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ON EXISTENCE OF SOLUTION OF IMPLICIT VECTOR EQUILIBRIUM PROBLEMS FOR TRIFUNCTION

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ABSTRACT. In this work, we introduce and study an extended version of implicit vector equilibrium problems for trifunction in real Hausdorff topological vector spaces. We prove some new existence results for the solution of these problems by using KKM lemma in real Hausdorff topological vector spaces. Some special cases are also discussed.

KEYWORDS: Equilibrium problem, upper semicontinuity, KKM-map, trifunction.
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1. Introduction

The area of an equilibrium theory is dynamic and has been experiencing an explosive growth in both theory and applications, as a consequences of research techniques and problems drawn from various fields. This theory of equilibrium problem is being intensively studied by Blum and Oettli [2], where they proposed it as a generalization of optimization and variational inequality problem to study a wide class of problems. It has been extended to vector equilibrium problems, vector optimization problems and vector saddle point problems, see [2, 6, 8, 12]. In 2005, Kazmi and Raouf [10] introduced a class of operator equilibrium problems and from this there are plentiful problems for equilibrium problems with operator solutions, see for example [7, 10, 14, 15, 16, 17, 18] and the references therein. Implicit vector equilibrium problem is a generalization of implicit vector variational inequality, for more detail, we refer to [1, 3, 4, 13].

In 1929, Knaster, Kuratowski and Mazurkiewicz [11] established the well-known KKM theory, which is one of the few areas among the subjects from nonlinear analysis that could provide an easy and convenient forms and tools for the study of problems from applied sciences, such as economics, optimization and game theory.

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We have employed KKM theorems to prove the existence of solutions of implicit vector equilibrium problems.
Throughout this paper, Let Z be an ordered topological vector space with an ordering cone C in Z. Note that the cone C in Z defines a partial ordering $\leq_C$ as follows:

$$x \leq y \;\Leftrightarrow\; y - x \in C,\; \forall x, y \in Z,$$
$$x \nleq y \;\Leftrightarrow\; y - x \notin C,\; \forall x, y \in Z.$$

If the $intC \neq \phi$, then the weak ordering in Z is defined as follows:

$$x < y \;\Leftrightarrow\; y - x \in intC,\; \forall x, y \in Z,$$
$$x \nless y \;\Leftrightarrow\; y - x \notin intC,\; \forall x, y \in Z.$$

Now we will work under the following settings:

Let X, Y and Z be real Hausdorff topological vector spaces, and let $K \subseteq X$ and $D \subseteq Y$ be nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be set-valued mappings. In this paper, we consider the following implicit vector equilibrium problem:
Find $x^* \in K,\; y \in T(x^*)$ such that

$$F(h(x^*), y, u) \notin -intC,\; \forall u \in S(x^*), \tag{1.1}$$

where $F : K \times D \times K \to Z$ be a trifunction and $h : K \to K$ be a map from K into itself.

Throughout the paper, 2^X denotes the set of all nonempty subsets of X.

Some special cases

(i) If $h : K \to K$ is the identity map on K, then (1.1) reduces to the problem of finding $x^* \in K$ such that

$$F(x^*, y, u) \notin -intC,\; \text{for all } y \in T(x^*), u \in S(x^*), \tag{1.2}$$

which is called the implicit vector variational inequality studied by Chiang et al. [3].

(ii) If $S(x) = K,\; \forall x \in K$, then (1.2) reduces to the problem of finding $x^* \in K$ such that

$$F(x^*, y, u) \notin -intC,\; \text{for all } u \in K,$$

which is extensively studied in [4].

(iii) If $Z = \mathbb{R},\; C = [0, \infty)$, and X as well as Y are finite dimensional spaces, then (1.2) reduces to implicit variational inequality studied by Cubiotti and Yao [5].

(iv) If $Z = \mathbb{R},\; C = [0, \infty)$, and $Y = X^*$, the topological dual of X and

$$F(x^*, y, u) = \langle y, u - x^* \rangle,$$

where $\langle ., . \rangle$ is the dual pairing between X^* and X, then (1.2) reduces to generalized quasi variational inequality studied by Shih and Tan [19], and Yao [20].

The main aim of this paper is to study the existence of solution of implicit vector equilibrium problems for trifunction in real Hausdorff topological vector spaces by using KKM -lemma. In section- 2, we recall some necessary definitions and results which are needed in the latter section. Some new existence results for the solution of implicit vector equilibrium problems for trifunction have been established in section-3.

2. Preliminaries

Now we give some definitions and preliminary results needed in the next sections.

Definition 2.1. A set-valued map $T : K \to 2^Y$ is called a KKM-map, if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $co\{x_1, x_2, ...x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i)$, where $co\{x_1, x_2, ...x_n\}$ denotes the convex hull of the set $\{x_1, x_2, ...x_n\}$.

Definition 2.2. A set-valued map $T : X \to 2^Y$ is called upper semicontinuous (for short, u.s.c) at $x_0 \in X$, if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and for any net $\{y_\lambda\}$ in Y with $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$ in Y, we have $y_0 \in T(x_0)$. T is called upper semicontinuous on X if it is upper semicontinuous at every point of X.

To prove the existence results for the solutions of problem (1.1), we shall use the following lemmas:

Lemma 2.3. [9] *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^X$ be a KKM-map, such that for any $y \in K$, $T(y)$ is closed and $T(y^*)$ is contained in a compact set $B \subseteq X$ for some $y^* \in K$. Then, there exist $x^* \in B$ such that $x^* \in T(y)$, for all $y \in K$, that is, $\bigcap_{y \in K} T(y) \neq \phi$.*

Lemma 2.4. [13] *Let (Z, C) be an ordered topological vector space with a closed and convex cone C. Then for any $x, y, z \in Z$, we have*

(i) $x - y \in -intC$ *and* $x \notin -intC \implies y \notin -intC$.
(ii) $x + y \in -C$ *and* $x + z \notin -intC \implies z - y \notin -intC$.
(iii) $x + z - y \notin -intC$ *and* $-y \in -C \implies x + z \notin -intC$.
(iv) $x + y \notin -intC$ *and* $y - z \in -C \implies x + z \notin -intC$.

3. Existence Results

In this section, we prove some new existence results for the solutions of implicit vector equilibrium problem for trifunction.

Theorem 3.1. *Let $K \subseteq X$ be a nonempty convex set and $D \subseteq Y$ be a nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be continuous set-valued mappings. Let $h : K \to K$ be a continuous mapping. Let $F : K \times D \times K \to Z$ be a continuous mapping with respect to the first argument. Suppose that the following assumptions holds:*

(1) *the map $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{-intC\}$, $\forall x \in K$ is upper semicontinuous on K,*
(2) *there exists a set-valued map $G : K \times D \times K \to Z$ such that*
  (i) *$G(h(x), y, x) \notin -intC$, for all $x \in K, y \in T(x)$,*
  (ii) *$G(h(x), y, u) - F(h(x), y, u) \notin -intC$, for all $x \in K, y \in T(x), u \in S(x)$,*
  (iii) *$\{u \in K : G(h(x), y, u) \in -intC\}$ is convex, for all $x \in K, y \in T(x), u \in S(x)$,*

(3) *furthermore, suppose that there exists a nonempty compact and convex subset M of K such that for each $x \in K \setminus M,\ y \in T(x)$, there exists $u \in M$ such that $F(h(x), y, u) \in -intC$.*

Then, there exists $x^* \in K,\ y \in T(x)$ such that $F(h(x^*), y, u) \notin -intC,$ for all $u \in S(x^*)$.

Proof. For each $u \in K$, define a set-valued map $P : K \to 2^M$ as

$$P(u) = \{x \in M : F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}.$$

We first prove that $P(u)$ is closed,for all $u \in K$. For this, let $\{x_\alpha\}$ be a net in $P(u)$ such that $x_\alpha \to x$. Then $x \in M$ (as M is compact). It follows from $x_\alpha \in P(u)$ that

$$F(h(x_\alpha), y, u) \notin -intC,\ \forall y \in T(x_\alpha),\ \forall u \in S(x_\alpha).$$

$$\text{So, } F(h(x_\alpha), y, u) \in W(x_\alpha) = Z \setminus \{-intC\},\ \forall y \in T(x_\alpha),\ \forall u \in S(x_\alpha).$$

Again, since $F(x, y, u)$ is continuous with respect to x and h, S, T are also continuous, we have

$$F(h(x_\alpha), y, u) \to F(h(x), y, u).$$

Therefore by the upper semicontinuity of W, we have

$$F(h(x), y, u) \in W(x),\ \forall y \in T(x),\ \forall u \in S(x).$$

$$\text{Therefore, } F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x).$$

Hence $P(u)$ is closed, for all $u \in K$.
Next, we will show that

$$\bigcap_{u \in K} P(u) \neq \phi.$$

Since M is compact, it is sufficient to show that the family $\{P(u)\}_{u \in K}$ has the finite intersection property. For this, let $\{u_1, u_2, \cdots, u_n\}$ be a finite subset of K. Set $N = co[M \cup \{u_1, u_2, \cdots, u_n\}]$. Clearly, N is compact and convex subset of K. Next, for each $u \in K$, we define two set-valued mappings, $T_1, T_2 : K \to 2^N$ as follows:

$$T_1(u) = \{x \in N : F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}$$

and

$$T_2(u) = \{x \in N : G(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}.$$

By assumption (i), (ii) of (2), we have

$$G(h(u), y, u) \notin -intC$$

and

$$G(h(u), y, u) - F(h(u), y, u) \in -intC.$$

It follows from Lemma 2.4(i), $F(h(u), y, u) \notin -intC$. and so $T_1(u) \neq \phi$.
Since $T_1(u)$ is a closed subset of a compact set N. Therefore $T_1(u)$ is compact. Now we will show that T_2 is a KKM-map. Suppose there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of N and $\lambda_i \geq 0,\ i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\bar{x} = \sum_{i=1}^{n} \lambda_i x_i \notin \bigcup_{j=1}^{n} T_2(x_j).$$

Then $G(h(\bar{x}), y, x_j) \in -intC\ ,\ j = 1, 2, \cdots, n.$
From assumption (2)(iii), we have

$$G(h(\bar{x}), y, \bar{x}) \in -intC,$$

which is a contradiction to (2)(i). Hence T_2 is a KKM-map.
From assumption (2)(ii) and Lemma 2.4(i), we have

$$T_2(u) \subseteq T_1(u) \ , \forall u \in K.$$

Infact, $x \in T_2(u)$ implies $G(h(x), y, u) \notin -intC$ and by assumption (2)(ii), we have,

$$G(h(x), y, u) - F(h(x), y, u) \in -intC$$

$$\text{or } F(h(x), y, u) \notin -intC \text{ and hence } x \in T_1(u).$$

So, T_1 is also a KKM -map.

From Lemma 2.3, there exists $x^* \in N$ such that $x^* \in T_1(u),$ for all $u \in K$.

This implies that there exists $x^* \in N$ such that

$$F(h(x^*), y, u) \notin -intC.$$

Therefore by assumption (3), we have $x^* \in M$ and moreover $x^* \in P(u_i)$ $, i = 1, 2, \cdots, n$. Hence $\{P(u)\}_{u \in K}$ has the finite intersection property.This completes the proof. □

Corollary 3.2. *Let $K \subseteq X$ be a nonempty convex set and $D \subseteq Y$ be a nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be a continuous set-valued mapping. Let $F : K \times D \times K \to Z$ be continuous mappings with respect to the first argument. Suppose that the following assumptions holds:*

1. *the map $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{-intC\}$, for all $x \in K$ is upper semicontinuous on K,*
2. *there exists a set-valued map $G : K \times D \times K \to Z$ such that*
   - (i) *$G(x, y, x) \notin -intC,$ for all $x \in K, y \in T(x)$,*
   - (ii) *$G(x, y, u) - F(x, y, u) \notin -intC,$ for all $x \in K, y \in T(x), u \in S(x)$,*
   - (iii) *$\{u \in K : G(x, y, u) \in -intC\}$ is convex, for all $x \in K, y \in T(x), u \in S(x)$,*
3. *furthermore, suppose that there exists a nonempty compact and convex subset M of K such that for each $x \in K \setminus M,$ $y \in T(x)$, there exists $u \in M$ such that $F(x, y, u) \in -intC$.*

Then there exists $x^* \in K,$ $y \in T(x)$ such that

$$F(x^*, y, u) \notin -intC, \text{ for all } u \in S(x^*).$$

Proof. If $h : K \to K$ be the identity map in the above Theorem 3.1, then it can be easily checked that all the assumptions of Theorem 3.1 are satisfied. □

Remark 3.3. The above corollary gives the existence results for the solution of implicit vector equilibrium problem for trifunction in Chiang et al. [3] without the compactness of K, closedness of D if we replace assumptions (i)-(vi) in [3] by the hypotheses of above Corollary 3.2 of this paper.

4. Conclusion

In this work, implicit vector equilibrium problems for trifunction in real Hausdorff topological vector space is considered, and established some existence results for the solution of the problems by using KKM-lemma. Some special cases have also been discussed to show that our results are generalization of several authors.

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References

1. M. O. Aibinu and J.K.Kim, On the rate of convergence of viscosity implicit iterative algorithms, Nonlinear Funct. Anal. Appl.,**25**(1)(2020),135–152. doi.org/10.22771/nfaa2020.25.01.10.
2. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Maths. Student, **63**(1994), 123–145.
3. Y. Chiang, O. Chadli and J.C.Yao, Existence of solutions to implicit vector variational inequalities, J. Optim. Theory and Appl., **116**(2)(2003), 251–264.
4. Y. Chiang, O.Chadli and J.C.Yao, Generalized vector equilibrium problems for trifunctions, J. Global Optim., **30**(2004), 135–154.
5. P. Cubiotti and J.C. Yao, Necessary and sufficient conditions for the existence of implicit variational inequality, Appl. Math. Letters, **10** (1997), 83–87.
6. X. P. Ding, The generalized vector quasi variational-like inequality, Computers Math. Appl., **37**(1999), 57–67.
7. A. Domokos and J. Kolumban, Variational inequalities with operator solutions, J. Global Optim.,**23**(2002), 99–110.
8. M. Durea, Some existence and stability results of solutions for vector equilibrium problems, Nonlinear Funct. Anal. Appl.,**11**(1) (2006), 125–137. doi.org/10.22771/nfaa2006.11.01.11.
9. K. Fan, A generalization of Tychnoff's fixed point theorems, Math. Ann.,**142**(1961), 305–310.
10. K. R. Kazmi and A. Raouf, A class of operator equilibrium problems, J. Math. Anal. Appl. **308**(2005), 554–564.
11. H. Knaster, C. Kuratowski and S. Mazurkiewicz, Ein Bewis des fix-punktasatzes fiir n-dimensional simplex, Fundamenta Mathematicae,**41**(1929), 132–137.
12. G. M. Lee, D. S. Kim and B. S. Lee, On non-cooperative vector equilibrium indian, J. Pure Appl. Math.,**27**(1996), 735–739.
13. J. Li., N. J. Huang, and J. K. Kim, On implicit vector equilibrium problems, J. Math. Anal. Appl.,**283**(2003), 501–512.
14. T. Ram, On existence of operator solutions of generalized-vector quasi-variational inequalities, Commun. Optim. Theory, **2015** (2015), 1–7.
15. T. Ram and A. K. Khanna, On perturbed quasi-equilibrium problems with operator solutions, Nonlinear Func. Anal. Appl.,**22**(2)(2017), 385–394.
16. T. Ram and A. K. Khanna, On generalized weak operator quasi equilibrium problems, Glob. J. Pure and Appl. Math.,**13** (8) (2017), 4189–4198.
17. T. Ram, P. Lal and J. K. Kim, Operator solutions of generalized equilibrium problems in Hausdorff topological vector spaces, Nonlinear Funct. Anal. Appl., **24**(1) (2019), 61–71. doi.org/10.22771/nfaa2019.24.01.05
18. A. Raouf and J. K. Kim, A class of generalized operator equilibrium problems, Filomat,**31**(1)(2017), 1–8.
19. M. H. Shih, and K. K. Tan, Generalized quasi variational inequalities in locally-convex topological vector spaces, J. Math. Anal. and Appl., **108** (1985), 333–343.
20. J. C. Yao, Generalized quasi variational inequalities problems with discontinuous mapping, Math. Oper. Res., **20**(1995), 465–478.