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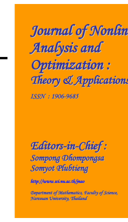

ON PARANORMED I-CONVERGENT SEQUENCE SPACES OF INTERVAL NUMBERS

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ABSTRACT. In this article we introduce and study the paranormed I-convergent sequence spaces $\mathcal{C}^I(\bar{\mathcal{A}}, p)$, $\mathcal{C}_0^I(\bar{\mathcal{A}}, p)$, $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p)$ on the sequence of interval numbers with the help of a bounded sequence $p = (p_k)$ of strictly positive real numbers. We study some topological and algebraic properties and some inclusion relations on these spaces.

KEYWORDS: Interval numbers; Ideal; Filter; I-convergent sequence; Solid and monotone space; Banach space.

AMS Subject Classification: 46A40

1. INTRODUCTION AND PRELIMINARIES

It is an admitted fact that the real and complex numbers are playing a vital role in the world of mathematics. Many mathematical structures have been constructed with the help of these numbers. In recent years, since 1965 fuzzy numbers and interval numbers also managed their place in the world of mathematics and credited into account some alike structures . Interval arithmetic was first suggested by P.S.Dwyer [5] in 1951. Further development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R.E.Moore

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[13] in 1959 and Moore and Yang [14] and others and have developed applications to differential equations.

Recently, Chiao [4] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryılmaz [19] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

Here after, we give the notation and definitions that will be used in the paper.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $I\mathbb{R}$. Any element of $I\mathbb{R}$ is called a closed interval and it is denoted by $\bar{A} = [x_l, x_r]$. An interval number is closed subset of real numbers [4]. The algebraic operations for interval numbers can be found in [19].

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{A}_1, \bar{A}_2) = \max\{| x_{1_l} - x_{2_l} |, | x_{1_r} - x_{2_r} |\}, (\text{see})[14, 19] \tag{1.1}$$

where x_l and x_r be first and last points of $\bar{A}$, respectively.

In a special case, $\bar{A}_1 = [a, a]$, $\bar{A}_2 = [b, b]$, we obtain the usual metric of $\mathbb{R}$ with

$$d(\bar{A}_1, \bar{A}_2) =| a - b | .$$

Let us define transformation f from $\mathbb{N}$ to $I\mathbb{R}$ by $k \to f(k) = \bar{A}$, $\bar{\mathcal{A}} = (\bar{A}_k)$. The function f is called sequence of interval numbers, where $\bar{A}_k$ is the k^{th} term of the sequence $(\bar{A}_k)$.

Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{\mathcal{A}}) = \{\bar{\mathcal{A}} = (\bar{A}_k) : \bar{A}_k \in I\mathbb{R}\}. \tag{1.2}$$

The algebraic properties of $\omega(\bar{\mathcal{A}})$ can be found in [4, 19].

The following definitions were given by Şengönül and Eryılmaz in [19].

A sequence $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ of interval numbers is said to be convergent to an interval number $\bar{A}_0 = [x_{0_l}, x_{0_r}]$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that $d(\bar{A}_k, \bar{A}_0) < \epsilon$, for all $k \geq n_0$ and we denote it as $\lim_k \bar{A}_k = \bar{A}_0$.

Thus, $\lim_k \bar{A}_k = \bar{A}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$ and it is said to be Cauchy sequence of interval numbers if for each $\epsilon > 0$, there exists a positive integer k_0

such that $d(\bar{A}_k, \bar{A}_m) < \epsilon$, whenever $k,\ m \geq k_0$.

Let us denote the space of all convergent, null and bounded sequences of interval numbers by $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$, respectively. The sets $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are complete metric spaces with the metric

$$\widehat{d}(\bar{A}_k, \bar{B}_k) = \sup_k \max\{|x_{kl} - y_{kl}|, |x_{rl} - y_{rl}|\}. \tag{1.3}$$

If we take $\bar{B}_k = \bar{O}$ in (3) then, the metric $\widehat{d}$ reduces to (see,[19])

$$\widehat{d}(\bar{A}_k, \bar{O}) = \sup_k \max\{|x_{kl}|, |x_{rl}|\}. \tag{1.4}$$

In this paper, we assume that a norm $||\bar{A}_k||$ of the sequence of interval numbers $(\bar{A}_k)$ is the distance from $(\bar{A}_k)$ to $\bar{O}$ and satisfies the following properties:
$\forall \bar{A}_k, \bar{B}_k \in \lambda(\bar{\mathcal{A}})$ and $\forall \alpha \in \mathbb{R}$
(N_1). $\forall \bar{A}_k \in \lambda(\bar{\mathcal{A}}) - \{\bar{O}\}, ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} > 0$;
(N_2). $||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = 0 \Leftrightarrow \bar{A}_k = \bar{O}$;
(N_3). $||\bar{A}_k + \bar{B}_k||_{\lambda(\bar{\mathcal{A}})} \leq ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} + ||\bar{B}_k||_{\lambda(\bar{\mathcal{A}})}$
(N_4). $||\alpha \bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = |\alpha|||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})}$, where $\lambda(\bar{\mathcal{A}})$ is a subset of $\omega(\bar{\mathcal{A}})$.

Let $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ be the element of $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ or $\ell_\infty(\bar{\mathcal{A}})$. Then, with respect to the above discussion the classes of sequences $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are normed interval spaces normed by

$$\| \bar{\mathcal{A}} \| = \sup_k \max\{| x_{k_l} |, | x_{k_r} |\} \text{ (see[19])}. \tag{1.5}$$

Throughout, $\bar{O} = [0, 0]$ and $\bar{I} = [1, 1]$ represent zero and identity interval numbers according to addition and multiplication, respectively.

As a generalisation of usual convergence for the sequences of real or complex numbers, the concept of statistical convergence was first introduced by Fast [6] and also independently by Buck [3] and Schoenberg [18]. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [7], Šalát [15], Tripathy [20] and many others. The notion of statistical convergence has been extended to interval numbers by Esi as follows in [1, 2].

Let us suppose that $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$. If, for every $\epsilon > 0$,

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon,\ n \leq k\}| = 0 \tag{1.6}$$

then the sequence $\bar{\mathcal{A}} = (\bar{A}_k)$ is said to be statistically convergent to an interval number $\bar{A}_0$, where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\epsilon)) = 0$, where $A(\epsilon) = \{k \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\}$.

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyński [11, 12]. Later on, it was studied by Šalát, Tripathy and Ziman [16, 17], Esi and Hazarika [1], Tripathy and Hazarika [21], Khan *et al* [8, 9, 10] and many others.

Theorem 1.1. *Let suppose that I be an ideal.*

Then, a sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) \subset \omega(\bar{\mathcal{A}})$
(i) is said to be I-convergent to an interval number$\bar{A}_0$ if for every $\epsilon > 0$, the set $\{k \in N : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\} \in I$.

In this case, we write $I - \lim \bar{A}_k = \bar{A}_0$. If $\bar{A}_0 = \bar{O}$ then the sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is said to be I-null. In this case, we write $I - \lim \bar{A}_k = \bar{O}$.
(ii) is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : \| \bar{A}_k - \bar{A_m} \| \geq \epsilon\} \in I$.
(iii) is said to be I-bounded if there exists some $M > 0$ such that $\{k \in N : \| \bar{A}_k \| \geq M\} \in I$.

Let us denote the classes of I-convergent, I-null, bounded I-convergent and bounded I-null sequences of interval numbers with $\mathcal{C}^I(\bar{\mathcal{A}})$, $\mathcal{C}^I_\circ(\bar{\mathcal{A}})$, $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}})$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}})$, respectively.

We know that for each ideal I, there is a filter $\pounds(I)$ corresponding to I, i.e $\pounds(I)$ = $\{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Theorem 1.2. *A sequence space $\lambda(\bar{\mathcal{A}})$ of interval numbers*
(iv) is said to be solid(normal) if $(\alpha_k \bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and for any sequence (α_k) of scalars with $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$,
(v) is said to be symmetric if $(\bar{A}_{\pi(k)}) \in \lambda(\bar{\mathcal{A}})$ whenever $\bar{A}_k \in \lambda(\bar{\mathcal{A}})$, where π is a permutation on $\mathbb{N}$,
(vi) is said to be sequence algebra if $(\bar{A}_k) * (\bar{B}_k) = (\bar{A}_k.\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k)$, $(\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$,
(vii) is said to be convergence free if $(\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and $\bar{A}_k = \bar{O}$ implies $\bar{B}_k = \bar{O}$, for all k.

Theorem 1.3. *Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$. The K-step space of the $\lambda(\bar{\mathcal{A}})$ is a sequence space $\mu_K^{\lambda(\bar{\mathcal{A}})} = \{(\bar{A}_{k_n}) \in \omega(\bar{\mathcal{A}}) : (\bar{A}_k) \in \lambda(\bar{\mathcal{A}})\}$.*

Theorem 1.4. *A canonical pre-image of a sequence* $(\bar{A}_{k_n}) \in \mu_K^{\lambda(\bar{\mathcal{A}})}$ *is a sequence*

$(\bar{B}_k) \in \omega(\bar{\mathcal{A}})$ *defined by*

$$\bar{B}_k = \begin{cases} \bar{A}_k, & \text{if } k \in K, \\ \bar{O}, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\mu_K^{\lambda(\bar{\mathcal{A}})}$ *is a set of canonical preimages of all elements in* $\mu_K^{\lambda(\bar{\mathcal{A}})}$*, i.e.* $\bar{\mathcal{B}}$ *is in the canonical preimage of* $\mu_K^{\lambda(\bar{\mathcal{A}})}$ *iff* $\bar{\mathcal{B}}$ *is the canonical preimage of some* $\bar{\mathcal{A}} \in \mu_K^{\lambda(\bar{\mathcal{A}})}$.

Theorem 1.5. *A sequence space* $\lambda(\bar{\mathcal{A}})$ *is said to be monotone if it contains the canonical preimages of its step space.*

Now, we give some important Lemmas:

Lemma 1.6. *Every solid space is monotone.*

Lemma 1.7. *Let* $K \in \pounds(I)$ *and* $M \subseteq N$*. If* $M \notin I$*, then* $M \cap K \notin I$*, where* $\pounds(I) \subseteq 2^N$ *filter on* N.

Lemma 1.8. *If* $I \subseteq 2^N$ *and* $M \subseteq N$*. If* $M \notin I$*, then* $M \cap N \notin I$.

2. MAIN RESULTS

Let us give an instrumental and important definition for this paper:

Theorem 2.1. *Let* $\bar{X}$ *be a set of interval numbers. A function* $g : \bar{X} \longrightarrow \mathbb{R}$ *is called paranorm on* $\bar{X}$ *, if for all* $A, \ B \in \bar{X}$,
(P_1) $g(A) = 0 \; if \; A = \bar{0}$,
(P_2) $g(-A) = g(A)$,
(P_3) $g(A + B) \leq g(A) + g(B)$,
(P_4) *If* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda \ (n \to \infty)$ *and* $(A_n), A_0 \in \bar{X}$ *with* $A_n \to A_0 \ (n \to \infty)$ *in the sense that* $g(A_n - A_0) \to 0 \ (n \to \infty)$ *, then* $g(\lambda_n A_n - \lambda A_0) \to 0 \ (n \to \infty)$.

In this article, we introduce and study the following paranormed classes of sequences of interval numbers

$$\mathcal{C}^I(\bar{\mathcal{A}}, p) = \Big\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{ k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon \big\} \in I, \Big\}, \tag{2.1}$$

$$\mathcal{C}_0^I(\bar{\mathcal{A}}, p) = \Big\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{ k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon \big\} \in I \Big\}, \tag{2.2}$$

$$\ell_\infty(\bar{\mathcal{A}}, p) = \Big\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \sup_k (\parallel \bar{A}_k \parallel)^{p_k} < \infty \Big\}, \tag{2.3}$$

where $p = (p_k)$ is a bounded sequence of strictly positive real numbers.

We also denote

$$\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p) = \ell_\infty(\bar{\mathcal{A}}, p) \cap \mathcal{C}^I(\bar{\mathcal{A}}, p) \text{ and } \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p) = \ell_\infty(\bar{\mathcal{A}}, p) \cap \mathcal{C}^I_0(\bar{\mathcal{A}}, p).$$

Theorem 2.2. *The classes of sequences $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ are paranormed spaces paranormed by*

$$g(\bar{\mathcal{A}}) = \sup_k \parallel \bar{A}_k \parallel^{\frac{p_k}{M}}, \text{ where } M = \max\{1, \sup_k p_k\}.$$

Proof. Let $\bar{\mathcal{A}} = (\bar{A}_k),\ \bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$.
(P_1) It is Clear that $g(\bar{\mathcal{A}}) = 0$ if $\bar{\mathcal{A}} = \bar{\theta}$.
(P_2) $g(\bar{\mathcal{A}}) = g(-\bar{\mathcal{A}})$ is obvious.
(P_3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$\begin{aligned} g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) = g(\bar{A}_k + \bar{B}_k) &= \sup_k \parallel \bar{A}_k + \bar{B}_k) \parallel^{\frac{p_k}{M}} \\ &\leq \sup_k \parallel \bar{A}_k \parallel^{\frac{p_k}{M}} + \sup_k \parallel \bar{B}_k \parallel^{\frac{p_k}{M}} \\ &= g(\bar{A}_k) + g(\bar{B}_k) = g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}}) \end{aligned}$$

Therefore, $g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) \leq g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}})$.
(P_4) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda$ $(k \to \infty)$ and $(\bar{A}_k),\ \bar{A}_0 \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k \to \bar{A}_0$ $(k \to \infty)$, in the sense that $g(\bar{A}_k - \bar{A}_0) \to 0$ $(k \to \infty)$. Then, since the inequality

$$g(\bar{A}_k) \leq g(\bar{A}_k - \bar{A}_0) + g(\bar{A}_0)$$

holds by subadditivity of g, the sequence $\{g(\bar{A}_k)\}$ is bounded.
Therefore,

$$\begin{aligned} g\big[(\lambda_k \bar{A}_k - \lambda \bar{A}_0)\big] &= g\big[(\lambda_k \bar{A}_k - \lambda \bar{A}_k + \lambda \bar{A}_k - \lambda \bar{A}_0)\big] \\ &= g\big[(\lambda_k - \lambda)\bar{A}_k + \lambda(\bar{A}_k - \bar{A}_0)\big] \\ &\leq g\big[(\lambda_k - \lambda)\bar{A}_k\big] + g\big[\lambda(\bar{A}_k - \bar{A}_0)\big] \\ &\leq \mid (\lambda_k - \lambda) \mid^{\frac{p_k}{M}} g(\bar{A}_k) + \mid \lambda \mid^{\frac{p_k}{M}} g(\bar{A}_k - \bar{A}_0) \to 0 \end{aligned}$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous.
Hence $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ is a paranormed space. For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$, the proof is similar. □

Theorem 2.3. *The set $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ is closed subspace of $\ell_\infty(\bar{\mathcal{A}}, p)$.*

Proof. Let $(\bar{A}_k^{(n)})$ be a Cauchy sequence in $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k^{(n)} \to \bar{A}$. We show that $\bar{A} \in \mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$. Since $(\bar{A}_k^{(n)}) \in \mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ there exists $\bar{A}_n$ such that

$$\{k \in \mathbb{N} : \| \bar{A}_k^{(n)} - \bar{A}_n \|^{p_k} \geq \epsilon\} \in I.$$

We need to show that
(1) $(\bar{A}_n)$ converges to $\bar{A}_0$.
(2) If $U = \{k \in \mathbb{N} : \big(\| \bar{A}_k - \bar{A}_0 \| \big)^{p_k} < \epsilon\}$, then $U^c \in I$.
(1) Since $(\bar{A}_k^{(n)})$ is Cauchy sequence in $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p) \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_k \big(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \| \big)^{\frac{p_k}{M}} < \frac{\epsilon}{3}$, for all $n,\ q \geq k_0$. For $\epsilon > 0$, we have

$$B_{nq} = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_q = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q)} - \bar{A}_q \| \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_n = \left\{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_n \| \big)^{p_k} < (\frac{\epsilon}{3})^M\right\}.$$

Then, B_{nq}^c, B_q^c and $B_n^c \in I$. Let $B^c = B_{nq}^c \cup B_q^c \cup B_n^c$, where $B = \{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_n \| \big)^{p_k} < \epsilon\}$. Then $B^c \in I$.
If we choose a integer $k_0 \in B^c$ then for each $n,\ q \geq k_0$, we have

$$\begin{aligned} &\big\{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_n \| \big)^{p_k} < \epsilon\big\} \\ &\supseteq \Big[\{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_k^{(q)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q)} - \bar{A}_k^{(n)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_n \| \big)^{p_k} < (\frac{\epsilon}{3})^M\}\Big] \end{aligned} \tag{2.4}$$

Then $(\bar{A}_n)$ is a Cauchy sequence of interval numbers, so there exists some interval number $\bar{A}_0$ such that $\bar{A}_n \to \bar{A}_0$ as $n \to \infty$.
(2) Let $0 < \delta < 1$ be given. Then, we show that if

$$U = \{k \in \mathbb{N} : \big(\| \bar{A}_k - \bar{A}_0 \| \big)^{p_k} < \delta\},$$

then $U^c \in I$. Since $(\bar{A}_k^{(n)}) \to \bar{A}$ then, there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_k \| \big)^{p_k} < (\frac{\delta}{3D})^M\} \tag{2.5}$$

implies $P^c \in I$. The number q_0 can be chosen that together with (11), we have

$$Q = \{k \in \mathbb{N} : \big(\| \bar{A}_{q_0} - \bar{A}_0 \| \big)^{p_k} < (\frac{\delta}{3D})^M\}$$

such that $Q^c \in I$. Since $\{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \| \big)^{p_k} \geq \delta\} \in I$ we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \| \big)^{p_k} < (\frac{\delta}{3D})^M\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_0 \parallel \big)^{p_k} < \delta\}$. Therefore, for each $k \in U^c$, we have

$$\begin{aligned}
&\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_0 \parallel \big)^{p_k} < \delta\} \\
&\supseteq [\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_k^{(q_0)} \parallel \big)^{p_k} < (\frac{\delta}{3})^M\} \\
&\cap \{k \in \mathbb{N} : \big(\parallel \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \parallel \big)^{p_k} < (\frac{\delta}{3})^M\} \\
&\cap \{k \in \mathbb{N} : \big(\parallel \bar{A}_{q_0} - \bar{A}_0 \parallel \big)^{p_k} < (\frac{\delta}{3})^M\}] \qquad (2.6)
\end{aligned}$$

Then, the result follows from (12). □

Since the inclusions $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p) \subset \ell_\infty(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p) \subset \ell_\infty(\bar{\mathcal{A}}, p)$ are strict so in view of Theorem (2.3) we have the following result.

Theorem 2.4. *The spaces $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p)$ are nowhere dense subsets of $\ell_\infty(\bar{\mathcal{A}}, p)$.*

Theorem 2.5. *The spaces $\mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p)$ are both solid and monotone.*

Proof. We shall prove the result for $\mathcal{C}_0^I(\bar{\mathcal{A}}, p)$. For $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p)$, the result follows similarly.

For, let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ and (α_k) be a sequence of scalars with $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$.

Since $\mid \alpha_k \mid^{p_k} \leq \max\{1, \mid \alpha_k \mid^G\} \leq 1$, for all $k \in \mathbb{N}$, where $G = \sup_k p_k$

we have

$$\big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \leq \big(\parallel \bar{A}_k \parallel \big)^{p_k}, \text{ for all k} \in \mathbb{N}.$$

which further implies that

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \supseteq \{k \in \mathbb{N} : \big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\}.$$

But

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \in I$$

Therefore,

$$\{k \in \mathbb{N} : \big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

Thus, $\alpha_k(\bar{A}_k) \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$.
Therefore, the space $\mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ is solid and hence by Lemma (1.6), it is monotone. □

Here, we will give a definition that will be used in the following theorem.

Theorem 2.6. *A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.*

Theorem 2.7. *Let $G = \sup_k p_k < \infty$ and I be an admissible ideal. Then, the following are equivalent:*
(1) $(\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$;
(2) there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I.
(3) there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ and $\bar{C}_k \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k + \bar{C}_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : \big(\parallel \bar{B}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I$
(4) there exists a subset $K = \{k_1 < k_2 < k_3 < k_4 ...\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel \big)^{p_{k_n}} = 0$.

Proof. (1) implies (2).
Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$. Then, there exists interval number $\bar{A}$ such that the set

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : \big(\parallel \bar{A}_k - \bar{A} \parallel)^{p_k} \geq t^{-1}\} \in I$$

Define a sequence $(\bar{B}_k)$ as

$$\bar{B}_k = \bar{A}_k, \text{ for all k} \leq m_1.$$

For $m_t < k \leq m_{t+1},\ t \in \mathbb{N}$

$$\bar{B}_k = \begin{cases} \bar{A}_k, \text{ if } \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} < t^{-1}, \\ \bar{A}, otherwise. \end{cases}$$

Then $\bar{B}_k \in \mathcal{C}(\bar{\mathcal{A}}, p)$ and from the inclusion

$$\{k \leq m_t : \bar{A}_k \neq \bar{B}_k\} \subseteq \{k \leq m_t : f\big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

we get $\bar{A}_k = \bar{B}_k$ for a.a.k.r.I.
(2) implies (3). For $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$, then, there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : \bar{A}_k \neq \bar{B}_k\}$, then $K \in I$. Define $\bar{C}_k$ as follows:

$$\bar{C}_k = \begin{cases} \bar{A}_k - \bar{B}_k, \text{if } k \in K, \\ O, \text{ if } k \notin K. \end{cases}$$

Then $\bar{C}_k \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ and $\bar{B}_k \in \mathcal{C}(\bar{\mathcal{A}}, p)$.
(3) implies (4). Suppose (3) holds. Let $\epsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : \big(\parallel \bar{C}_k \parallel)^{p_k} \geq \epsilon\} \in I.$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < k_4 ...\} \in \pounds(I).$$

Then we have

$$\lim_{k\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel \big)^{p_{k_n}} = 0.$$

(4) implies (1). Let $K = \{k_1 < k_2 < k_3 < k_4 ...\} \in \pounds(I)$ and

$$\lim_{k\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel)^{p_{k_n}} = 0.$$

Then for any $\epsilon > 0$, and Lemma (1.7), we have

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\}.$$

Thus, $(\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$ □

Theorem 2.8. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p) \supseteq \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$ if and only if $\lim_{k\in K} \inf \frac{p_k}{q_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.*

Proof. Let $\lim_{k\in K} \inf \frac{p_k}{q_k} > 0$ and $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$ for sufficiently large $k \in K$.
Since $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$. For a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{q_k} \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \subseteq \{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{\beta q_k} \geq \epsilon\} \in I.$$

Therefore, $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$. The converse part of the result follows obviously. □

Theorem 2.9. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q) \supseteq \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ if and only if $\lim_{k\in K} \inf \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.*

Proof. The proof follows similarly as that of the proof Theorem (2.8). □

Theorem 2.10. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q) = \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ if and only if $\lim_{k\in K} \inf \frac{p_k}{q_k} > 0$ and $\lim_{k\in K} \inf \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.*

Proof. On combining Theorem (2.8) and (2.9), we get the desired result. □

Theorem 2.11. *The spaces $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ are not seperable.*

Proof. By a counter example we prove the result for the space $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$
For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$, the result follows similarly.

Counter Example.

Let M be an infinite subset of increasing natural numbers such that $M \in I$.
Let

$$p_k = \begin{cases} 1, \text{if k} \in M, \\ 2, \text{otherwise.} \end{cases}$$

Let $P_0 = \{(\bar{A}_k) : \bar{A}_k = [0,0]$ or $[1,1],$ for $k \in M$ and $\bar{A}_k = [0,0],$ otherwise$\}$.
Since M is infinite, so P_0 is uncountable. Consider the class of open balls $\mathcal{B}_1 = \{B(\bar{Z}, \frac{1}{2}) : \bar{Z} \in P_0\}$. Let $\mathcal{C}_1$ be an open cover of $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ containing $\mathcal{B}_1$. Since $\mathcal{B}_1$ is uncountable, so $\mathcal{C}_1$ cannot be reduced to a countable subcover for $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$. Thus $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ is not seperable. Hence the result. □

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