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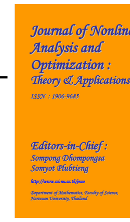

A COMMON FIXED POINT THEOREM VIA FAMILY OF R-WEAKLY COMMUTING MAPS

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ABSTRACT. In the present paper we prove a unique common fixed point theorem for a family of R-weakly commuting maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the result of Khan and Sumitra [5] and few others, also suggest a path to a new inequality containing rational, product and minimum of some terms under implicit relation.

KEYWORDS : Non-Archimedean Menger PM-spaces; R-weakly commuting maps; Common fixed point.
AMS Subject Classification:$47H10, 54H25$.

1. INTRODUCTION

Non-Archimedean probabilistic metric space and some topological preliminaries on them were first studied by Istratescu and Babescu [9] and Istratescu and Crivat [10]. Some fixed point theorems for mappings on non-Archimedean Menger spaces have been proved by Istratescu ([11],[12]) as a result of the generalizations of some of the results of Sehgal and Bharucha-Reid [13] and Sherwood [2]. Achari [3] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istratescu [11]. In 1994, Pant [6] introduced the concept of R-weakly commuting maps in metric spaces. Later on Cho et al. [14] generalised this this idea and gave the concept of R-weakly commuting maps of type A_g. Vasuki [7] proved some common fixed point theorem for R-weakly commuting maps in fuzzy metric spaces. Recently Khan and Sumitra [5] introduced the concept of R-weakly commuting maps in non-Archimedean menger PM-spaces and proved a common fixed point theorem for three point wise

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R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. In the present paper we prove a unique common fixed point theorem for a family of R-weakly commuting maps in non-Archimedean Menger PM-spaces without using the notion of continuity. our result generalizes and extends the result of Khan and Sumitra [5] and other, also suggest a path to a new inequality containing rational, product and minimum of some terms under implicit relation.

2. PRELIMINARIES

Definition 2.1. [10],[11] Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be non-Archimedean probabilistic metric space (N.A. PM-space) if F is a mapping from $X \times X$into D satisfying the following conditions, where the value of Fat $(x, y) \in X \times X$ is represented by $F_{x,y}$ or $F\,(x, y)$ for all $x, y \in X$ such that

(i) $F(x, y; t) = 1$ for all $t > 0$ if and only if $x = y$;
(ii) $F(x, y; t) = F(y, x; t)$;
(iii) $F(x, y; 0) = 0$;
(iv) if $F(x, y; t_1) = F(y, z; t_2) = 1$ then $F(x, z; \max\{t_1, t_2\}) = 1$ for all $x, y, z \in X$.

Definition 2.2. [4] A t-norm is a function $\Delta : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta\,(a, 1) = a$ for all $a \in [0, 1]$.

Definition 2.3. [1],[9] A non-Archimedean Menger PM-space is an ordered triplet (X, F, Δ) where Δ is a t-norm and (X, F) is a N.A. PM-space satisfying the following condition: $F(x, z; \max\{t_1, t_2\}) \geq \Delta(F(x, y; t_1),\ F(y, z; t_2))$ for all $x, y, z \in X, t_1, t_2 \geq 0$. For details of topological preliminaries on non-Archimedean Menger PM-spaces, we refer to Cho et al.[15].

Definition 2.4. [8],[15] An N.A. Menger PM-space (X, F, Δ) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that
$g(F(x, z; t)) \leq g(F(x, y; t)) + g(F(y, z; t))$ for all $x, y, z \in X, t \geq 0$, where $\Omega = \{g|g : [0, 1] \longrightarrow [0, \infty)$ is continuous, strictly decreasing with $g(1) = 0$ and $g(0) < \infty$.

Definition 2.5. [8],[15] A N.A. Menger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists a, $g \in \Omega$ such that $g(\Delta(t_1, t_2)) \leq g(t_1) + g(t_2)$ for all $t_1, t_2 \in [0, 1]$.

Remark 2.6. [8],[15] (i) If N.A. Menger PM-space is of type $(D)_g$then $(X, F, \Delta)$is of type $(C)_g$.
(ii) If (X, F, Δ) is N.A. Menger PM-space and $\Delta \geq \Delta(r, s) = \max(r + s - 1, 1)$ then (X, F, Δ) is of type $(D)_g$ for $g \in \Omega$ and $g(t) = 1 - t$.

Throughout this paper (X, F, Δ) is a complete N.A. Menger PM-space with a continuous strictly increasing t-norm Δ. Let $\emptyset : [0, \infty) \longrightarrow [0, \infty)$ be a function satisfying the condition (Φ); (Φ) (ϕ) is upper semi-continuous from the right and $\phi(t) < t$ for $t > 0$.

Definition 2.7. [8],[15] A sequence $\{x_n\}$ in the N.A. Menger PM-space (X, F, Δ) converges to x if and only if for each $\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x; \epsilon)) < g(1 - \lambda)$ for all $n > M$.

Definition 2.8. [15] A sequence x_n in the N.A. Menger PM-space is a Cauchy sequence if and only if for each$\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x_{n+p}; \epsilon)) < g(1 - \lambda)$ for all $n > M$ and $p \geq 1$.

Example 2.9. [15] Let X be any set with at least two elements. If we define $F(x,x;t) = 1$ for all $x \in X, t > 0$ and

$$F\left(x,y;t\right) = \begin{cases} 0\ if\ t \leq 1 \\ 1\ if\ t > 1 \end{cases}$$

where $x, y \in X, x = y$, then (X, F, Δ) is the N.A. Menger PM-space with $\Delta(a,b) = \min(a,b) or (a.b)$.

Proof: condition (i),(ii) and (iii) are trivial. Let us go for condition (iv) Suppose that $F(x,y;t_1) = 1 = F(y,z;t_2), x \neq y, y \neq z$ then $t_1, t_2 > 1$ implies $\max(t_1,t_2) > 1 \Rightarrow F(x,z;\max(t_1,t_2) = 1, x \neq z$. Also, Menger inequality $F(x,z;\max(t_1,t_2)) \geq \triangle(F(x,y;t_1), \triangle F(x,y;t_1))$ is obvious.Thus $(X, F; \triangle)$ is an N.A. Menger space.

Example 2.10. [15] Let $X = R$ be the set of real numbers equipped with metric defined as $d(x,y) = |x - y|$ Set

$$F\left(x,\ y;\ t\right) = \frac{t}{t + d\left(x,y\right)}$$

Then (X, F, Δ) is a N.A. Menger PM-space with Δ as continuous t-norm Satisfying $\Delta(r,s) = min(r,s) or (r,s)$.

Lemma 2.11. [15] *If a function $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfies the condition (Φ)then we*
(i) For all $t \geq 0, \lim_{\mathrm{n}\longrightarrow\infty} \phi^{\mathrm{n}}(t) = 0$, where $\phi^{\mathrm{n}}\left(t\right)$ is the nth iteration of $\phi(t)$.
(ii) If $\{t_n\}$ is a non decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)\ n = 1, 2, \ldots$ then $\lim_{\mathrm{n}\longrightarrow\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$, than $t = 0$.

Lemma 2.12. [15] *Let $\{y_n\}$ be a sequence in X such that $lim_{n\longrightarrow\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$. If the sequence $\{y_n\}$ is not a Cauchy sequence in X, then there exists $\in_0 > 0, t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive inte-gers such that*
(i) $m_i \geq n_{i+\,1}$ and $n_i \longrightarrow \infty$ as $i \ \longrightarrow\ \infty$
(ii) $F(y_{mi}, y_{ni};\ t_0\ ) < 1 - \in_0$ and $F\left(y_{m_i - 1}, y_{ni};\ t_0\ \right) \geq 1 - \in_0, i = 1, 2, \ldots$

Definition 2.13. [5] Two maps A and S of a Non-Archimedean Menger PM space (X, F, Δ) into itself are said to be R-weakly commuting if there exists some $R > 0$ such that $g(F(ASx, SAx; t) \leq g(F(ASx, SAx; t/R)$ for every $x \in X, t > 0$.

Theorem 2.14. *Let $(X,\ F, *)$ be a complete fuzzy metric space and let f and g be R-weakly commuting self mappings of X satisfying the condition: $M\left(fx, fy, t\right) \geq r.M(gx, gy, t)$ where $r : [0,1] \longrightarrow [0,1]$ is a continuous function such that $r\left(t\right) > t$ for each $0 \leq t < 1$ and $r\left(1\right) = 1$ and the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\{x_n\} \longrightarrow x, \{y_n\} \longrightarrow y$ implies $M(x_n,\, y_n, t) \longrightarrow M(x,\, y, t)$. If the range of g contains the range of f and either f or g is continuous, then f and g have a unique common fixed point.*

3. MAIN RESULTS

Theorem 3.1. *Let S and T be a complete N. A. Menger PM-space $(X, F, \triangle)$. Let $\{R_n\}_{n=1}^{\infty}$ be a family of self mappings satisfying:*

(i) *$R_i(X) \subseteq T(X), R_j(X) \subseteq S(X)$ and the pair $\{R_i, S\}$ and $\{R_j, T\}$ are point wise R-weakly commuting;*

(ii)

$$g(F(R_ix,R_jy;t)) \leq \phi[\max\{g(F(Sx,Ty;t)), g(F(Sx,R_ix;t)), g(F(Ty,R_jy;t)), \frac{1}{2}(g(F(Sx,R_jy;t))+g(F(Ty,R_ix;t))), \min\{g(F(R_jy,Ty;t)), g(F(R_ix,Sx;t))\}, \sqrt{g(F(Ty,R_jy;t)).g(F(Ty,R_ix;t))}, \frac{g(F(Sx,R_jy;t)).g(F(Ty,R_jy;t))}{g(F(Sx,Ty;t))}\}]$$

for every $x,\ y \in X,\ i = 2n-1,\ j = 2n,\ (n \in N)$ *and* $i \neq j$, *where* ϕ *satisfies the condition* (Φ). *Then* $\{R_n\}$,S *and* T *have a unique common fixed point in* X.

Proof:- Since $R_i(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $R_1(x_0) = Tx_1$.
Since $R_j(X) \subseteq S(X)$, for this x_1 we can choose a point $x_2 \in X$ such that $R_2(x_1) = Sx_2$ and so on. Inductively, We can define a sequence $\{y_n\}$in X,

$$y_{2n} = R_{2n+1}(x_{2n}) = T_{x_{2n+1}}, y_{2n-1} = R_{2n}(x_{2n-1}) = Sx_{2n} \qquad n = 1,2,3... \tag{3.1}$$

Let $M_n = g(F(R_{2n+1}(x_n), R_{2n}(x_{n-1});t)) = g(f(y_n, y_{n-1};t))$ $n = 1,2,3,...$ then

$$\begin{aligned} M_{2n} =& g(F(R_{2n+1}(x_{2n}), R_{2n}(x_{2n-1});t)) \\ \leq& \phi[max\{g(F(Sx_{2n}, Tx_{2n-1};t)), g(F(Sx_{2n}, R_{2n+1}(x_{2n});t)), \\ & g(F(Tx_{2n-1}, R_{2n}(x_{2n-1});t)), \\ & \frac{1}{2}(g(F(Sx_{2n}, R_{2n}(x_{2n-1});t)) + g(F(Tx_{2n-1}, R_{2n+1}(x_{2n});t))), \\ & \min\{g(F(R_{2n}(x_{2n-1}), Tx_{2n-1};t)), g(F(R_{2n+1}(x_{2n}), Sx_{2n};t))\}, \\ & \sqrt{g(F(Tx_{2n-1}, R_{2n}(x_{2n-1});t)).g(F(Tx_{2n-1}, R_{2n+1}(x_{2n});t))}, \\ & \frac{g(F(R_{2n}(x_{2n-1}), Sx_{2n};t)).g(F(Tx_{2n-1}, R_{2n}(x_{2n-1});t))}{g(F(Sx_{2n}, Tx_{2n-1};t)))}\}] \end{aligned}$$

$$\begin{aligned} M_{2n} =& \phi[max\{g(F(y_{2n-1}, y_{2n-2};t)), g(F(y_{2n-1}, y_{2n};t)), g(F(y_{2n-2}, y_{2n-1};t)), \\ & \frac{1}{2}(g(F(y_{2n-1}, y_{2n-1};t)) + g(F(y_{2n-2}, y_{2n};t))), \\ & \min\{g(F(y_{2n-1}, y_{2n-2};t)), g(F(y_{2n}, y_{2n-1};t))\}, \\ & \sqrt{g(F(y_{2n-1}, y_{2n-2};t)).g(F(y_{2n-1}, y_{2n-2};t))}, \\ & \frac{g(F(y_{2n-1}, y_{2n-1};t)).g(F(y_{2n-1}, y_{2n-2};t))}{g(F(y_{2n-1}, y_{2n-2};t))}\}] \end{aligned}$$

i.e.

$$\begin{aligned} M_{2n} \leq& \phi[Max\{M_{2n-1}, M_{2n}, M_{2n-1}, \frac{1}{2}(M_{2n-1}+M_{2n}), \min\{M_{2n-1}, M_{2n}\}, \\ & \sqrt{(M_{2n-1})^2}, g(1)\}] \end{aligned} \tag{3.2}$$

Case I : If $M_{2n} > M_{2n-1}$then by (3.2) $M_{2n} \leq \phi(M_{2n})$Which is contradiction.
Case II : If $M_{2n-1} > M_{2n}$ then by (3.2) gives $M_{2n} \leq \phi M_{2n-1}$
Then by lemma (2.12) we get $lim_{n \longrightarrow \infty} M_{2n} = 0$ i.e.
$lim_{n \longrightarrow \infty} g(F(Rx_{2n+1}(x_{2n}), R_{2n}(x_{2n-1});t)) = 0$ or
$lim_{n \longrightarrow \infty} g(F(y_{2n}, y_{2n-1};t)) = 0$ Similarly, we can show that
$\lim_{n \longrightarrow \infty} g(F(R_{2n}(x_{2n+1}), R_{2n+1}(x_{2n+2});t)) = 0$ or

$\lim_{n\longrightarrow\infty} g(F(y_{2n+1}, y_{2n+2}; t)) = 0$ Thus we have
$\lim_{n\longrightarrow\infty} g(F(R_{2n+1}(x_n), R_{2n}(x_{2n+1}); t)) = 0$ For all $t > 0$ or

$$\lim_{n\longrightarrow\infty} g(F(y_n, y_{n+1}; t)) = 0 \tag{3.3}$$

Before preceding the proof of the theorem, we first prove the following claim
Claim: Let $\{R_n\}_{n=1}^{\infty}$,S and $T : X \longrightarrow X$ be maps satisfying equations(3.1), (3.2), (3.3) and $\{y_n\}$ defined by (3.1) such that

$$\lim_{n\longrightarrow\infty} g(F(y_n, y_{n+1}; t)) = 0 \tag{3.4}$$

for all n, is a Cauchy sequence.
Proof of the Claim :- Since $g \in \Omega$, it follows that $\lim_{n\longrightarrow\infty} F(y_n, y_{n+1}) = 1$ for each $t > 0$ iff $\lim_{n\longrightarrow\infty} g(f(y_n, y_{n+1}; t)) = 1$ for each $t > 0$. By Lemma (2.12) If $\{y_n\}$ is not a Cauchy sequence In X, there exists $\in_0 > 0, t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that:
 (a) $m_i > n_i + 1$ and $n_i \longrightarrow \infty$ as $i \longrightarrow \infty$
 (b) $g(F(y_{m_i}, y_{n_i}; t_0) > g(1- \in_0)$ and

$$g(F(y_{m_i-1}, y_{n_i}; t_0) \leq (1- \in_0), i = 1, 2, ...$$

Since $g(t) = 1 - t$, we have

$$\begin{aligned} g(1- \in_0) &< g(f(y_{m_i}, y_{n_i}; t_0)) \\ g(1- \in_0) &\leq g(F(y_{m_i}, y_{m_i-1}; t_0)) + g(F(y_{m_i-1}, y_{n_i}; t_0)) \\ g(1- \in_0) &\leq g(F(y_{m_i}, y_{m_i-1}; t_o)) + g(1- \in_0) \end{aligned} \tag{3.5}$$

As $i \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} g(F(y_{m_i}, y_{n_i}; t_0)) = g(1- \in_0) \tag{3.6}$$

on the other hand , we have

$$\begin{aligned} g(1- \in_0) &< g(f(y_{m_i}, y_{n_i}; t_0)) \\ g(1- \in_0) &\leq g(F(y_{n_i}, y_{n_i+1}; t_0)) + g(F(y_{m_i}, y_{n_i+1}; t_0)) \end{aligned} \tag{3.7}$$

Now consider $g(F(y_{m_i}, y_{n_i+1}; t_0)$ in (3.7) and assume that both m_i and n_i are even. Then by (ii) of Theorem 3.1, we have

$$\begin{aligned} g(F(y_{m_i}, y_{n_i+1}; t_0)) \;=\; & g(F(R_{2n+1}(x_{mi}), R_{2n}(x_{n_i+1}); t_0)) \\ & \leq \phi[max\{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0)), g(F(Sx_{m_i}, R_{2n+1}(x_{m_i}); t_0)), \\ & g(F(Tx_{n_i+1}, R_{2n}(x_{n_i+1}); t_0)), \\ & \frac{1}{2}(g(F(Sx_{m_i}, R_{2n}(x_{n_i+1}); t_0)) + g(F(Tx_{n_i+1}, R_{2n+1}(x_{m_i}); t_0))), \\ & \min\{g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0)), g(F(R_{2n+1}(x_{m_i}), Sx_{m_i}; t_0))\}, \\ & \sqrt{g(FR_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0)).g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0))}, \\ & \frac{g(F(R_{2n}(x_{n_i+1}), Sx_{m_i}; t_0)).g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0))}{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0))}\}] \end{aligned}$$

$$\begin{aligned} \leq \;& \phi[\max\{g(F(y_{m_i-1}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_o)), \\ & g(f(y_{n_i}, y_{n_i+1}; t_0)), \\ & \frac{1}{2}(g(F(y_{m_i-1}, y_{n_i+1}; t_0)) + g(F(y_{n_i}, y_{m_i}; t_0)))\}, \\ & \min\{g(F(y_{n_i+1}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_o))), \\ & \sqrt{g(F(y_{n_i+1}, y_{n_i}; t_0)), g(F(y_{n_i}, y_{n_i+1}; t_o))}, \end{aligned}$$

$$\frac{g(F(y_{n_i+1}, y_{m_i-1}; t_0)), g(F(y_{n_i+1}, y_{n_i}; t_o))}{g(f(y_{n_i}, y_{m_i-1}; t_o))}\}]$$

Which on letting $n \longrightarrow \infty$, reduces to

$$g(1- \in_0) \le \phi[max\{g(1- \in_0), 0, 0, g(1- \in_0), \min\{0,0\}, 0, 0\}]$$
$$g(1- \in_0) \le \phi g(1- \in_0)$$

which is contradiction. Hence the sequence $\{y_n\}$ defined by (3.1) is a cauchy sequence, which concludes the proof of the claim.

Since X is complete, then the sequence $\{y_n\}$ converges to a point z in X and so the Sub-sequences $\lim_{n\longrightarrow\infty} R_{2n+1}(x_{2n})$, $\lim_{n\longrightarrow\infty} R_{2n}(x_{2n+1})$, $\lim_{n\longrightarrow\infty} Sx_{2n}$ and $\lim_{x\longrightarrow\infty} Tx_{2n+1}$ of seq. $\{y_n\}$ also Converge to the limit z.
Since the pair (R_i, S) are R-weakly commuting, So by definition (2.13)

$$g(F(R_iSx_{2n+1}, SR_ix_{2n+1}; t)) \le g(F(R_ix_{2n+1}, Sx_{2n+1}; t/R))$$

which gives

$$\lim_{n\longrightarrow\infty} R_iSx_{2n+1} = \lim_{n\longrightarrow\infty} SR_ix_{2n+1} = Sz \text{ as } S \text{ is continuous}$$

Implies $\lim_{n\longrightarrow\infty} R_iSx_{2n+1} = Sz$ and $\lim_{n\longrightarrow\infty} SR_ix_{2n+1} = Sz$.
Now we claim that $Sz = z$. Contrary suppose contrary that $Sz \ne z$ then by (ii) of Theorem (3.1)

$$\begin{aligned} g(F(R_iSx_{2n+1}, R_jx_{2n}; t)) \le& \phi[max\{g(F(SSx_{2n+1}, Tx_{2n}; t)),\\ & g(F(SSx_{2n+1}, R_ix_{2n+1}; t)), g(F(Tx_{2n}, R_jx_{2n}; t)),\\ & \frac{1}{2}(g(F(SSx_{2n+1}; R_jx_{2n}; t)) + g(F(Tx_{2n}, R_iSx_{2n+1}; t))),\\ & \min\{g(F(R_jx_{2n}, Tx_{2n}; t)), g(F(R_iSx_{2n+1}, SSx_{2n+1}; t))\},\\ & \sqrt{g(F(R_jx_{2n}, Tx_{2n}; t)).g(F(Tx_{2n}, R_jx_{2n+1}; t))},\\ & \frac{g(F(R_jx_{2n}, SSx_{2n+1}; t)).g(F(Tx_{2n}, R_jx_{2n}; t))}{g(F(SSx_{2n+1}, Tx_{2n}; t))}\}] \end{aligned}$$

Which on letting limit $n \longrightarrow \infty$

$$\begin{aligned} g(F(Sz, z; t)) \le& \phi[max\{g(F(Sz, z; t)), g(F(Sz, z; t))g(F(z, z; t)),\\ & \frac{1}{2}(g(F(Sz, z; t)) + g(F(z, Sz; t))),\\ & \min\{g(F(z, z; t)), g(F(Sz, Sz; t))\},\\ & \sqrt{g(F(z, z; t)).g(F(z, z; t))},\\ & \frac{g(F(z, Sz; t))g(F(z, z; t))}{g(F(Sz, z; t))}\}]\\ =& \phi[max\{g(F(Sz, z; t)), g(F(Sz, z; t)), g(1), g(F(Sz, z, t)),\\ & g(1), g(1), g(1)\}]\\ g(F(Sz, z; t)) \le& \phi(g(F(Sz, z; t))) < g(F(Sz, z; t)) \end{aligned}$$

Thus z is a fixed point of S Similarly we can show that z is a fixed point of R_i

Again pair (R_j, T) is R-weakly commuting so by definition of (2.13)

$$g(F(R_jTx_{2n+1}, TR_jx_{2n+1}; t)) \le g(F(R_jx_{2n+1}, Tx_{2n+1}; t/R))$$

which gives

$$\lim_{n\longrightarrow\infty} R_jTx_{2n+1} = \lim_{n\longrightarrow\infty} TR_jx_{2n+1} = Tz \text{ as } T \text{ is continuous}$$

Implies $\lim_{n\longrightarrow\infty} R_jTx_{2n+1} = Tz$ and $\lim_{n\longrightarrow\infty} TR_jx_{2n+1} = Tz$.
We have to show that $Tz = z$, to do this contrary suppose that $Tz \neq z$ then by (ii) of Theorem (3.1)

$$\begin{aligned} g(R_iz, R_jTx_{2n}; t) \leq \phi[max\{& g(F(Sz, TTx_{2n}; t)), g(F(Sz, R_iz; t)), \\ & g(F(TTx_{2n}, R_j(Tx_{2n}); t)), \\ & \frac{1}{2}(g(F(Sz, R_jTx_{2n}; t)) + g(F(TTx_{2n}, R_iz; t))) \\ & \min\{g(F(R_j(Tx_{2n}, TTx_{2n}; t))), g(F(R_iz, Sz; t))\} \\ & \sqrt{g(F(R_j(Tx_{2n}), TTx_{2n}; t)).g(F(R_j(Tx_{2n}), TTx_{2n}; t))}, \\ & \frac{g(F(R_j(Tx_{2n}, Sz; t))).g(F(R_j(Tx_{2n}), TTx_{2n}; t))}{g(F(Sz, , TTx_{2n}; t))}\}] \end{aligned}$$

which on letting limit $n \longrightarrow \infty$

$$\begin{aligned} g(F(Rz, Tz; t)) \leq & \phi[max\{g(F(z, Tz; t)), g(F(z, z; t)), g(F(Tz, Tz; t)), \\ & \frac{1}{2}g(F(z, Tz; t)) + g(F(Tz, z; t)), \\ & \min\{g(F(Tz, Tz; t)), g(F(z, z; t))\}, \\ & \sqrt{(g(F(Tz, Tz; t)))^2}, \\ & \frac{g(F(Tz, z; t)).g(F(Tz, Tz; t))}{g(F(z, Tz; t))}\}] \end{aligned}$$

i.e. $g(F(z, Tz; t) \leq \phi(g(F(z, Tz; t)) < g(F(z, Tz; t))$.
Which is a contradiction, Thus z is a fixed point of T. Similarly we can show that z is a fixed point of R_j. Hence $R_iz = R_jz = Sz = Tz = z$. Thus z is a common fixed point of R_i,R_j,S & T. The uniqueness of the common fixed point follows from inequality (ii) of Theorem (3.1).

In the paper Khan & Sumitra [5], obtained a common fixed point theorem in 2 non Archimedean Menger spaces for R-weakly commuting maps inspired by this result we motivate to prove more generalized version in the setting of non-Archimedean Menger PM spaces.

Corollary 3.2. *Let R_1, R_2, S and T be four continuous self maps of a complete N.A Menger PM spaces (X, F, Δ) ,satisfying*
(i) $R_1(X) \subseteq T(X), R_2(X) \subseteq S(X)$, and $\{R_1, S\}$ and $\{R_2, S\}$ are R-weakly Commuting
(ii)

$$\begin{aligned} g(F(R_1(x), R_2(y); t)) \leq & \phi[max\{g(F(Sx, Ty; t)), g(F(Sx, R_1x; t)), g(F(Ty, R_2y; t)) \\ & \frac{1}{2}(g(F(Sx, R_2y; t)) + g(F(Ty, R_1x; t))), \\ & \min\{g(F(R_2y, Ty, t)), g(F(R_1x, Sx, t))\}, \\ & \sqrt{g(F(R_2y, Sy; t)).g(F(R_2y, Ty; t))}, \\ & \frac{g(F(R_2y, Sx; t)).g(F(R_2y, Ty; t))}{g(F(Sx, Ty; t))}\}] \end{aligned}$$

for every $x, y \in X$, where ϕ satisfies the condition (Φ) . Then R_1, R_2, S and T have a unique common fixed point in X.

Corollary 3.3. *Let R, S, T be three continuous self maps of a complete N.A Menger PM- spaces (X, F, Δ) satisfies;*
(i) $R(x) \subseteq S(x) \cap T(x)$, and pair $\{R, S\}$ and $\{R, T\}$ are R-weakly commuting
(ii)

$$\begin{aligned} g(F(Rx, Ry; t)) \leq & \phi[max\{g(F(Sx, Ty; t)), g(F(Sx, Rx; t)), g(F(Ty, Ry; t)), \\ & \frac{1}{2}(g(F(Sx, Ry; t)) + g(F(Ty, Rx; t))), \\ & \min\{g(F(Ry, Ty, t)), g(F(Rx, Sx, t))\}, \\ & \sqrt{g(F(Ry, Sy; t)).g(F(Ry, Ty; t))}, \\ & \frac{g(F(Ry, Sx; t)).g(F(Ry, Ty; t))}{g(F(Sx, Ty; t))}\}] \end{aligned}$$

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then R, S and T have a unique common fixed point in X.

Corollary 3.4. *Let R, S be two continuous self maps of a complete N.A Menger PM space (X, F, Δ) satisfying;*
(i) $R(X) \subseteq S(X)$ and the pair $\{R, S\}$ is R-weakly commuting
(ii)

$$\begin{aligned} g(F(Rx, Ry; t)) \leq & \phi[man\{g(F(Sx, Sy; t)), g(F(Sx, Rx; t)), g(F(Sy, Ry; T)), \\ & \frac{1}{2}(g(F(Sx, Ry; t)) + g(F(Sy, Rx; t))), \\ & \min\{g(F(Ry, Sy, t)), g(F(Rx, Sx, t))\}, \\ & \sqrt{g(F(Ry, Sy; t)).g(F(Ry, Sy; t))}, \\ & \frac{g(F(Ry, Sx; t)).g(F(Ry, Sy; t))}{g(F(Sx, Sy; t))}\}] \end{aligned}$$

for every $x, y \in X$, Where ϕ satisfies the condition (Φ). Then $R and\ S$ have a unique common fixed point in X.

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