced Tabu Search Approach To The Airlift Loading Problem. Military Oper. Res. J., 15 (2010), no. 1, 5-29.
9. A.G. Roesener, S.N. Hall, and Y.G. Ozen, An Integer Programming Formulation for a Single Pallet in the Airlift Loading Problem with Insufficient Aircraft. Conference Preceding: World Congress on Computer Science, Computer Engineering and Applied Computing. Las Vegas, NV, 12-15 July 2010.