Sehie Park, J. Nonlinear Anal. Optim. Vol. 13(2) (2022), 75 -84

Journal of Nonlinear Analysis and Optimization
Volume 13(2) (2022)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

APPLICATIONS OF GENERALIZED ZORN'S LEMMA

SEHIE PARK*[1]

[1] The National Academy of Sciences, Republic of Korea, Seoul 06579;
Department of Mathematical Sciences, Seoul National University, Seoul 08826,
Korea

ABSTRACT. In the present article, by applying our 2013 Metatheorem and the Brøndsted-Jachymski Principle, we obtain various forms generalizations of Zorn's Lemma and their applications. Such examples are our version of the Zermelo fixed point theorem, equivalent formulations of the Caristi fixed point theorem, and Jachymski's 2003 theorem on equivalent conditions when fixed point sets are same to periodic point sets.

KEYWORDS: fixed point theorem, preorder, metric space, fixed point, stationary point, maximal element

AMS Subject Classification: 03E04, 03E25, 06A06, 06A75, 47H10, 54E35, 54H25, 58E30, 65K10

1. Introduction

There are several fields in the fixed point theory. *Analytical* fixed point theory is originated from Brouwer in 1912 and concerns mainly with topological vector spaces. *Metric* fixed point theory is originated from Banach in 1922 and deals with generalizations of contractions and nonexpansive maps. *Topological* fixed point theory relates mainly originated works of Lefschetz, Nielsen, and Reidemeister.

Now the *Ordered* fixed point theory began by Zermelo [36](1908) implicitly and was developed mainly by Knaster [16](1928), Zorn [37](1935), Bourbaki [3](1949-50), Tarski [30, 31](1949, 1955), Ekeland [10, 11](1972,1974), Caristi [7](1976), Brézis-Browder [4](1976), Takahashi [29](1991), and many others. Moreover, in 1985-86 [18, 19], we discovered a Metatheorem stating that any maximum elements in ordered sets can be fixed points, stationary points, collectively fixed points, collectively stationary points, and conversely. Consequently, Ordered fixed point theory is a rich source of information on fixed points of families of multimaps on ordered sets.

* *Corresponding author.*
Email address : park35@snu.ac.kr; sehiepark@gmail.com; website:parksehie.com.
Article history : Received 4 August 2022; Accepted 21 September 2022.

Recently in 2022 [20, 21, 22, 24], we obtained an extended form of Metatheorem and applied it to a large number of known or new results. Moreover in 2022 [24], we found the Brøndsted-Jachymski Principle on ordered sets showing the equalities of maximal elements, fixed point sets and periodic point sets of progressive selfmaps. Later we rearranged the order of statements to the 2023 Metatheorem in [25], which will be the basis of future study in various fields of mathematics.

In the present article, by applying Metatheorem and its particular Brøndsted-Jachymski Principle, we obtain various forms of Zorn's Lemma and their applications.

In Section 2, we introduce our 2023 Metatheorem and the Brøndsted-Jachymski Principle. Section 3 devotes various types of generalizations of Zorn's Lemma. We show their important applications in Sections 4-6. In Section 4, we introduce our version of the Zermelo fixed point theorem and its usefulness. Section 5 devotes to equivalent formulations of the Caristi fixed point theorem. In Section 6, we improve Jachymski's 2003 theorem [13] on equivalent conditions when fixed point sets are same to periodic point sets. Finally, Section 7 devotes to some conclusion.

2. Our Metatheorem and The Brøndsted-Jachymski Principle

In order to give some equivalents of the Ekeland variational principle, we introduced a metatheorem in 1985-86 [18, 19] on equivalent statements in the Ordered fixed point theory. Later we found some more additional equivalent statements and, consequently, we obtain an extended version of the metatheorem in 2022 [20-22,24] as follows in [25]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that $G(v,w)$ for any $w \in X\backslash\{v\}$.*

($\beta 1$) *If $f : A \longrightarrow X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $\neg G(x,y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\{x\}$ satisfying $\neg G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : A \longrightarrow X$ is a map such that $\neg G(x, f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $\neg G(x, f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A \setminus F(x)$ there exists $y \in X\backslash\setminus\{x\}$ satisfying $\neg G(x,y)$, then F has a fixed element $v \in A$, that is, $v \in F(v)$.*

($\delta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \setminus F(x)$ there exists $y \in X \setminus \{x\}$ satisfying $\neg G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

($\epsilon 1$) *If $F : A \multimap X$ is a multimap satisfying $\neg G(x,y)$ for any $x \in A$ and any $y \in F(x) \multimap \{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = F(v)$.*

($\epsilon 2$) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $\neg G(x,y)$ holds for any $x \in A$ and any $y \in F(x) \setminus \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

($\zeta 1$) *If a multimap $F : A \multimap X$ satisfy, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x, y)$ holds, then there exists $v \in A$ such that $F(v) = \emptyset$.*

($\zeta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x, y)$ holds. Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ satisfying $\neg G(x, z)$, then there exists a $v \in A \cap Y$.*

Here, $\neg$ denotes the negation. We give the proof for completeness.

Proof. Note that each of $(\beta), (\gamma), (\epsilon)$ implies (δ). and that $(\beta 2) - (\zeta 2)$ imply $(\beta 1) - (\zeta 1)$, respectively. We adopt our previous proof for $(\alpha) \Longrightarrow (\gamma 1)$ as follows:

$(\alpha) \Longrightarrow (\delta 1)$: Suppose $v \notin F(v)$ in $(\delta 1)$. Then there exists a $y \in X \setminus \{v\}$ satisfying $\neg G(v, y)$. This contradicts (α).

$(\delta 1) \Longrightarrow (\beta 1)$: Clear.

$(\beta 1) \Longrightarrow (\gamma 1)$: Clear.

We prove $(\gamma 1) \Longrightarrow (\alpha)$ as follows:

$(\gamma 1) \Longrightarrow (\epsilon 1)$: Suppose F has no stationary element, that is, $F(x) \setminus \{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{F(x) \setminus \{x\} : x \in A\}$. Then f has no fixed element by its definition. However, $\neg G(x, f(x))$ for any $x \in A$. Therefore, by $(\gamma 1)$, f has a fixed element, a contradiction.

$(\epsilon 1) \Longrightarrow (\gamma 2)$: Define a multimap $F : A \multimap X$ by $F(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by $(\epsilon 1)$, F has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

$(\gamma 2) \Longrightarrow (\alpha)$: Suppose that for any $x \in A$, there exists a $y \in X \setminus \{x\}$ satisfying $\neg G(x, y)$. Choose $f(x)$ to be one of such y. Then $f : A \longrightarrow X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By $(\gamma 2)$, f has a fixed element, a contradiction.

Consequently, we showed equivalency of $(\alpha) - (\gamma 2)$.

We show that $(\alpha) \Longleftrightarrow (\epsilon 2)$ as follows:

$(\alpha) + (\epsilon 1) \Longrightarrow (\epsilon 2)$: By (α), there exists a $v \in A$ such that $G(v, w)$ for all $w \in X \setminus \{v\}$. For each $F \in \mathfrak{F}$, by $(\epsilon 1)$, we have a $v_F \in A$ such that $\{v_F\} = F(v_F)$. Suppose $v \neq v_F$. Then $G(v, v_F)$ holds by (α) and $\neg G(v, v_F)$ holds by assumption on $(\epsilon 2)$. This is a contradiction. Therefore $v = v_F$ for all $F \in \mathfrak{F}$.

$(\epsilon 2) \Longrightarrow (\epsilon 1) \Longrightarrow (\alpha)$: Already shown.

$(\alpha) \Longrightarrow (\zeta 2)$: By (α) there exists $v \in A$ such that $G(v, x)$ holds for all $x \in X \setminus \{v\}$. Suppose to the contrary, there exists $F \in \mathfrak{F}$ such that $F(v) \neq \emptyset$. By hypothesis, there exists $w \in X$ with $w \neq v$ and $\neg G(v, w)$ holds. Therefore it leads a contradiction and $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.

$(\zeta 2) \Longrightarrow (\alpha)$: Suppose that, for each $x \in A$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x, y)$ holds. For each $x \in A$, define a multimap $F : A \multimap X \setminus \{x\}$ by

$$F(x) = \{y \in X : \neg G(x, y)\} \neq \emptyset \text{ for all } x \in A.$$

Then, by $(\zeta 2)$, there exists $v \in A$ such that $F(v) = \emptyset$. This is a contradiction.

$(\alpha) \Longrightarrow (\eta)$: By (α), there exists a $v \in A$ such that $G(v, w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

$(\eta) \Longrightarrow (\alpha)$: For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ \neg G(x,y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (η) is satisfied. Therefore, by (η), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v,w)$ for all $w \neq v$. Hence (α) holds.

This completes our proof. □

Remark 2.1. All of the elements v's in Metatheorem are same as we have seen in the proof. We adopted the Axiom of Choice in $(\gamma 1) \Longrightarrow (\epsilon 1)$.

Example 2.2. Khamsi [15]: Let A be an abstract set partially ordered by $\prec$. We will say that $a \in A$ is a minimal element of A if and only if $b \prec a$ implies $b = a$. The concept of minimal element is crucial in the proofs given for Caristi's fixed point theorem.

(K) *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.*

(1) *A contains a minimal element.*

(2) *Any multimap T defined on A, such that for any $x \in A$ there exists $y \in T(x)$ with $y \prec x$, has a fixed point, i.e., there exists a in A such that $a \in T(a)$.*

This follows from Metatheorem (α) and $(\delta 1)$.

For a partially ordered set $(X, \preccurlyeq)$ and a map $f : X \longrightarrow X$, we define

Max$(\preccurlyeq)$: the set of maximal elements;
Fix(f) : the set of fixed points of f;
Per(f) : the set of periodic points $x \in X$; that is, $x = f^n(x)$ for some $n \in \mathbb{N}$.

In our previous work [25], we established the following based on Brøndsted [5] in 1976 and Jachymski [13] in 2003:

Brøndsted-Jachymski Principle. *Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \longrightarrow X$ be a progressive map (that is, $x \preccurlyeq f(x)$ for all $x \in X$). Then X admits a maximal element $v \in X$ if and only if v is a fixed point of f if and only if v is a periodic point, that is,*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f).$$

This is a particular form of Metatheorem and not claiming the non-emptiness of three sets. We noticed that, in most applications of this principle, the existence of a maximal element or a fixed point is achieved by the upper bound of a chain in $(X, \preccurlyeq)$ as we can see examples in Section 4.

3. Generalized Zorn's Lemma

The following is a useful consequence of Metatheorem as in [25] without listing $(\beta 1) - (\zeta 1)$.

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ have an upper bound (resp. $A = T(x_0) = \{z \in X : z \preccurlyeq x_0\}$ have a lower bound) $v \in A$.*

Then the following equivalent statements hold:

(α) *There exists a maximal (resp. minimal) element $v \in A$ such that $v \not\preccurlyeq w$ (resp. $w \not\preccurlyeq v$) for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \setminus F(x)$ there exists $y \in X \setminus \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x) \setminus \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(ζ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $x \preceq y$ (resp. $y \preceq x$) holds. Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.*

Proof. (α) Since A has an upper bound $v \in A$, for each $x \in A$, we have $x_0 \preccurlyeq x \preccurlyeq v$. If $v \preccurlyeq w$ for some $w \in X$, then $w \in S(x_0) = A$ and $w \preccurlyeq v$. Since $(X, \preccurlyeq)$ is partially ordered, we have $w = v$. Hence v is maximal. Therefore, the maximal case of (α) holds. Similarly, the minimal case of (α) also holds.

Let $G(x, y)$ be $x \not\preceq y$ (resp, $y \not\preceq x$). Then (α)$-$(η) are equivalent by Metatheorem. □

Remark 3.2. (1) All the elements v's in Theorem 3.1 are same as we have seen in the proof of Metatheorem.

(2) Note that (α) $\Longleftrightarrow$ (γ1) is a new proof of the Brøndsted-Jachymski Principle.

(3) Theorem 3.1 improves the Abian-Brown fixed point theorem, the Tarski-Kantorovitch theorem, and Zorn's lemma. See [25].

(4) In Sections 4 and 5, we can see some other theorems which are closely related to Theorem 3.1

A partially ordered set $(X, \preccurlyeq)$ is said to be *inductive* (*complete*, resp.) if every non-empty chain in X has an upper bound (a least upper bound, resp.).

From Theorem 3.1, we have the following:

Corollary 3.3. *Let $(X, \preccurlyeq)$ be a partially ordered set satisfying one of the following:*

(a) *all nonempty chain in X has an upper bound ($\Leftrightarrow$ X is inductive),*
(b) *all nonempty chain in X has a least upper bound ($\Leftrightarrow$ X is complete),*
(c) *all nonempty well-ordered subset of X has an upper bound,*
(d) *all nonempty well-ordered subset of X has a least upper bound,*

Then the equivalent statements in Theorem 3.1 for the maximum case hold.

From the Brøndsted-Jachymski Principle and Corollary 3.3, we have the following generalization of Zorn's lemma:

Corollary 3.4. *Let $(X, \preccurlyeq)$ be a partially ordered set satisfying one of (a)–(d) in Corollary 3.3. If $f : X \longrightarrow X$ is progressive, then we have*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f) \neq \emptyset.$$

Example 3.5. For complete partially ordered sets, particular results of Corollary 3.4 are known as follows; see Kang ([14], p.20):

Tarski [30] and Davis [8] proved that the completeness of a lattice is equivalent to the existence of fixed points of increasing selfmap. And Tascović [32] proved that a partially ordered set is complete iff every progressive selfmap has a fixed point.

Smarzenski [26] obtained a result related to $(\gamma 1)$. Smithson [27, 28] obtained some fixed point theorems for a partially ordered space satisfying (d) and multimaps satisfying $(\delta 1)$.

Example 3.6. Recall that Tasković [32] showed that Zorn's lemma is equivalent to the following Theorem 3.1(γ):

(T) *Let $\mathcal{F}$ be a family of self-maps defined on a partially ordered set A such that $x \preccurlyeq f(x)$ [resp. $f(x) \preccurlyeq x$] for all $x \in A$ and all $f \in \mathcal{F}$. If each chain in A has an upper bound (resp. lower bound), then the family $\mathcal{F}$ has a common fixed point.*

4. Extending Zermelo's theorem

The following is known the Zermelo fixed point theorem by Dunford-Schwartz ([9], p.5, Theorem I.2.5.) :

Theorem 4.1. (Zermelo) *Let $(P, \preccurlyeq)$ be a partially ordered set in which every chain has a supremum. Assume that $f : P \longrightarrow P$ is such that f is progressive, that is,*

$$p \preccurlyeq f(p) \text{ for all } p \in P.$$

Then f has a fixed point.

Amann [1] derived several fixed point theorems from Theorem 4.1. For example, Tarski's fixed point theorem, fixed point theorems for condensing maps and nonexpansive maps.

Jachymski [13] noted: "The above theorem attributed to Zermelo although it does not appear *explicitly* in any of his papers. However, a proof of it can be derived from Zermelo's proof [36] of the well-ordering principle. This observation is due to Bourbaki [3], who was the first to formulate the theorem in the above form. (Actually, Bourbaki used well-ordered subsets of P instead of chains so his assumption is formally weaker than that of Theorem 4.1. However it is more convenient for us to work with chains as will be seen in the sequel. The proof of Zermelo's theorem does not depend on the Axiom of Choice (AC). If, however, we allow the use of Zorn's Lemma, then the proof is straightforward; moreover, the assumption on $(P, \preccurlyeq)$ can be weakened then to 'every chain has an upper bound'. This is Kneser's [17] fixed point theorem which turns out to be equivalent to the AC as shown by Abian [1]. In the literature, Zermelo's Theorem is sometimes called the Bourbaki-Kneser theorem (cf. Zeidler [35]. p.504)."

Recently Toyoda [33] also noted: "The Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem or the Bourbaki-Kneser fixed point theorem. It implies the Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma."

We have generalizations of the Zermelo theorem from Section 3, see Theorems 3.1(γ1), 3.3(γ1), Corollaries 3.4(γ1), 3.5 for single-valued case and their multi-valued versions in Section 3. Therefore, many generalizations of Zorn's Lemma also extends the Zermelo theorem.

As an example, from Theorem 3.1(α) and (γ1), we have the following:

Theorem 4.2. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ has an upper bound. If $f : A \longrightarrow X$ is a map such that $x \preccurlyeq f(x)$ for any $x \in A$, then*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f) \neq \emptyset.$$

Recall that Theorem 3.1(γ1) follows from (α) under the Axiom of Choice.

From Theorem 3.1(α) and (γ1), we have the following:

Theorem 4.3. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, $\varphi : X \longrightarrow X$ a map, and let $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$ have upper bounds and $A = B \cup \{$its upper bounds$\}$ such that $x \preccurlyeq \varphi(x)$ for all $x \in A$. Then we have*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(\varphi) = \mathrm{Per}(\varphi) \neq \emptyset.$$

5. Equivalent formulations of Caristi theorem

In this section, we consider a particular case of Theorem 3.1 as follows:

Theorem 5.1. *Let $(X, \preccurlyeq)$ be a partially ordered metric space, and a function $\varphi : X \longrightarrow \mathbb{R}^+$ be lower semicontinuous such that*

$$x \preccurlyeq y \quad \text{iff} \quad d(x, y) \leq \varphi(x) - \varphi(y) \quad \text{for} \quad x, y \in X.$$

Let $x_0 \in X$ and $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ have an upper bound.

Then the following equivalent statements (α) − (η) of Theorem 3.1 hold.

(α) *There exists a maximal element $v \in A$, that is, $d(v, w) > \varphi(v) - \varphi(w)$ for any $w \in X \setminus \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $d(x, y) \leq \varphi(x) - \varphi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that, for any $x \in A \setminus F(x)$, there exists $y \in X \setminus \{x\}$ satisfying $d(x, y) \leq \varphi(x) - \varphi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$ holds for any $x \in A$ and any $y \in F(x) \setminus \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F_i(v)$ for all $F \in \mathfrak{F}$.*

(ζ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for all $x \in A$ with $T(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$ holds. Then there exists $v \in A$ such that $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ such that $d(x, z) \leq \varphi(x) - \varphi(z)$, then there exists an element $v \in A \cap Y$.*

From Theorem 5.1 we have equivalent formulations of the Caristi theorem as follows:

Theorem 5.2. *Let (X,d) be a complete metric space, and a function $\varphi : X \longrightarrow \mathbb{R}^+$ be lower semicontinuous such that*

$$x \preccurlyeq y \quad \textit{iff} \quad d(x,y) \leq \varphi(x) - \varphi(y) \quad \textit{for} \quad x,y \in X.$$

Then the equivalent statements $(\alpha) - (\eta)$ of Theorem 5.1 hold where we include

$(\gamma 1)$ (Caristi) *If $f : X \longrightarrow X$ is a map such that $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for any $x \in X$, then f has a fixed and periodic element $v \in X$, that is, $v = f(v)$.*

Proof. Since $(\gamma 1)$ holds by the Caristi fixed point theorem and (α) holds by Brunner [**?**] in 1987, so do the others. This completes our proof. □

There are possibly dual equivalent formulations of the Caristi theorem.

6. Jachymski's 2003 Theorem

In this article, we introduced many examples of maps $f : X \longrightarrow X$ satisfying $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$. Such sets X can have more rich properties by applying the following main theorem of Jachymski ([13], Theorem 2):

Theorem 6.1. *Let X be a nonempty abstract set and $T : X \longrightarrow X$. The following statements are equivalent:*

(a) $\mathrm{Per}(T) = \mathrm{Fix}(T) \neq \emptyset$.
(b) (Zermelo) *There exists a partial ordering $\preccurlyeq$ such that every chain in $(X, \preccurlyeq)$ has a supremum and T is progressive with respect to $\preccurlyeq$.*
(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \longrightarrow \mathbb{R}^+$ such that T satisfies the Caristi condition.*
(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \longrightarrow \mathbb{R}^+$ such that T satisfies the Caristi condition and T is nonexpansive with respect to d; i.e.*

$$d(Tx, Ty) \leq d(x,y) \quad for\ all \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0,1)$, there exists a complete metric d such that T is nonexpansive with respect to d and*

$$d(Tx, T^2x) \leq \alpha d(x, Tx) \quad for\ all \quad x \in X.$$

(f) *There exists a complete metric d such that T is continuous with respect to d and for each $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*
(g) *There exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, such that all the sets X_γ are nonempty, T-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $T|_{X_\gamma}$ has a unique periodic point.*
(h) *For each $\alpha \in (0,1)$, there exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; T-invariant and pairwise disjoint; and*

$$d_\gamma(Tx, Ty) \leq \alpha d_\gamma(x,y) \quad for\ all\ x, y \in X.$$

Remark 6.2. ([13]) Implication (a) $\Longrightarrow$ (b) is a converse to Zermelo's theorem. Implication (a) $\Longrightarrow$ (c) is a reciprocal to Caristi's theorem; in fact, a stronger result, (a) $\Longrightarrow$ (d) can be obtained here. Implication (a) $\Longrightarrow$ (e) is a converse to a fixed point theorem of Hicks-Rhoades. Finally (a) $\Longrightarrow$ (f) answers a question posed by Matkowski.

Comments 6.3. Each of (a)–(h) seems to be order theoretic fixed point theorems. For them, we state our own comments.

(a) This could be $\mathrm{Fix}(T) = \mathrm{Per}(T) = \mathrm{Max}(\preccurlyeq) \neq \emptyset$ by defining $\preccurlyeq$ on X.
(b) Zermelo's theorem is improved in Section 4 and has many equivalents there. Note that its conclusion should be as above in (a).
(c) Caristi's theorem is improved by Theorem 5.2 and its conclusion should be as in (a).
(d) This is a variant of Caristi's theorem and its conclusion should be as in (a).
(e) Here nonexpansiveness is redundant in view of Theorem H(iv) in [25].

7. Conclusion

Zermelo's fixed point theorem suggested in 1904 and 1908 more than one hundred years ago, Kuratowski-Zorn's Lemma or Zorn's Lemma in 1935, Bourbaki's fixed point theorem in 1949-50, and some other classical results are all improved in the present article. Moreover, their equivalent formulations based on our 2023 Metatheorem and the Brøndsted-Jachymski Principle should be reflected in the most of classical works on Ordered fixed point theory.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works [20-25] and the present article. Therefore, Metatheorem is a machine to expand our knowledge easily. In this article we presented relatively old and well-known examples.

References

1. A. Abian, A fixed point theorem equivalent to the axiom of choice, Arch. Math. Logik 25 (1985), 173-174.
2. H. Amann, Order structures and fixed points, Seminario di Analisi Funzionali e Applicazioni, Universit‘a della Calabria, 1977.
3. N. Bourbaki, Sur le theoreme de Zorn, Arch. Math. 2 (1949-50), 434–437.
4. H. Brézis and F.E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Adv. Math. 21 (1976) 355–364.
5. A. Brøndsted, Fixed point and partial orders, Shorter Notes, Proc. Amer. Math. Soc. 60 (1976), 365–366.
6. N. Brunner, Topologische Maximalprinzippen, Zeitschr. f. math. Logik und Grundlagen d. Math. 33 (1987), 135–139.
7. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241–251.
8. A.C. Davis, A characterization of complete lattice, Pacific J. Math. 5 (1955), 311–319.
9. N. Dunford, J. Schwartz, Linear Operators I: General Theory, Wiley Interscience, New York. 1957.
10. I. Ekeland, Sur les problèmes variationnels, C.R. Acad. Sci. Paris 275 (1972) 1057–1059; 276 (1973), 1347–1348.
11. I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.
12. J. Jachymski, Order-theoretic aspects of metric fixed point theory, Chapter 18 in: Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, eds.), Kluwer Academic Publ. (2001), 613–641.
13. J. Jachymski, Converses to fixed point theorems of Zermelo and Caristi, Nonlinear Analysis 52 (2003), 1455–1463.
14. B.G. Kang, Fixed point property and completeness of ordered sets, J. Korea Soc. Math. Ed. Series B: The Pure and Applied Math. 4(1) (1997), 19–26.
15. M.A. Khamsi, Remarks on Caristis fixed point theorem, Nonlinear Anal. 71(1-2) (2009), 227-231
16. B. Knaster, Un theoreme sur les fonctions d'ensembles, Ann. Soc. Polon. Math. 6 (1928), 133-134.

17. H. Kneser, Eine direkte Ableitung des Zornschen lemmas aus dem Auswahlaxiom, Math. Z. 53 (1950), 110–113.
18. S. Park, Some applications of Ekeland's variational principle to fixed point theory, in Approximation Theory and Applications (S.P. Singh, ed.), Pitman Res. Notes Math. 133 (1985), 159–172.
19. S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems (S.P. Singh et al., eds.), MSRI-Korea Publ. 1 (1986), 55–68.
20. S. Park, Equivalents of various maximum principles, Results in Nonlinear Analysis 5(2) (2022), 169–174.
21. S. Park, Applications of various maximum principles, J. Fixed Point Theory (2022), 2022:3, 1–23. ISSN:2052–5338.
22. S. Park, Equivalents of maximum principles for several spaces, Top. Algebra Appl. 10 (2022), 68–76. 10.1515/taa-2022-0113
23. S. Park, Equivalents of generalized Brøndsted principle, J. Informatics Math. Sci., to appear.
24. S. Park, Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others, Adv. Th. Nonlinear Anal. Appl. 6(4) (2022), 420–432.
25. S. Park, Foundations of ordered fixed point theory, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61(2) (2022)
26. R. Smarzewski, A characterization of fixed point property, Nonlinear Anal. TMA 19 (1992), 197–199.
27. R.E. Smithon, Fixed points of order preserving multifunctions, Proc. Amer. Math. Soc. 28 (1971), 304–310.
28. R.E. Smithon, Fixed points in partially ordered set, Pacific J. Math. 45 (1973), 363–367.
29. W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser. 252, Longman Sci. Tech., Harlow, (1991), 397–406.
30. A. Tarski, A fixpoint theorem for lattices and its applications (preliminary report), Bull. Amer. Math. Soc. 55 (1949), 1051-1052 and 1192.
31. A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific J. Math. 5 (1955), 285–309.
32. M.R. Tasković, On an equivalent of the axiom of choice and its applications, Math. Jpn. 31(6) (1986), 979-991.
33. M. Toyoda, Caristi fixed point theorem and Bourbaki-Kneser fixed point theorem, RIMS Kôkyûroku, Kyoto Univ. 2194 (2020), 97–107; II, 2214 (2021) 7pp.
34. M. Turinici, A maximality principle on ordered metric space, Rev. Colombiana Math. 16 (1982), 115–124.
35. E. Zeidler, Nonlinear Functional Analysis and Its Applications I, Fixed-Point Theorems, Springer, 1986.
36. E. Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung, Math. Ann. 65 (1908), 107–128.
37. M. Zorn, A remark on method in transfinite algebra, Bull. Amer. Math. Soc. 41 (1935), 667–670.