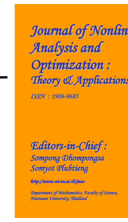
**J**ournal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 1, (2014), 81-87
**ISSN : 1906-9685**
**http://www.math.sci.nu.ac.th**




# A METHOD FOR APPROXIMATING SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY USING FUZZY TRANSFORMS

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**ABSTRACT.**Fuzzy transforms developed by Irina Perfilieva is a novel, mathematically well founded soft computing tool with many applications. These techniques are based on mainly two transforms, direct fuzzy transform and inverse fuzzy transform. However a lot of works has still to be done on it for e.g. the extension of fuzzy transform and its applications in ordinary differential equations. In this paper we develop an approximating model based on fuzzy transform and apply the model for numerical solution of ordinary differential equations.

**KEYWORDS** : Fuzzy transforms; Fuzzy partitions; Ordinary Differential Equations.
**AMS Subject Classification**:

## 1. INTRODUCTION

In classical Mathematics, various types of transforms are introduced (e.g. Laplace transform, Fourier transform, wavelet transform etc.) by various researchers. In 2001 Irina Perfilieva introduced fuzzy transform in her paper [2]. Latter on fuzzy transform is applied in to various fields, like image processing, data mining etc in the papers [5, 10]. The fuzzy transform provides a relation between the space of continuous functions defined on a bounded domain of real line $R$ and $R^n$.Similarly inverse fuzzy transform identified each vector of $R^n$ with a continuous map. The central idea of the fuzzy transform is to partition the domain of the function by fuzzy sets.

**Definition1.1([5]):** Let $[a, b]$ be an interval of real numbers and $x_1, x_2, \dots x_n$ be fixed nodes within $[a, b]$ such that $x_1 = a, x_n = b$ and $n \geq 2$.We say that fuzzy sets $A_1, A_2, \dots A_n$, identified with their membership functions $A_1(x), A_2(x), \dots A_n(x)$ and defined on $[a, b]$ form a fuzzy partition of $[a, b]$ if they fulfill the following conditions for$i = 1, 2, \dots n$.

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*Article history* : Received ...... Accepted .....

1. $A_i : [a,b] \to [0,1]$, for $i = 1,2,\ldots,n$.

   $A_i(x_i) = 1$, for $i = 1,2,\ldots,n$
2. $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$
3. $A_i(x)$is continuous.
4. $A_i(x)$is monotonically increasing on $x_{i-1}, x_i$and monotonically decreasing on $x_i, x_{i+1}$.
5. $\sum_{i=1}^{n} A_i(x) = 1$, for all $x \in [a,b]$.
6. $A_i(x_i - x) = A_i(x_i + x)$,$\forall x \in [0,h], i = 2,3,\ldots,n-1, n > 2$.
7. $A_{i+1}(x) = A_i(x-h)$,$\forall x \in [a+h,b]$,for $i = 2,3,\ldots,n-2, n > 2$

Where $h$ is the uniform distance between two nodes.
Let us remark that the shape of basic functions is specified by a set of nodes $x_1, x_2, \ldots x_n$ and the properties 1-7. The shape of basic functions is not predetermined and therefore it can be chosen additionally according to further requirements.
Figure 1 shows a fuzzy partition of the interval $[-4,4]$, with triangular membership functions.
The following expression gives the formal representation of such triangular membership functions.

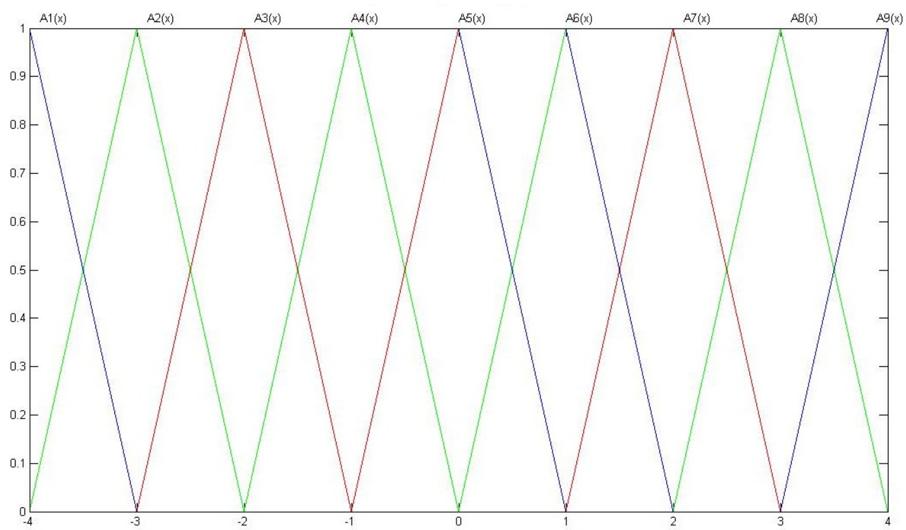


FIGURE 1. An example fuzzy partition of [-4,4].

$$A_1(x) = \begin{cases} -3 - x & \text{if} x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}$$

and for $i = 2, 3, , n-1$.

$$A_i(x) = \begin{cases} x - x_{i-1} & \text{if} x \in [x_{i-1}, x_i] \\ 1 - x + x_i & \text{if} x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

$$A_n(x) = \begin{cases} x - x_{n-1} & \text{if} x \in [x_{n-1}, x_n] \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 1.1.** [6]: *Let a uniform fuzzy partition of $[a, b]$ is given by basic functions $A_1, A_2, \cdots, A_n, n \geq 3$, then*

$$\int_{x_1}^{x_2} A_1(x)dx = \int_{x_{n-1}}^{x_n} A_n(x)dx = \frac{h}{2} \tag{1.1}$$

*and for $i = 2, 3, \cdots, n-1$*

$$\int_{x_{i-1}}^{x_{i+1}} A_i(x)dx = h \tag{1.2}$$

The above lemma shows that the integral of the basic functions does not depend upon the particular shape of the basic functions.

## 2. FUZZY TRANSFORM

In this section we first give the definition of fuzzy transform given by Irina Perfilieva in 2006.

**Definition2.1([6]):**Let $f(x)$ be a continuous function on $[a, b]$ and $A_1(x), A_2(x), \cdots, A_n(x)$ be basis functions determining a uniform fuzzy partition of $[a, b]$.Then the n-tuple of real numbers $[F_1, F_2, \cdots, F_n]$ such that

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}, i = 1, 2, \cdots, n \tag{2.1}$$

will be called the F- transform of $f$ w.r.t. the given basis functions. Real's $F_i$ are called components of the F-transform.

**Lemma 2.1.** [5]: *Let $f$ be any continuous function defined on $[a, b]$, but function $f$ is twice continuously differentiable in $(a, b)$ and let $A_1(x), A_2(x), \cdots, A_n(x)$ be basis functions determining a uniform fuzzy partition of $[a, b]$ .Then for each $i = 1, 2, \cdots, n$*

$$F_i = f(x_i) + O(h^2)$$

*Proof.* Perfilieva (2004). □

Now a question arises in the minds that can we get back the original function by its fuzzy transform. The answer is we can reconstruct an approximate function to the original function. For that purpose Perfilieva define inverse fuzzy transform.

**Definition2.2**[6]: Let $A_1, A_2, \cdots, A_n$ be basic functions which form a uniform fuzzy partition of $[a, b]$ and $f$ be a function from $c([a, b])$ . Let $F_n[f] = [F_1, F_2, \cdots, F_n]$ be the fuzzy transform of $f$ with respect

to $A_1, A_2, \cdots, A_n$ . Then the function defined by

$$f_{F,n}(x) = \sum_{i=1}^{n} F_i.A_i(x)$$

is called the inverse fuzzy transform of $f$ with respect to $A_1, A_2, \cdots, A_n$ .

The following theorem shows that the inverse fuzzy transform can approximate the original continuous function $f$ with a very small precision.

**Theorem 2.2.** [6]:*Let $f$ be a continuous functions defined on $[a,b]$. Then for any $\epsilon > 0$ there exist $n_\epsilon$ and a uniform fuzzy partition $A_1, A_2, \cdots, A_n$ of [a, b] such that for all $x \in [a,b]$*

$$|f(x) - f_{F,n_\epsilon}| \leq \epsilon$$

## 3. APPROXIMATE SOLUTIONS OF SECOND ORDER O.D.E. BY USING FUZZY TRANSFORM

In this section, we show that how the fuzzy transform can be used for solution of second order ordinary differential equation. Consider the following differential equation:

$$y''(x) = f(x,y) \tag{3.1}$$
$$y'(x_1) = c, y(x_1) = d.$$

Here we show that this differential equation can be solved by using fuzzy transform.

For solving the above equation we need a uniform fuzzy partition of the domain. Let $a = x_1 < x_2 < \cdots < x_n = b$ be fixed nodes within the domain and consider the fuzzy partition, $A_1, A_2, \cdots, A_n$ defined on these domain. Here we also assume that all the node points are equidistant, i.e. $x_i - x_{i-1} = h$ (say).

Now we approximate $y\prime(x)$ and $y''(x)$ by the following formula

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{3.2}$$

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2) \tag{3.3}$$

Denote

$$y_1(x) = y(x+h)$$

as a new function and

$$y_2(x) = y(x-h)$$

as another new function. Now if we apply the F-transform on both sides of equation (3.3), then by using these new functions and by linearity of F-transform, we obtain the relation between the fuzzy transform components of $y, y_1, y_2$ and $y''$ as follows

$$F_n[y''] = \frac{F_n[y_1] - 2F_n[y] + F_n[y_2]}{h^2}$$

where, $F_n[y''] = [Y_2'', Y_3'', \cdots, Y_{n-1}'']$, $F_n[y_1] = [Y1_2, Y1_3, \cdots, Y1_{n-1}]$, $F_n[y_2] = [Y2_2, Y2_3, \cdots, Y2_{n-1}]$ and $F_n[y] = [Y_2, Y_3, \cdots, Y_{n-1}]$ are the fuzzy transform components of $y'', y_1, y_2$ and $y$ respectively. Note that these vectors are two components shorter since $y_1$ may not be defined on $[x_{n-1}, x_n]$ and $y_2$ may not be defined on $[x_1, x_2]$.

Now by using the definition of fuzzy transform it can be easily proved that, $Y1_k = Y_{k+1}$ and $Y2_k = Y_{k-1}$, for $k = 2, 3, \cdots, n-1$. Indeed,for values of $k = 2, 3, \cdots, n-2$,

$$Y1_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} y(x+h)A_k(x)dx = \frac{1}{h}\int_{x_k}^{x_{k+2}} y(t)A_{k+1}(t)dt$$

For, $k = n-1$ the proof is similar.The proof of $Y2_k = Y_{k-1}$,for $k = 2, 3, \cdots, n-1$ is analogous. Therefore, we can write the components of the F-transform of $y''$ via components of the F-transform of y. So, we can write the equation (3.1) component wise as

$$Y_k'' = \frac{Y_{k+1} - 2Y_k + Y_{k-1}}{h^2}, fork = 2, 3, \cdots, n-1.$$

Now for $k = 2, 3, \cdots, n-1$, we introduce the following system of linear equations.

$$\begin{aligned} Y_2'' &= \frac{Y_3 - 2Y_2 + Y_1}{h^2} \\ Y_3'' &= \frac{Y_4 - 2Y_3 + Y_2}{h^2} \\ &\vdots \\ Y_{n-1}'' &= \frac{Y_n - 2Y_{n-1} + Y_{n-2}}{h^2} \end{aligned}$$

The above system can be written in matrix form as

$$[Y_2'', Y_3'', \cdots, Y_{n-1}'']^T = D[Y_1, Y_2, \cdots, Y_n]^T \tag{3.4}$$

where, D is the $(n-2) \times n$ matrix, given by

$$\mathsf{D} = \frac{1}{h^2} \times \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}$$

Now, by using equation (3.4) and equation (3.1), we can write

$$D[Y_1, Y_2, \cdots, Y_n]^T = [F_2, F_3, \cdots, F_{n-1}]^T$$

where, $F_2, F_3, \cdots, F_{n-1}$ are the corresponding fuzzy transform components of $f(x, y)$. Now we use the initial conditions and make the matrix $D$ as $n \times n$ matrix. The initial conditions are given as,

$$y(x_1) = c, \Rightarrow Y_1 = c \text{ (by using lemma2.1)}$$

and

$$y'(x_1) = d, \Rightarrow y(x_2) - y(x_1) = dh \Rightarrow Y2 - Y1 = dh, \text{ (by using lemma2.1)}$$

By using the above initial conditions, we make the matrix $D$ as square matrix of order $n \times n$ and also write the system of linear equations as

$$D^c[Y_1, Y_2, \cdots, Y_n]^T = [\frac{c}{h^2}, \frac{d}{h}, F_2, \cdots, F_{n-1}]^T \tag{3.5}$$

where,

$$\mathsf{D^c} = \frac{1}{h^2} \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}$$

Now, we solve the system of linear equation (3.5) by using any numerical techniques. Note here that the solution must exist since $D^c$ is a invertible matrix.

**Note3.1:** Here for solving both the systems of linear equations we need the value of $F_i$, which is the fuzzy transform components of the function $f(x, y)$, with respect to $x$, therefore $F_i$ will be dependent on $y$ also. For overcoming these difficulties we approximate $F_i$ as

$$F_i' = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x, Y_k) A_k(x) dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx}$$

That means here we have assumed the value of the function $y$ as constant for the interval $(x_{k-1}, x_{k+1})$. Here we omit the proof that

$$F_i - F_i' = O(h^2).$$

Now after evaluating all the fuzzy transform components of y, we will find an approximation of the function y by using inverse fuzzy transform. For illustration of the above method we give the following example.

**Example3.1:** Consider the following initial value problem

$$y''(x) = 2x + y, -4 \le x \le 4$$
$$y(-4) = 8, y'(-4) = -2$$

For solving the above system we consider the fuzzy partition given in figure above. Now here the system of linear equations (3.5) can be written as

$$D^c[Y_1, Y_2, \cdots, Y_n]^T = [8, -2, F_2, \cdots, F_{n-1}]^T$$

Where $D^c$ is as given above with $h = 1$.Now after solving this system of linear equations we find

$Y1 = 8$, $Y2 = 6$, $Y3 = 4 + F_2$, $Y4 = 2 + 2F_2 + F_3$ , $Y5 = 3F_2 + 2F_3 + F_4$
$Y6 = -2 + 4F_2 + 3F_3 + 2F_4 + F_5$, $Y7 = -4 + 5F_2 + 4F_3 + 3_F4 + 2F_5 + F_6$
$Y8 = -6 + 6F_2 + 5F_3 + 4F_4 + 3F_5 + 2F_6 + F_7$ and
$Y9 = -8 + 7F_2 + 6F_3 + 5F_4 + 4F_5 + 3F_6 + 2F_7 + F_8$

Now, for finding

$$Y3, Y4, Y5, Y6, Y7, Y8 \text{ and } Y9$$

, we need the value of

$$F_2, F_3, F_4, F_5, F_6, F_7 \text{ and } F_8$$

, which are the 2nd ,3rd and 4th, 5th ,6th, 7th and 8th fuzzy transform components of the function $f(x, y)$. Now by using the note (3.1), we evaluate $F_2, F_3, F_4, F_5, F_6, F_7$ and $F_8$ by using software "MATLAB" and then subsequently we find $Y1 = 8, Y2 = 6, Y3 = 4, Y4 = 2, Y5 = 0, Y6 = -2, Y7 = -4, Y8 = -6$ and $Y9 = -8$. Now by using these fuzzy transform components of $y$ we approximate $y$ by using inverse fuzzy transform, which graph is shown below. Since the exact solution is $y = -2x$. Therefore from the graph given in Figure 2, we can say that in this case our approximate solution coincides with the exact solution.

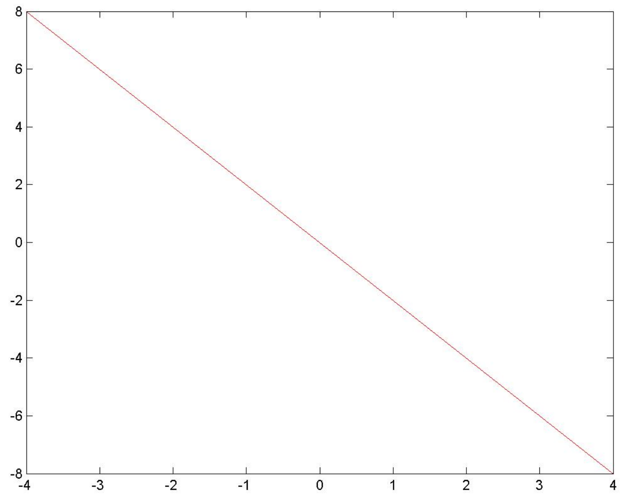


FIGURE 2. The approximation graph.

## 4. CONCLUSION

We have introduced a new numerical technique by using fuzzy transform, which enables us to construct various approximating models depending on the choice of basic functions. With this new technique (fuzzy transform) we solved second order initial value ordinary differential equations.

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