S. Park, J. Nonlinear Anal. Optim. Vol. 13(1) (2022), 1-24

Journal of Nonlinear Analysis and Optimization
Volume 13(1) (2022)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

FROM *KKM* CLASS TO $\mathfrak{KC}$ CLASS OF MULTIMAPS

SEHIE PARK*[1]

[1] The National Academy of Sciences, Republic of Korea; Seoul 06579 and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea

---

ABSTRACT. This is to give a short history of our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KD}$ in the KKM theory. We show that many authors adopted or imitated the inadequate definition of the KKM class due to Chang and Yen in 1996. We list such works in chronological order and also introduce other works which extended the class properly. Our study on such history will improve the KKM theory in the new millennium.

KEYWORDS:Abstract convex space, KKM theorem, KKM class of multimaps, $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KD}$
AMS Subject Classification: :47H10, 49J53, 54C60, 54H25, 90A14, 90C76, 91A13, 91A10.

---

1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Since the theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions of a lot of problems in pure and applied mathematics, many scholars have been studying its further extensions and applications.

The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde, to c-spaces by Horvath and others, to G-convex spaces mainly by the present author. Since then a large number of authors introduced imitations, modifications, and generalizations of G-convex spaces and published hundreds of papers. Motivated by this, in 2006-09, we proposed new concepts of abstract convex spaces and partial

---

* *Corresponding author.*
Email address : park35@snu.ac.kr; sehiepark@gmail.com.
Article history : Received 16 March 2022; Accepted 22 March 2022.

KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory properly.

Now the KKM theory becomes the study of abstract convex spaces due to ourselves in 2006 and we obtained a large number of new results in such frame. For the history of the KKM theory, see our previous article [Park 2017]. Moreover, applications of the KKM theory to equilibrium theory, variational inequalities, best approximations, economic theory, and many others can be seen in the references therein.

More early, motivated our works, Chang and Yen [1996] introduced the so-called KKM class of multimaps. According to Google Scholar in 2022, more than 171 papers have quoted the paper, and we noticed that most of these papers adopted or imitated the inadequate definition of the KKM class. This is mainly because such authors were ignorant of the fact that *the KKM theorem also holds for open-valued KKM maps*. This was discovered by W. K. Kim [1987] and Shih and Tan [1987]. Motivated by this, in order to improve the KKM theory, we replaced the KKM class by the modern classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ of multimaps.

This survey article is to give a history of the multimap classes KKM and $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ in the KKM theory, and to let the readers know that the usage of KKM class is not adequate. We show that many authors adopted or imitated the inadequate definition of the KKM class due to Chang and Yen [1996]. We list such works in chronological order and also introduce other works which extended the class properly. Our study on such history will improve the KKM theory in the new millennium.

Section 2 deals with preliminaries on abstract convex spaces. In Section 3, we introduce a short history of our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$. Section 4 deals with the literature on the KKM class due to Chang and Yen [1996] and many of its modifications or imitations appeared mainly in the new millennium. In Section 5, we introduce some articles concerned with our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$. Finally, Section 6 deals with some conclusion.

2. Preliminary on abstract convex spaces

The following is given by Park [2006], where $\langle D\rangle$ denotes the collection of nonempty finite subsets of a set D:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D\rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Later we always assumed that E is a topological space in an abstract convex space $(E, D; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{−map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

From now on, in this section, we give examples of abstract convex spaces $(E, D; G)$ in the chronological order. For the most of the examples, we have $1_E \in \mathfrak{KC}(E, E)$ [or $1_E \in \mathfrak{KO}(E, E)$].

1. If $E = \Delta_n$ is an n-simplex, D is the set of its vertices, $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM principle [1929] says that $1_E \in \mathfrak{KC}(E, E)$. In this case, note that $1_E \notin \mathfrak{K}(E, E)$. A simple example for $n = 1$ is as follows: Let $\Delta_1 := [0, 1]$, $D := \{0, 1\}$, and $G(0) := [0, \frac{1}{2}]$, $G(1) := (\frac{1}{2}, 1]$. Then G is a KKM map, but $G(0) \cap G(1) = \emptyset$.
2. If D is a nonempty subset of a topological vector space E (not necessarily Hausdorff), Fan's KKM lemma [1961] says that $1_E \in \mathfrak{KC}(E, E)$.
3. Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be *almost convex* [Jeng et al. 2006] if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \cdots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \cdots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \cdots, n$ and $\mathrm{co}\, B \subset X$. By choosing $\Gamma_A := B$ for each $A \in \langle X\rangle$, $(X; \Gamma)$ becomes a G-convex space and hence an abstract convex space.
4. If X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology, then $(X, D; \Gamma)$ becomes a *convex space* generalizing the one due to Lassonde [1983]. Note that any convex subset of a t.v.s. is a convex space, but not conversely. For a convex space (X, co), Lassonde showed that $1_X \in \mathfrak{KC}(X, X)$.
5. In the same year, Kim [1989] and Shih and Tan [1989] showed that $1_E \in \mathfrak{KO}(E, E)$ when E is an n-simplex. Therefore, in general, we have
$$\mathfrak{K}(E, E) \subsetneq \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E).$$
6. A well-known subclass of G-convex spaces due to Horvath [1987-1993] can be generalized as follows: A G-convex space $(X, D; \Gamma)$ is called a *c-space* (or an *H-space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D\rangle$. For a c-space (X, Γ), Horvath showed that $1_X \in \mathfrak{KC}(X, X)$. In particular, Khamsi [1996] obtained $1_X \in \mathfrak{KC}(X, X)$ for a hyperconvex metric space X.
7. In early 1990's, the author [1993] introduced the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of multimaps $X \multimap Y$ between topological spaces and showed that this class is contained in the class $\mathfrak{KC}(X, Y)$ when X is a convex space and Y is a Hausdorff space [1994]. Motivated by this, Chang and Yen [1996] defined the KKM class of maps on convex subsets of topological vector spaces, and further, Chang et al. [1999] extended the KKM-class to S-KKM class. On the other hand, the author extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, in the class of compact closed multimaps from convex spaces

to Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide [1997]. Moreover, H. Kim [2005] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is slightly meaningful].

8. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

   Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. It is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

   For details on G-convex spaces, see Park [2000-2003] and Park et al. [1993-2005], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

9. For a G-convex space $(X, D; \Gamma)$ and a topological space Z, we defined the classes $\mathfrak{K}, \mathfrak{KC}, \mathfrak{KO}$ of multimaps $F : X \multimap Z$, and showed that $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$. Moreover, we noted that if $F : X \longrightarrow Z$ is a continuous single-valued map or if $F : X \multimap Z$ has a continuous selection, then $F \in \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$. Furthermore, for a Hausdorff space Z, it is shown that $\mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$ by H. Kim and the author [2005].

10. Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see Sortan [1984], where the bibliography lists 283 papers. For any subset $X \subset E$, its *$\mathcal{C}C$-convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap \{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is *$\mathcal{C}$-convex* if $X{=}\mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A{:=}\mathrm{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

    Notice that our abstract convex space $(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E.

11. For any metric space (M, d), Amini et al. [2005] introduced a convexity structure similar to the one for hyperconvex metric space; see Khamsi [1996]. They defined an $\mathcal{NR}$-metric space (M, d) and showed that, for any subadmissible subset X of M, $1_X \in \mathfrak{KC}(X, X)$ holds. Here, subadmissible subsets are simply Γ-convex subsets.

    Recall that, for a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y, Park and Kim [1997] showed that an acyclic map $F : X \multimap Y$ or, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. Amini et al. [2005] repeatedly claimed that they obtained this result in 2005. More early in Park [1994], the result was obtained for convex spaces and this is the origin of the study of the so-called KKM-class of multimaps.

12. Imitating the original definition of S-KKM maps of Chang et al. [1999], Amini et al. [2007] defined the S-KKM class for a classical convexity space $(X, \mathcal{C})$ with a nonempty set Z and a topological space Y as follows: If $S : Z \multimap X$, $F : X \multimap Y$, and $G : Z \multimap Y$ are three multimaps satisfying

    $$F(\mathrm{Co}_{\mathcal{C}}(S(A)) \subset G(A) \quad \text{for each} \quad A \in \langle Z \rangle,$$

then G is called a $\mathcal{C}$-S-KKM map with respect to F. If the map $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM map G with respect to F, the family $\{\overline{G(z)} \mid z \in Z\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. Amini et al. defined

$$S\text{-KKM}_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y \mid F \text{ has the } S\text{-KKM property with respect to } \mathcal{C}\}.$$

It should be noted that, by putting $\Gamma_A := \mathrm{Co}_{\mathcal{C}}(S(A))$ for each $A \in \langle Z \rangle$, S-KKM$_{\mathcal{C}}(Z, X, Y)$ becomes simply $\mathfrak{KC}(X, Y)$. Therefore, it should be eliminate the S-KKM class.

13. There were many imitations, modifications, or fake extensions of G-convex spaces like the so-called L-spaces, spaces having property (H), M-spaces, MC-spaces, FC-space, GFC-space, simplicial spaces, FWC-spaces, etc. They were all destroyed now.

3. Multimap classes in the KKM theory

We already introduced certain broad classes $\mathfrak{KO}$ and $\mathfrak{KC}$ of maps in several papers; see Park [2018]. Note that $\mathfrak{KC}$ includes the KKM class introduced by Chang and Yen [1996] as a special case. With these concepts, some coincidence theorems and fixed point theorems were proved in abstract convex spaces by ourselves; see Park [2018].

Subclasses of multimaps in the KKM theory were appeared as follows:

1929 identity function —- KKM
1961 identity function —- Fan
1989 continuous function —- Park
1991 acyclic map —- Shioji, Park
1993 admissible map $\mathfrak{A}_c$ —- Park
1994 admissible map $\mathfrak{A}_c^\kappa$ —- Park
1996 the class *KKM* —- Chang and Yen
1997 better admissible map $\mathfrak{B}$ —- Park
1997 KKM family $\mathfrak{K}$ —- Park
1998 $\mathfrak{B}^\kappa$ —- Park
2003 $\mathfrak{KC}$, $\mathfrak{KO}$ —- Park
2004 $\mathfrak{B}^p$ —- Park

Recall that W. K. Kim [1987] and Shih and Tan [1987] discovered that the well-known KKM theorem in 1929 also holds for open-valued multimaps. This open-valued KKM theorem has been generalized to various types of abstract convex spaces by the present author. Consequently, we introduced multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$ in 2003.

Mutual relations of the classes $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, and $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ depend on the nature of the related abstract convex spaces; see the previous works of Park in 2006–2021.

Even after we defined these classes, many authors imitated or adopted the class *KKM* for two decades. Most of their results are mere copies of the classical ones; see Park [2021].

4. On the KKM class of multimaps

In this section, we introduce several ones among 171 papers stated in Google Scholar in 2021. Key statements in each paper are quoted in their original expressions, and, in most cases, certain comments by the present author are added.

Chang and Yen [1996] — JMAA203

Assume that X is a convex subset of a linear space and Y is a topological space. If $S, T : X \longrightarrow 2^Y$ are two set-valued mappings such that $T(coA) \subset SA$ for each finite subset A of X, then we call S a generalized KKM mapping w.r.t. T, where coA denotes the convex hull of A. Let $T : X \longrightarrow 2^Y$ be a set-valued mapping such that if $S : X \longrightarrow 2^Y$ is a generalized KKM mapping w.r.t. T then the family $\overline{Sx} : x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of Sx), then we say that T has the KKM property. Denote KKM$(X, Y) = \{T : X \longrightarrow 2^Y \mid T$ has the KKM property$\}$

Remark 4.1. Generalized KKM mappings were first introduced by Park [1989], and followed by some others.

Comments: Note that the KKM class contains the admissible class $\mathfrak{A}_c^\kappa$ due to Park. Later, we denoted KKM class by $\mathfrak{K}$ in [Park 1997] and, extended it to the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ for abstract convex spaces; see Section 3 of the present paper.

Chang-Yen's paper has 171 citations (see Google Scholar in 2022) and many peoples still use their obsolete generalized KKM classes. In Section 2 we gave some examples in the new millennium.

Chang-Yen's definition of generalized KKM mapping seems to be not elegant, and works only for closed-valued maps. Recall that more early there have appeared open-valued KKM maps in 1987. This is why we defined the classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ later.

Lin, Ko, and Park [1998] — Discuss. Math. Diff. Incl. 18

In this paper, a set-valued map with G-KKM property is defined and a minimax theorem for set-valued maps with G-KKM property on G-convex space is established. As a consequence of these results we verify coincidence theorem for set-valued maps with G-KKM property on G-convex spaces. Finally, we apply our results to the best approximation problem and fixed point problem.

Chang, Huang, Jeng, and Kuo [1999] — JMAA229

Definition 4.2. Let X be a nonempty set, Y a nonempty convex set of a linear space, and Z a topological space. If $S : X \longrightarrow 2^Y$, $T : Y \longrightarrow 2^Z$, and $F : X \longrightarrow 2^Z$ are three multifunctions satisfying

$$T(coS(A)) \subseteq F(A)$$

for any $A \in \langle X \rangle$, then F is called a generalized S-KKM mapping with respect to T. If the multifunction $T : Y \longrightarrow 2^Z$ satisfies the requirement that for any generalized S-KKM mapping F with respect to T the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X, Y, Z) is defined to be the set $\{T : Y \longrightarrow 2^Z : T$ has the S-KKM property$\}$

Comments: Note that F can be assumed closed-valued, and no consideration on open-valued mappings is given.

Agarwal and O'Regan [2002] — DSA21

We also discuss KKM maps in this paper. Here again X is a convex subset of a Hausdorff topological vector space and Y a topological space. If $S,\ T : X \longrightarrow 2^Y$ are two set valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say S is a generalized KKM map w.r.t. T. T is said to have the KKM property if for any generalized KKM w.r.t. T map $S : X \longrightarrow 2^Y$, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$KKM(X,Y) = \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Comments: The Hausdorffness is redundant and S can be assumed closed-valued, and no consideration on open-valued mappings is given. From now on we will not repeat such comments.

Lin, Ansari, and Wu [2003] — JOTA117

Let X be a convex space, and let Y be a Hausdorff topological space. If $S,\ T : X \longrightarrow 2^Y$ are multivalued maps such that

$$T(coN) \subseteq S(N), \text{ for each } N \in \langle X \rangle,$$

then S is said to be generalized KKM mapping w.r.t. T (Chang-Yen [1996]). The multivalued map $T : X \longrightarrow 2^Y$ is said to have the KKM property (Chang-Yen [1996]) if $S : X \longrightarrow 2^Y$ is a generalized KKM mapping w.r.t. T such that the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property.

Chen, Chang, and Yen [2004] — JKMS41

We generalized the KKM property to the following form for a nearly-convex set X. Assume that X is a nearly-convex subset of a linear space and Y is a topological space. If $T,\ S : X \longrightarrow 2^Y$ are two set-valued mapping such that $T(coA \cap X)\overline{S(A)}$ for each finite subset A of X, then we call S a generalized KKM mapping with respect to T, where $co(A)$ denotes the convex hull of A. Let $T : X \longrightarrow 2^Y$ be a set-valued KKM mapping with respect to T then the family $\overline{Sx} : x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of Sx), then we say that T has the KKM property. Denote

$$KKM(X.Y) = \{T : X \longrightarrow 2^Y \mid T\text{has the KKM property}\}$$

Remark 4.3. Generalized KKM mappings were first introduced by Park [1989], and followed by some others.

Comments: Note that the above is the same one to Chang-Yen [1996] just replacing a convex subset by a nearly-convex subset.

Shahzad [2004] — NA56

Definition 4.4. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S,\ T : X \longrightarrow 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \longrightarrow 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. We let

$$KKM(X,Y) = \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Comments: Similarly, the author defined generalized S-KKM map w.r.t. some T.

Zafarani [2004] — Liège73

Chang and Yen [1996] made a systematic study of the class of the KKM mappings: Let X be a nonempty convex subset of a topological vector space and Y a topological space. If $G : X \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multivalued maps such that for any $A \in \langle X\rangle,\ F(co(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping respect to F. Let $F : X \longrightarrow 2^Y$ be a multivalued mapping such that if $G : X \longrightarrow 2^Y$ is a generalized KKM mapping with respect to F, then the family $\{clG(x) : x \in X\}$ has the finite intersection property. In this case, we say that F has the KKM property. We define $KKM(X,Y) = \{F : X \longrightarrow 2^Y : T$ has the KKM property$\}$

Comments: Many results on the KKM theory on various types of spaces are introduced. Zafarani's Γ-convex spaces are motivated by our G-convex spaces and the same to our original abstract convex spaces in 2006. But he did not establish any theory on his spaces. He adopted a KKM theorem, a very particular form of our KKM theorems. Many terms in this paper are obsolete; for example, the KKM class, the S-KKM class, and the generalized S-KKM class belong to our $\mathfrak{KC}$ class in our abstract convex space theory. Zafarani introduced NR-metric spaces, which generalize hyperconvex metric spaces and are G-convex spaces.

Amini, Fakhar, and Zafarani [2005] — NA60

Let (M,d) be a metric space and X a subadmissible subset of M. A multifunction $G : X \multimap M$ is called a KKM mapping, if for each $A \in \langle X\rangle X,\ co(A) \subset G(A)$. More generally, if Y is a topological space and $G : X \multimap Y,\ F : X \multimap Y$ are two multifunctions such that for any $A \in \langle X\rangle,\ F(co(A)) \subseteq G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any generalized KKM mapping $G : X \multimap Y$ with respect to F the family $\{cl\, G(x) : x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$KKM(X,Y) := \{F : X \multimap Y : F \text{ has the KKM property}\}$$

.

Comments: On the surface, this is a very nice paper. However, the authors adopted inadequate terminology of Chang-Yen. For example, the class of KKM type mappings is $\mathfrak{KC}$ in our works.

Fakhar and Zafarani [2005] — JOTA126

Let X be a convex subset of a t.v.s. E and let Y be a topological space. If $\Gamma : X \longrightarrow 2^Y,\ T : X \longrightarrow 2^Y$ are two multivalued mappings such that, for any $A \in \langle X\rangle,\ T(\mathrm{co}A) \subseteq \Gamma(A)$, then Γ is said to be a generalized KKM mapping with respect to T. Let $T : X \longrightarrow 2^Y$ be a multivalued mapping such that, if $\Gamma : X \longrightarrow 2^Y$ is a generalized KKM mapping with respect to T, then the family $\{\mathrm{cl}\Gamma(x) : x \in X\}$ has the finite intersection property; in this case, we say that T has the KKM property. Denote

$$KKM(X,Y) := \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Fakhar and Zafarani [2005a] — Belgium 12

Let $(X,D;\Gamma)$ be a G-convex space and Y be a topological space. A multivalued map $F : D \longrightarrow 2^X$ is called a KKM map if for each $A \in \langle D\rangle,\ \Gamma(A) \subset \bigcup_{x\in A} F(x)$. More generally if $G : D \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multivalued maps such that for any $A \in \langle D\rangle,\ F(\Gamma(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping with respect to F. Let $F : X \longrightarrow 2^Y$ be a multivalued mapping such that if $G : D \longrightarrow 2^Y$ is a generalized KKM mapping with respect to F, then the family

$\{clG(x) : x \in D\}$ has the finite intersection property. In this case we say that F has the KKM property. We define

$$\mathfrak{K}(X, Y) := \{F : X \longrightarrow 2^Y : F \text{ has the generalized KKM property}\}.$$

When X is a convex subset of a topological vector space, the class $\mathfrak{K}(X, Y)$ was introduced and studied by Chang and Yen [1996]. This concept is further extended to G-convex spaces by Lin, Ko, and Park [1998].

Comments: The notation $\mathfrak{K}$ was originally introduced by Park in 1997.

H. Kim [2005] — NA63

Let X be a convex subset of a vector space and Y a topological space. In 1996, Chang and Yen defined the following: A multimap $T : X \multimap Y$ is said to have the KKM property if, for any map $F : X \multimap Y$ with closed values satisfying

$T(coN) \subset F(N)$ for all $N \in \langle X \rangle X$,

the family $\{F(x)\}_{x\in X}$ has the finite intersection property. We denote $\mathrm{KKM}(X, Y) := \{T : X \multimap Y \mid T \text{ has the KKM property}\}$.

Comments: This is the first paper assuming closed-valued multimaps F in KKM.

Chen [2006] —JMAA323

Comments: Motivated by Amimi-Fakhar-Zafarni [2005], Chen established some fixed point theorems with domain as a nearly-subadmissible subset of a complete metric space (M, d) for a k-set contraction map, which does not need to be a compact map. He also deduces a generalization of the approximate fixed point theorem for the lower semicontinuous mappings on a metric space.

He defines a generalized KKM mapping with respect to T, a slightly modified definition of Chang-Yen [1996].

The main result is the following fixed point theorem for the k-set contraction.

Theorem 4.1. *Let (M, d) be a complete metric space and X be a nonempty bounded nearly subadmissible subset of M. If $T \in KKM(X, X)$ is a k-set contraction, $0 < k < 1$ and closed with $T(X) \subset X$, then T has a fixed point in X.*

Since the KKM families $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ were introduced in 2003, $KKM(X, X)$ can be replaced by more correct $\mathfrak{KC}$.

Recall that any article adopting or imitating the *KKM* class of Chang-Yen [1996] is obsolete.

Jeng, Hsu, and Huang [2006]— JMAA319

Similar to Chang-Yen [1996] we now extend the concept of generalized KKM mapping in the following manner.

Definition 4.5. Suppose X and Y are two nonempty subsets of a linear space E, and $T,\ F : X \multimap Y$. We say that F is a generalized KKM mapping with respect to T if for any $A = \{x_1, \dots, x_n\} \in \langle X \rangle$ there is $B = \{y_1, \dots, y_n\} \in \langle X \rangle$ satisfying

(a) $\mathrm{co}(B) \subseteq X$, and

(b) $T(co\{y_i : i \in I\}) \subseteq \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{1, \cdots, n\}$.

Definition 4.6. Let X and Y be two nonempty subsets of a topological vector space E. If a multifunction $T : X \multimap Y$ satisfies that for any generalized KKM mapping $F : X \multimap Y$ with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The class

KKM(X, Y) is defined to be the set

$$\{T : X \multimap Y : T \text{ has the KKM property}\}.$$

Kuo, Huang, Jeng, and Shih [2006]—FPTA2006

The concept of S-KKM property of Chang et al. [1999] can be extended to G-convex spaces.

Definition 4.7. Let X be a nonempty set, $(Y, D; \Gamma)$ a G-convex space and Z a topological space. If $S : X \multimap D,\ T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\Gamma_{S(A)}) \subseteq F(A)$$

for any $A \in \langle X \rangle$, then F is called a S-KKM mapping with respect to T. If the multimap $T : Y \multimap Z$ satisfies that for any S-KKM mapping F with respect to T, the family $\{\overline{F}(x) : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X, Y, Z) is defined to be the set $\{T : X \multimap Y : T$ has the S-KKM property$\}$.

When $D = Y$ is a nonempty convex subset of a linear space with $\Gamma_B = co(B)$ for $B \in \langle Y \rangle$, the S-KKM(X, Y, Z) is just that as in Chang et al. [1999].

Comments: Now the S-KKM class is obsolete.

Shahzad [2006]—Simon Stevin

Definition 4.8. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \longrightarrow 2^Y,\ T : Y \longrightarrow 2^Z,\ F : X \longrightarrow 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X , then F is called a generalized S-KKM map w.r.t. T . If the map $T : X \longrightarrow 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then F is said to have the S-KKM property. The class

$$S - KKM(X, Y, Z) = \{T : Y \longrightarrow 2^Z : T\text{has the S-KKM property}\}.$$

Comments: Now the S-KKM class is obsolete.

Amini, Fakhar, and Zafarani [2007] — NA66

Like the work of Chang et al. [1999], we introduce the family of multifunctions with the S-KKM property as follows. Let Z be a nonempty set, $(X, \mathcal{C})$ an abstract convex space, and Y a topological space. If $S : Z \multimap X,\ F : X \multimap Y$ and $G : X \multimap Y$ are three multifunctions satisfying

$$F(co_{\mathcal{C}}(S(A))) \subseteq \bigcup_{x \in A} G(x)$$

for each $A \in \langle Z \rangle$, then G is called a $\mathcal{C}$-S-KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM mapping G with respect to F, the family $\{clG(x) : x \in X\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. We define

$$S - KKM_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y : F \text{ has } S\text{-KKM property with respect to } \mathcal{C}\}.$$

Comments: In this paper, we notice the following: (1) Here abstract convex spaces mean spaces having the routine convexity structure, (2) Chang-Yen's KKM class

[1996] should be replaced by the $\mathfrak{KC}$ class. (3) The S-KKM class of Chang et al. [1999] is simply a $\mathfrak{KC}$ class.

Chen [2007] — Sci. Math. Jpn. 2007

A G-convex space $(X, D; \Gamma)$, where D is a nonempty subset of X, is called an L-convex space in this paper.

Definition 4.9. Let X be an L-convex space, Y a topological space such that for each $N \in \langle X \rangle$ with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_N \longrightarrow X$. If $T, F : X \longrightarrow 2^Y$ are two set-valued function satisfying that $T(\psi_N(\Delta_N)) \subset F(N)$ for each $N \in \langle X \rangle$ with $|N| = n+1$, then F is said to be a generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N. Moreover, if the set-valued function $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N the family $\{\overline{Fx} \mid x \in X\}$ has the finite intersection property, then T is said to have the R_{ψ_N}-KKM property. The class R_{ψ_N}-$KKM(X, Y)$ is defined to be the set $\{T : X \longrightarrow 2^Y \mid T$ has the R_{ψ_N}-KKM property$\}$. (* This ψ_N may be different from the ϕ_N of the definition for the L-convex space.)

Comments: In the above definition of a class R_{ψ_N}-$KKM(X, Y)$, the role of G-convex spaces (its author's L-convex spaces) is not clear. This remark also works all of Theorems 4–20 in this paper. Moreover, Theorems 21-23 in this paper are concerned with G-convex spaces and follow easily from the known results in the G-convex space theory.

Chen and Chang [2007] — JMAA329

Abstract: We first establish a fixed point theorem for a k-set contraction map on the family KKM(X, X), which not needs to be a compact map. Next, we establish the matching theorems, coincidence theorems and minimax theorems on the family KKM(X, Y) and the Φ-mapping.

Al-Thagafi and Shahzad [2008] — FPTA2008

Let $T : A \longrightarrow 2^B$. We say that (e) T is an $\mathfrak{A}_c^\kappa$-multimap if for every compact set K in A, there exists an $\mathfrak{A}_c$-multimap $f : K2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$, (f) T is a **K**-multimap (or Kakutani multimap) if T is upper semicontinuous with compact and convex values, (g) $S : A \longrightarrow 2^B$ is a generalized **KKM**-multimap with respect to T if $T(coD) \subseteq S(D)$ for each finite subset D of A, (h) T has the **KKM** property if, whenever $S : A \longrightarrow 2^B$ is a generalized **KKM** multimap w.r.t. T, the family $\{\overline{S(x)} : x \in A\}$ has the finite intersection property; (i) T is a **PK**-multimap if there exists a multimap $g : A \longrightarrow 2^B$ satisfying $A = \bigcup\{\operatorname{int} g^{-1}(y) : y \in B\}$ and $\operatorname{co}(g(x)) \subseteq T(x)$ for every $x \in A$.

Balaj [2008]—NA68

Definition 4.10. (See Chang-Yen [1996]) Let X be a convex subset of a topological vector space and Y be a topological space. If $S,\ T : X \multimap Y$ are two maps such that

$$T(coA) \subseteq S(A) \text{ for each nonempty finite subset } A \text{ of } X,$$

then S is said to be *generalized* KKM *w.r.t.* T. The map $T : X \multimap Y$ is said to have the KKM *property* if for each $S : X \multimap Y$ which is a generalized KKM map w.r.t. T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property.

We denote by KKM(X, Y) the family of maps having the KKM property.

Comments: The author still follows the work of Chang-Yen [1996].

Chang, Chen, and Huang [2008]—TJM12

Recently, Amini, Fakhar and Zafarani [2005] introduced the class KKM(X, Y) in metric space, and get some results about fixed point theorems and matching theorem. In this work, we use the conception of J. C. Jeng, H. C. Hsu and Y. Y. Huang [2006] to define the KKM family on metric space. We establish a generalized KKM theorem in a hyperconvex metric space, and then we use this theorem to get a fixed point theorem, the matching theorem, the coincidence theorem, minimax inequality theorems and the variational inequality theorems.

Suppose X is a bounded subset of a metric space (M, d). Then the admissible hull of X is defined by

$$ad(X) = \bigcap\{B \subset M : B \text{ is a closed ball in } M \text{ such that } X \subset B\}$$

Definition 4.11. Let X be a metric space, Y be a nonempty set, and Z be a hyperconvex metric space. If $T : X \longrightarrow 2^Z$, $F : Y \longrightarrow 2^Z$ are two set-valued mappings satisfying that for each $\{y_1, y_2, \ldots, y_n\} \in \langle Y\rangle$, there exists $\{x_1, x_2, \ldots, x_n\} \in \langle X\rangle$ such that $T(ad(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})) \subset \bigcup_{j=1}^{k} F(y_{i_j})$, for all $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Z$ satisfies the requirement that for any generalized KKM mapping $F : Y \longrightarrow 2^Z$ with respect to T, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, then T is said to have the KKM property. We denote

$$KKM(X, Z) = \{T : X \longrightarrow 2^Z \,|\, T \text{ has the KKM property}\}.$$

Chang, Chen, and Peng [2008] — NA69

Definition 4.12. Let (M, d) be a metric space. A subset X of M is called admissible if it is an intersection of closed balls in M. The collection of all admissible subsets in M is denoted by $\mathcal{A}(M)$. The smallest admissible set containing a bounded subset X of M is called the admissible hull of X and denoted by ad(X). So

$$ad(X) = \bigcap_{x \in M} B(x, r_x(X)),$$

where $B(x, r_x(X))$ is the closed ball centered at x with radius $r_x(X) \geq 0$ and $r_x(X) = \sup\{d(x, y) : y \in X\}$.

Definition 4.13. Let M be a metric space and $X \subset M$. A set-valued mapping $F : X \longrightarrow 2^M$ is called a KKM map if

$$ad(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i),$$

for any $x_1, x_2, \ldots, x_n \in X$.

Definition 4.14. Let X be a metric space and Y a topological space. If $F, T : X \longrightarrow 2^M$ are two set-valued mappings such that for each $A \in \langle X\rangle$, $T(\text{ad}(A)) \subset F(A)$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has finite intersection property, then T is said to have the KKM property. The class KKM(X, Y) is defined as the set $\{T : X \longrightarrow 2^Y \mid T \text{ has the KKM property}\}$.

Chen and Chang [2008] — NA69

We first define the generalized ${}_g$KKM mapping and the family ${}_g$KKM(X, Y).

Definition 4.15. Let X be a metric space, and Y a nonempty set. If $F : Y \longrightarrow 2^X$ is a set-valued mapping satisfying that for each $\{y_1, y_2, \dots, y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ such that $\mathrm{ad}\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subset \bigcup_{j=1}^k F(y_{i_j})$, for all $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, then F is called a generalized ${}_g$KKM mapping.

Definition 4.16. Let X be a metric space, Z a nonempty set, and Y a topological space. If $T : X \longrightarrow 2^Y$, $F : Z \longrightarrow 2^Y$ are two set-valued mappings satisfying that for each $\{z_1, z_2, \dots, z_n\} \in \langle Z \rangle$, there exists $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ such that $T(\mathrm{ad}(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})) \subset \bigcup_{j=1}^k F(z_{i_j})$, for all $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, then F is called a generalized ${}_g$KKM mapping with respect to T . If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized ${}_g$KKM mapping $F : Y \longrightarrow 2^Y$ with respect to T, the family $\{overlineF(z) : z \in Z\}$ has the finite-intersection property, then T is said to have the ${}_g$KKM property. We denote

$$ {}_g\mathrm{KKM}(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the } {}_g\mathrm{KKM} \text{ property}\}. $$

Agarwal, Balaj, and O'Regan [2009] — Appl.Anal.88

We introduce the concept of a family of set-valued mappings generalized KKM w.r.t. other family of set-valued mappings. We then prove that if X is a nonempty compact convex subset of a locally convex Hausdorff topological vector space and $\mathcal{T}$ and $\mathcal{S}$ are two families of self set-valued mappings of X such that $\mathcal{S}$ is generalized KKM w.r.t. $\mathcal{T}$, under some natural conditions, the set-valued mappings $S \in \mathcal{S}$ have a fixed point. Other common fixed point theorems and minimax inequalities of Ky Fan type are obtained as applications.

The following close concept is due to Lin and Chang [1998]: if X is a nonempty set, Y is a convex subset of a vector space and S, $T : X \multimap Y$ are two set-valued mappings, S is called a T-KKM mapping if $\mathrm{co}(\bigcup_{n=1}^n T(x_i)) \subseteq \bigcup_{n=1}^n S(x_i)$, for any nonempty finite subset $\{x_1, \dots, x_n\}$ of X. Inspired by these concepts we introduce a new one, concerning two families of set-valued mappings.

Definition 4.17. Let X be a nonempty set, Y be a convex subset of a vector space and $\mathcal{T}$ and $\mathcal{S}$ are two families of set-valued mappings with nonempty values from X into Y. We say that $\mathcal{S}$ is *generalized KKM* w.r.t. $\mathcal{T}$ if for any nonempty finite subfamily $\{S_1, \dots, S_n\}$ of $\mathcal{S}$ there exist $T_1, \dots, T_n \in \mathcal{T}$ such that $\mathrm{co}(\bigcup_{i\in I} T_i(x)) \subseteq \bigcup_{i\in I} S_i(x))$, for each nonempty subset I of $\{1, \dots, n\}$ and for all $x \in X$.

Remark 4.18. If Y is a convex subset of a topological vector space and $\mathcal{S}$ is generalized KKM w.r.t. $\mathcal{T}$, then for each $x \in X$, $\{\overline{S(x)} : S \in \mathcal{S}\}$ has the finite intersection property.

Comments: No consideration on open-valued maps is given.

Amini-Harandi, Farajzadeh, O'Regan, and Agarwal [2009] — NFAA14

Definition 4.19. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$ F(\Gamma(A)) \subseteq G(A), \text{ for all } A \in \langle X \rangle, $$

then G is called a *KKM* map with respect to F. A *KKM* map $G : D \multimap Z$ is a KKM map with respect to the identity map 1_E. A multimap $F : E \multimap Z$ is said to have the KKM property if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{\overline{G(x)}\}_{x\in X}$ has the finite intersection property. We denote

$$ KKM(E, Z) := \{F : E \multimap Z : F \text{ has the KKM property}\}. $$

Comments: The authors adopt abstract convex (uniform) spaces due to Park, but still imitate the KKM class of Chang-Yen [1996].

Chen [2009] — NA71

The generalized KKM property on a convex subset of a Hausdorff topological vector space that was introduced by Chang and Yen [1996], we now extended this class KKM(X, Y) to be the class KKM$^*(X, Y)$ for the almost convex set X.

Definition 4.20. Let X be a nonempty almost convex subset of a topological vector space E, and Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V} : A \longrightarrow X$ such that $T(co(h_{A,V}(A))) \subset F(A)$, then we call F a generalized KKM* mapping with respect to T.

If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM* mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the KKM* property. Denote

$$KKM^*(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the KKM}^* \text{ property } \}.$$

Comments: Chang-Yen's class KKM(X, Y) is extended to the class KKM$^*(X, Y)$ for the almost convex set X. No consideration on open-valued maps is given.

Chen and Chang [2009] — FPTA2009

In 1996, Chang and Yen introduced the family KKM(X, Y) on the topological vector spaces and got results about fixed point theorems, coincidence theorems, and its applications on this family. Later, Amini et al. [2005] introduced the following concept of the KKM(X, Y) property on a subadmissible subset of a metric space (M, d).

Let X be an nonempty subadmissible subset of a metric space (M, d), and let Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for any $A \in \langle X \rangle X,\ T((A)) \subset F(A)$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping F with respect to T, the family $\{overlineF(x) : x \in X\}$ has finite intersection property, then T is said to have the KKM property. The class KKM(X, Y) is denoted to be the set $\{T : X \longrightarrow 2^Y : T$ has the KKM property$\}$.

Chen, Chang, and Chung [2009] — TJM13

Chang and Yen [1996] introduced the family KKM(X, Y), and got some results about fixed point theorems, coincidence theorems and some applications on this family. In this paper, we establish some coincidence theorems, generalized variational inequality theorems and minimax inequality theorems for the family KKM$^*(X, Y)$ and the generalized Φ-mapping on a nonconvex set.

Definition 4.21. Let X be a nonempty almost-convex subset of a topological vector space E, and Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V} : A \longrightarrow X$ such that $T(\bar{(}h_{A,V}(A))) \subset F(A)$, then we call F a generalized KKM* mapping with respect to T.

If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM* mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the KKM* property. Denote

$$KKM^*(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the KKM}^* \text{ property}\}.$$

Chang, Chen, and Chen [2010] — NA72

Abstract: We first define the family 2-$_g$KKM(X, Y) in a hyperconvex metric space, and then we get a 2-$_g$KKM theorem and a fixed point theorem without compactness assumption. Next, by using the 2-$_g$KKM theorem, we get the matching theorems, coincidence theorems, variational inequality theorems and minimax inequality theorems.

Definition 4.22. Let X be a metric space, Z a nonempty set, and Y a topological space. If $T : X \longrightarrow 2^Y$, $F : Z \longrightarrow 2^Y$ are two set-valued mappings satisfying that for each $\{z_1, z_2, \dots, z_n\} \in \langle Z\rangle$, there exists $\{x_1, x_2, \dots, x_n\} \in \langle X\rangle$ such that $T(ad(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})) \subset \bigcup_{j=1}^{k} F(z_{i_j})$, for all $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, then F is called a generalized $_g$KKM mapping with respect to T . If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized $_g$KKM mapping $F : Y \longrightarrow 2^Y$ with respect to T, the family $\{\overline{F(z)} : z \in Z\}$ has the finite intersection property, then T is said to have the $_g$KKM property. We denote

$$_gKKM(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the } _g\text{KKM property}\}$$

Chen, Chang, and Huang [2010] — AML23

Abstract: We use the conception of the abstract convexity to define the almost S-KKM$_c$ mapping, al-S-KKM$_c(X, Y, Z)$ family, and almost Φ-spaces. In the setting of the almost Φ-spaces, we establish some new fixed point theorems for the al-S-KKM$_c$ type set-valued mapping which is a generalized set contraction mapping. Our results generalize the results of Amini et al. [2007].

Khamsi and Hussain [2010] — NA73

Let M be a metric type space and X a subadmissible subset of M. A multifunction $G : X \longrightarrow 2^M$ is called a KKM mapping, if for each $A \in \langle X\rangle$, we have $\overline{A} \subset G(A) = \bigcup\{G(a), a \in A\}$. More generally, if Y is a topological space and $G : X \longrightarrow 2^Y$, $F : X \longrightarrow 2^Y$ are two multifunctions such that for any $A \in \langle X\rangle$, we gave $F(\bar{(A)}) \subset G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \longrightarrow 2^Y$ with respect to F the family $\{\overline{G(x)}, x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$KKM(X, Y) = \{F : X \longrightarrow 2^Y,\ F \text{ has the KKM property}\}$$

Turkoglua, Abuloha, and Abdeljawad [2010] — NA72

Let (M, d) be a cone metric space and X a subadmissible subset of M. A multifunction $G : X \longrightarrow 2^M$, is called a KKM mapping, if for each $A \in \langle X\rangle$, $Co(A) \subset G(A)$.

If Y is a topological space and $G : X \longrightarrow 2^Y$, $F : X \longrightarrow 2^Y$ are two multifunctions such that for every $A \in \langle X\rangle$, $F(Co(A)) \subset G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \longrightarrow 2^Y$ with respect to F, the family $\{cl(G(x) : x \in X)\}$ has the finite intersection property (f.i.p.), then

F is said to have the KKM property. We define

$$KKM(X,Y) = \{F : X \longrightarrow 2^Y : F \text{ has the KKM property}\}.$$

Balaj and Coroianu [2011] — BKMS48

Assume that X is a convex subset of a vector space and Y is a topological space. If $S,\ T : X \multimap Y$ are two set-valued mappings such that $T(coA) \subseteq S(A)$ for each nonempty finite subset A of X, then we say that S is a KKM mapping with respect to T. A set-valued mapping $T : X \multimap Y$ is said to have the KKM property if for any $S : X \multimap Y$, KKM mapping with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. Denote $\mathbf{KKM}(X,Y) = \{T : X \multimap Y : T$ has the KKM property$\}$.

Cho, Delavar, Mohammadzadeh, and Roohi [2011] — JIneqAppl2011

Let Y be a nonempty set, Z be a minimal space, $s : Y \longrightarrow D$ be a function and (X, D, Γ) be an abstract convex space. Let $T : X \multimap Z$ and $F : Y \multimap Z$ be two multimaps. we say that F is generalized s-KKM with respect to T if

$$T(\Gamma(s(A))) \subseteq F(A) \text{ for any } A \in \langle Y \rangle.$$

Comments: Minimal spaces can be made into topological spaces.

Chaipunya and Kumam [2013] — JIA2013

Definition 4.23. Let M be a circular metric space, X be a subadmissible subset of M and Y be a topological space. Let $F,\ G : X \multimap Y$ be two multivalued maps. If for each $A \in \langle X \rangle$ we have $F(ad(A)) \subset G(A)$, then G is said to be a generalized KKM map with respect to F.

Definition 4.24. Let M be a circular metric space, X be a subadmissible subset of M and Y be a topological space. A multivalued map $F : X \multimap Y$ is said to satisfy the KKM property if for any generalized KKM map $G : X \multimap Y$ with respect to F, the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property. In general, we write

$$KKM(X,Y) := \{F : X \multimap Y : F \text{ satisfy the KKM property}\}.$$

Fakhar, Lotfipour, and Zafarani [2013] — JGO55

We assume that X is a convex space, Y a Hausdorff topological space and Z a Hausdorff topological vector space. . . . Suppose that $K \subseteq X$ and $S : K \rightrightarrows X$ is a set-valued map, then S is called to be a KKM map if

$$convA \subseteq \bigcup_{x \in A} S(x), \text{ for each } A \in \langle K \rangle.$$

Let $T,\ H : X \rightrightarrows Y$ be set-valued maps. The set-valued map H is said to be a generalized KKM map with respect to (w.r.t.) T if $T(\text{conv}A) \subseteq H(A)$, for each $A \in \langle X \rangle$. The set-valued map T has the KKM property if the following statement is satisfied:

If $S : X \rightrightarrows Y$ is a generalized KKM map w.r.t. T, then the family $\{\text{cl}S(x) : x \in X\}$ has the finite intersection property. The family of all set-valued maps $T : X \rightrightarrows Y$ having the KKM property is denoted by $\text{KKM}(X,Y)$. The class $\text{KKM}(X,Y)$ was introduced and studied by Chang and Yen [1996].

Tang and Zhang [2014] — AAA2014

Definition 4.25. Let X be a nonempty set, Y a nonempty convex subset of a linear space, and Z a topological space, and let $S : X \longrightarrow 2^Y,\ T : Y \longrightarrow 2^Z$, and F :

$X \longrightarrow 2^Z$ be three multivalued mappings. F is said to be a $\mathfrak{A}$-KKM mapping with respect to T if, for any $\{x_0, \ldots, x_n\} \in \langle X\rangle$, there exists $y_i \in S(x_i)(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, one has

$$T(co\{y_{i_0}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} F(x_{i_j}).$$

The multivalued mapping $T : Y \longrightarrow 2^Z$ is said to have the $\mathfrak{A}$-KKM property, if, for any $\mathfrak{A}$-KKM mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Let the set $\{T : T$ has the $\mathfrak{A}$-KKM property$\}$ be denoted by $\mathfrak{A}$-$KKM(X, Y, Z)$.

Comments: What is the role of $\mathfrak{A}$? Simply F could be closed-valued. Moreover, we can consider the case F is open-valued.

S. Huang [2021] — JNCA22(6)

Let M be a connected Riemannian manifold endowed with a Riemannian metric.

Definition 4.26. Let X and Z be nonempty sets, Y a convex set in M and $T : Y \longrightarrow 2^Z$ a multifunction. A multifunction $S : X \longrightarrow 2^Z$ is a generalized KKM mapping with respect to T if for any $\{x_1, \ldots, x_n\} \in \langle X\rangle$, there is $\{y_1, \ldots, y_n\} \in \langle Y\rangle$ such that

$$T(co\{y_i : i \in I\}) \subset \bigcup\{S(x_i) : i \in I\}, \quad \forall I \subset \{1, \ldots, n\},$$

If Z is a topological space, T is said to have the KKM property if for any generalized KKM mapping $S : X \longrightarrow 2^Z$ with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. The collection of all multifunctions $T : Y \longrightarrow 2^Z$ with KKM property is denoted by KKM(X, Y, Z).

Final Remark for Section 4. We can add up more and more examples on various types of modifications, imitations, or fake generalizations of G-convex spaces. But we will stop here.

5. On the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps

In 1987, W. K. Kim [1987] and Shih and Tan [1987] independently discovered that the KKM theorem also holds for open-valued multimaps; see also Lassonde [1990]. Consequently, all papers in Section 4 and many others can be modified to corresponding open-valued versions.

Moreover, the KKM class of multimaps can be extended to the $\mathfrak{KC}$-class of multimaps, and new $\mathfrak{KO}$-class and $\mathfrak{K}$-class are derived for abstract convex spaces due to Park in the new millennium.

In this section, we introduce articles concerning such new classes and related topics.

Park [1997] — NA30

Chang and Yen [1996] extended the class $\mathfrak{A}_c^\kappa$ to multimap class *KKM* having the KKM property and obtained some generalized results in the KKM theory and fixed point theory. We improve their definition as follows:

Let (X, D) be a convex space, Y a Hausdorff space, and $T : X \multimap Y$. We say that T has *the KKM property* provided that the family $\{Sx : x \in D\}$ has the finite intersection property whenever $S : D \multimap Y$ has closed values and $T(coN) \subset S(N)$

for each $N \in \langle D \rangle$. Let

$$T \in \mathfrak{K}(X,Y) \Longleftrightarrow T : X \multimap Y \text{ has the KKM property.}$$

We will denote their class by $\mathfrak{K}$.

Let X be a convex space and Y a Hausdorff space. In this paper, we define a new "better¡± admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \longrightarrow P$, $f(F|_P)$ has a fixed point.

Our new class contains the admissible class $\mathfrak{A}_c^\kappa$ due to the author and generalizes closed maps in KKM due to Chang and Yen [1996].

Comments: In this paper, we first used the notations $\mathfrak{K}$ and $\mathfrak{B}$. Later we changed their meanings. Note that our $\mathfrak{K}$ seems to be better than the KKM of Chang and Yen [1996].

Park [1997a] — MSR Hot-Line 1(9)

We give general Schauder type fixed point theorems for compact multimaps in the 'better' admissible class $\mathfrak{B}$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Our new theorems subsume a large number of particular forms, and generalize them in terms of the involving spaces and the multimaps as well. We apply our new results to condensing maps.

Park [1998] — JKMS35(4)

We give general fixed point theorems for compact multimaps in the 'better' admissible class $\mathfrak{B}^\kappa$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Those theorems are used to obtain results for Φ-condensing maps. Our new theorems subsume more than seventy known or possible particular forms, and generalize them in terms of the involving spaces and the multimaps as well. Further topics closely related to our new theorems are discussed and some related problems are given in the last section.

In 1998, we obtained the following:

Theorem 5.1. *Let E be a Hausdorff t.v.s. and X an admissible (in the sense of Klee) convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In [1998], it was shown that Theorem subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

Park [2003]— JNCA4

Let $(X,D;\Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have the *KKM property* if, for any map $G : D \multimap Y$ with closed [open] values satisfying

$$F(\Gamma_A) \subset G(A) \text{ for all } A \in \langle D \rangle,$$

Some authors use the notation KKM(X,Y). Note that $1_X \in \mathfrak{K}(X,X)$. $\cdots$

From now on, $\mathfrak{KC}$ denote the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G.

Park [2006] — NAF11

We introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. We study the nature of Kakutani type maps, $\mathfrak{B}$-maps, and $\mathfrak{KC}$-maps in G-convex spaces; and show that generalizations of the key results in known works are consequences of the G-convex space theory and the new abstract convex space theory.

Park [2007] — NAF12

We study the mutual relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ on abstract or generalized convex spaces. We show also that the examples given by Jeng, Huang, and Zhang [2002] can be used to deduce more examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps. Finally, some historical remarks on classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ are added.

Park [2007a] — NAF12(2)

Our principal aim is to introduce basic results in the KKM theory on abstract convex spaces and the map class $\mathfrak{K}$ as in Park [2008b]. These are applied to simplify various modifications of the concept of generalized convex spaces. We discuss the nature of these modifications and criticize recently appeared so-called generalizations of our previous works due to other authors.

In Section 2, we introduce our new abstract convex spaces, KKM maps, and the map class $\mathfrak{KC}$ [or $\mathfrak{KO}$] in [2008b], and, in Section 3, a few basic theorems in our KKM theory for those spaces given there. Section 4 deals with KKM type theorems for G-convex spaces, which are shown to be easily deduced from our new results on abstract convex spaces. Sections 5-8 are devoted to various modifications of G-convex spaces and KKM type maps appeared in the 21st century. We show that most of them are mere modifications without having any proper example or any applicability. Such modifications are, for examples, L-spaces, generalized R-KKM maps, pseudo H-spaces, and others.

Park [2008b] — JKMS45(1)

Definition 5.1. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \text{ for all} A \in \langle D \rangle$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{K}$-*map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

Park [2010] — Tamkang41(1)

Abstract: Recent results in the KKM theory on abstract convex spaces and the related multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are applied to deduce generalizations of results on KKM maps in metric spaces in Amini et al. [2007] and generalized KKM theorems on hyperconvex metric spaces in Chang et al. in [2008, 2008a].

Yang, Huang, and Lee [2011] — TJM15(1)

Park [2006] introduced a new concept of abstract convex spaces which include convex subsets of topological vector spaces, convex spaces, C-spaces and G-convex spaces as special cases. Park [2006] also introduced certain broad classes $\mathfrak{RO}$ and $\mathfrak{RC}$ of maps (having the KKM property). The class $\mathfrak{RC}(X,Y)$ includes the well-known class $\mathrm{KKM}(X,Y)$ introduced by Chang and Yen [1996] as a special case. With these new concepts, he obtained some coincidence theorems and fixed point theorems in abstract convex spaces. Very recently, Park [2008a-d] further studied KKM theory in abstract convex spaces with applications to fixed points, maximal elements, equilibria problems and other problems.

Comments: $\mathfrak{R}$ should be $\mathfrak{K}$.

Yang and Huang [2012] — BKMS49(6)

A coincidence theorem for a compact $\mathfrak{RC}$-map is proved in an abstract convex space. Several more general coincidence theorems for noncompact $\mathfrak{RC}$-maps are derived in abstract convex spaces. Some examples are given to illustrate our coincidence theorems. As applications, an alternative theorem concerning the existence of maximal elements, an alternative theorem concerning equilibrium problems and a minimax inequality for three functions are proved in abstract convex spaces."

Recently, Park [2006] introduced a new concept of abstract convex spaces which include convex subsets of topological vector spaces, convex spaces, C-spaces and G-convex spaces as special cases. Park [2006] also introduced certain broad classes $\mathfrak{RO}$ and $\mathfrak{RC}$ of maps (having the KKM property), which includes the well-known class $KKM(X,Y)$ introduced by Chang and Yen [1996] as a special case. With these new concepts, some coincidence theorems and fixed point. theorems were proved in abstract convex spaces by Park [2006]. Very recently, Park [2008a-d] further studied KKM theory in abstract convex spaces with applications to fixed points, maximal elements, equilibria problems and other problems. It is noted that, in the KKM theory, there have appeared a number of coincidence theorems with many significant applications.

Lu and Hu [2013] — JFSA2013

The main purpose of this paper is to establish a new collectively fixed point theorem in noncompact abstract convex spaces. As applications of this theorem, we obtain some new existence theorems of equilibria for generalized abstract economies in noncompact abstract convex spaces.

Comments: The authors followed our works faithfully, but $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ are denoted as $\mathfrak{R}$, $\mathfrak{RC}$, $\mathfrak{RO}$.

Lu, Zhang, and Li [2021] — AIMS Math.6(11)

Abstract: In this paper, by using the KKM theory and the properties of Γ-convexity and $\mathfrak{RC}$-mapping, we investigate the existence of collectively fixed points for a family with a finite number of set-valued mappings on the product space of noncompact abstract convex spaces.

6. Conclusion

In this paper, we followed the original sources faithfully.

Some one said that the progress of mathematics often follows a standard path: the discovery of a new theorem, followed by a systematic exploration of that theorem.

(1) Two standard ways of exploring theorems are by weakening the hypotheses and strengthening the conclusion. Here new hypothesis should be carefully and properly chosen.

(2) Another way is to find similar situation or similar hypotheses to known results and follow the same conclusion. Here we see certain parallelism.

Note that most of papers mentioned in Section 4 are of such types (1) or (2) and variants of the same ones for G-convex spaces.

The papers listed in Section 5 are relatively new and mainly related abstract convex spaces. However some of their authors misused $\mathfrak{R}$ instead of traditional $\mathfrak{K}$.

This survey is to help the authors to improve their works for the current KKM theory.

References

1. R. P. Agarwal, M. Balaj, and D. O'Regan, 2009. *Common fixed point theorems and minimax inequalities in locally convex Hausdorff topological vector spaces*, Applicable Anal. **88**(12): 1691– 1699.
2. R. P. Agarwal and D. O'Regan, 2002. *Fixed points for admissible multimaps*, Dynamic Systems Appl. **21**: 437–448.
3. M. A. Al-Thagafi and N. Shahzad, 2008. *Best proximity sets and equilibrium pairs for a finite family of multimaps*, Fixed Point Theory Appl. vol. 2008, Article ID 457069, 10pp.
4. A. Amini, M. Fakhar, and J. Zafarani, 2005. *KKM mappings in metric spaces*, Nonlinear Anal. **60**: 1045–1052.
5. A. Amini, M. Fakhar, and J. Zafarani, 2007. *Fixed point theorems for the class S-KKM mappings in abstract convex spaces*, Nonlinear Anal. **66**: 14–21.
6. A. Amini-Harandi, A. P. Farajzadeh, D. O'Regan, and R. P. Agarwal, 2009. *Fixed point theorems for condensing multimaps on abstract convex uniform spaces*, Nonlinear Funct. Anal. Appl. **14**(1): 109–120.
7. M. Balaj, 2008. *Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities*, Nonlinear Anal. **68**: 3962–3971.
8. M. Balaj and L. Coroianu, 2011. *Matching theorems and simultaneous relation problems*, Bull. Korean Math. Soc. **48**(5): 939–949.
9. P. Chaipunya and P. Kumam, 2013. *Topological aspects of circular metric spaces and some observations on the KKM property towards quasi-equilibrium problems*, J. Inequal. Appl. 2013: 343.
10. T.-H. Chang, C.-M. Chen, and C.-H. Chen, 2010. *Generalized 2_gKKM property in a hyperconvex metric space and its applications*, Nonlinear Anal. **72**: 50–56.
11. T.-H. Chang, C.-M. Chen, and H.-C. Huang, 2008. *Generalized KKM theorems on hyperconvex metric spaces with applications*, Taiwan. J. Math. **12**(9): 2363–2372.
12. T.-H. Chang, C.-M. Chen, and C.-Y. Peng, 2008a. *Generalized KKM theorems on hyperconvex metric spaces and some applications,* Nonlinear Anal. **69**: 530–535.
13. T.-H. Chang, Y.-Y. Huang, J.-C. Jeng, K.-H. Kuo, 1999. *On the S-KKM property and related topics*, J. Math. Anal. Appl. **229**: 212–227.
14. T.-H. Chang and C.-L. Yen, 1996. *KKM property and fixed point theorems,* J. Math. Anal. Appl. **203**: 224–235.
15. C.-M. Chen, 2006. *KKM property and fixed point theorems in metric spaces,* J. Math. Anal. Appl. **323**: 1231–1237.
16. , 2007. *R-KKM theorems on L-convex spaces and its applications,* Sci. Math. Jpn. **65**: 195–207.
17. C.-M. Chen, 2009. *Fixed point theorems for the Ψ-set contraction mapping on almost convex sets*, Nonlinear Anal. **71**: 2600–2605.
18. C.-M. Chen and T.-H. Chang, 2007. *Some results for the family KKM(X,Y) and the Φ-mapping*, J. Math. Anal. Appl. **329**: 92–101.
19. C.-M. Chen and T.-H. Chang, 2008. *Some results for the family $2-_g KKM(X,Y)$ and the Φ-mapping in hyperconvex metric spaces*, Nonlinear Anal. **69**: 2533–2540.
20. C.-M. Chen and T.-H. Chang, 2009. *Fixed point theorems for a weaker Meir-Keeler type Ψ-set contraction in metric spaces*, Fixed Point Theory Appl. 2009, Article ID 129124, 8pp.
21. C.-M. Chen, T.-H. Chang, and C.-W. Chung, 2009. *Coincidence theorems on nonconvex sets and its applications*, Taiwan. J. Math. **13**(2A): 501–513.

22. C.-M. Chen, T.-H. Chang, and Y.-H. Huang, 2010. *Approximate fixed point theorems for the generalized Ψ-set contraction mappings on an almost Φ-space*, Appl. Math. Lett. **23**: 152–155.
23. C.-M. Chen, T.-H. Chang, and C.-L. Yen, 2004. *Fixed point theory for k-set contraction*, J. Korean Math. Soc. **41**(2): 243–248.
24. Y. J. Cho, M. R. Delavar, S. A. Mohammadzadeh, and M. Roohi, 2011. *Coincidence theorems and minimax inequalities in abstract convex spaces*, J. Inequal. Appl. 2011:126
25. M. Fakhar, M. Lotfipour, and J. Zafarani, 2013. *On the Brezis Nirenberg Stampacchia-type theorems and their applications*, J. Glob. Optim. **55**: 751–770.
26. M. Fakhar and J. Zafarani, 2005. *Generalized vector equilibrium problems for pseudomonotone multivalued bifunctions,* J. Optim. Theory Appl. **126**(1): 109–124.
27. , 2005a. *Fixed points theorems and quasi-variational inequalities in G-convex spaces*, Bull. Belg. Math. Soc. **12**: 235–247.
28. K. Fan, 1961. *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142**: 305–310.
29. C. D. Horvath, 1987. *Some results on multivalued mappings and inequalities without convexity*, Nonlinear and Convex Analysis — Proc. in honor of Ky Fan (B. L. Lin and S. Simons, eds.), 99–106, Marcel Dekker, New York.
30. C. D. Horvath, 1990. *Convexité généralisée et applications*, Sém. Math. Supér. **110**: 81–99, Press. Univ. Montréal.
31. , 1991. *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156**: 341–357.
32. C. D. Horvath, 1993. *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2**: 253–269.
33. S. Huang, 2021. *Section theorems in Hadamard manifolds*, J. Nonlinear Convex Anal. **22**(6): 1189–1203.
34. J.-C. Jeng, H.-C. Hsu, and Y.-Y. Huang, 2006. *Fixed point theorems for multifunctions having KKM property on almost convex sets*, J. Math. Anal. Appl. **319**: 187–198.
35. J.-C. Jeng, Y.-Y. Huang, and H.-L. Zhang, 2002. *Characterization of maps having the KKM property*, Soochow J. Math. **28**: 329–338.
36. M.A. Khamsi, 1996. *KKM and Ky Fan theorems in hyperconvex metric spaces*, J. Math. Anal. Appl. **206**: 298–306.
37. M. A. Khamsi and N. Hussain, 2010. *KKM mappings in metric type spaces*, Nonlinear Anal. **73**: 3123–3129.
38. H. Kim, 2005. *KKM property, S-KKM property and fixed point theorems*, Nonlinear Anal. **63**: e1877–e1884.
39. H. Kim and S. Park, 2005. *Remarks on the KKM property for open-valued multimaps on generalized convex spaces,* J. Korean Math. Soc. **42**: 101–110.
40. W. K. Kim, 1987. *Some applications of the Kakutani fixed point theorem,* J. Math. Anal. Appl. **121**: 119–122.
41. B. Knaster, K. Kuratowski, und S. Mazurkiewicz, 1929 *Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe*, Fund. Math. 14: 132–137.
42. T.-Y. Kuo, Y.-Y. Huang, J.-C. Jeng, and C.-Y. Shih, 2006. *Coincidence and fixed point theorems for functions in S-KKM class on generalized convex spaces*, Fixed Point Theory Appl. vol. 2006, Article ID 72184, Pages 1–9.
43. M. Lassonde, 1983. *On the use of KKM multifunctions in fixed point theory and related topics,* J. Math. Anal. Appl. 97: 151–201.
44. M. Lassonde, 1990. *Sur le principe KKM,* C. R. Acad. Sci. Paris **310**: 573–576.
45. L. J. Lin, Q. H. Ansari, and J. Y. Wu, 2003. *Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems*, J. Optim. Theory Appl. **117**(1): 121–137.
46. L. J. Lin and T. H. Chang, 1998. *S-KKM theorems, saddle points and minimax inequalities*, Nonlinear Anal. **34**: 73–86.
47. L. J. Lin, C. J. Ko, and S. Park, 1998. *Coincidence theorems for set-valued maps with G-KKM property on generalized convex space*, Discuss. Math. Differential Incl. 18: 69–85.
48. H. Lu and Q. Hu, 2013. *A collectively fixed point theorem in abstract convex spaces and its applications*, J. Function Spaces Appl. vol. 2013, Article ID 517469, 10pp.
49. H. Lu, K. Zhang, and R. Li, 2021. *Collectively fixed point theorems in noncompact abstract convex spaces with applications*, AIMS Mathematics **6**(11): 12422–12459.
50. S. Park, 1989. *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. 141: 164–176.
51. S. Park, 1993. *Fixed point theory of multifunctions in topological vector spaces*, II, J. Korean Math. Soc. **30**: 413–431.

52. S. Park, 1994. *Foundations of the KKM theory via coincidences of composites of upper semicon- tinuous maps*, J. Korean Math. Soc. **31**: 493–519.
53. S. Park, 1997. *Coincidence theorems for the better admissible multimaps and their applications*, Nonlinear Anal. **30**: 4183–4191.
54. S. Park, 1997a. *Fixed points of the better admissible multimaps*, Math. Sci. Res. Hot-Line **1**(9): 1–6.
55. S. Park, 1998. *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35**: 803–829. *Corrections*, ibid. **36** (1999) 829–832.
56. S. Park, 2000. *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. Appl. Math. **7**: 1–28.
    S. Park S. Park, 2000a. *Remarks on topologies of generalized convex spaces*, Nonlinear Func. Anal. Appl. **5**: 67–79.
57. S. Park, 2003. *Coincidence, almost fixed point, and minimax theorems on generalized convex spaces*, J. Nonlinear Convex Anal. **4**: 151–164.
58. S. Park, 2006. *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11**: 67–77.
59. S. Park, 2006a. *Comments on some fixed point theorems on generalized convex spaces*, Fixed Point Theory and Applications (Y. J. Cho et al., eds.), vol. 6, Nova Sci. Publ., Huntington, NY.
60. S. Park, 2006b. *Remarks on concepts of generalized convex spaces*, Fixed Point Theory and Appli- cations (Y. J. Cho et al., eds.), vol. 7, Nova Sci. Publ., Huntington, NY.
61. S. Park, 2007. *Remarks on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps in abstract convex spaces*, Nonlinear Anal. Forum **12**(1): 29–40.
62. S. Park, 2007a. *Comments on some abstract convex spaces and the KKM spaces*, Nonlinear Anal. Forum **12**(2): 125–139.
63. S. Park, 2008a. *Comments on recent studies on abstract convex spaces*, Nonlinear Anal. Forum **13**(1): 1–17.
64. S. Park, 2008b. *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1): 1–27.
65. S. Park, 2008c. *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. **69**(12): 4352–4364.
66. S. Park, 2008d. *Remarks on fixed points, maximal elements, and equilibria of economies in abstract convex spaces*, Taiwanese J. Math. **12**(6): 1365–1383.
67. S. Park, 2010. *Comments on the KKM theory on hyperconvex metric spaces*, Tamkang J. Math. **41**(1): 1–14.
68. S. Park, 2017. *A history of the KKM Theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **56**(2): 1–51.
69. S. Park, 2018. *On various multimap classes in the KKM theory and their applications*, Adv. Theory Nonlinear Anal. Appl. **2**: 88–105.
70. S. Park, 2021. *Applications of the KKM family of multimaps*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **60**(2): 327–371.
71. S. Park and H. Kim, 1993. *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci., Seoul Nat. Univ. **18**: 1–21.
72. S. Park and H. Kim, 1996. *Coincidence theorems on admissible maps on generalized convex spaces*, J. Math. Anal. Appl. **197**: 173–187.
73. S. Park and H. Kim, 1997. *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209**: 551–571.
74. S. Park and H. Kim, 2005. *Remarks on the KKM property for open-valued multimaps on gener- alized convex spaces*, J. Korean Math. Soc. **42**: 101–110.
75. S. Park and W. Lee, 2001. *A unified approach to generalized KKM maps in generalized convex spaces,* J. Nonlinear Convex Anal. **2**: 157–166.
76. N. Shahzad, 2004. *Fixed point and approximation results for multimaps in S-KKM class*, Nonlinear Anal. **56**: 905–918.
77. N. Shahzad, 2006. *Approximation and Leray-Schauder type results for multmimaps in the S-KKM class,* Bull. Belg. Math. Soc. **13**: 113–121.
78. M.-H. Shih and K.-K. Tan, 1987. *Covering theorms of convex sets related to fixed-point theorems*, Nonlinear and Complex Analysis (Proc. in Honor of Ky Fan) pp. 235–244, Marcel Dekker, Inc., New York-Basel.
79. B. P. Sortan, 1984. *Introduction to Axiomatic Theory of Convexity*, Kishyneff. [Russian with English summary].

80. G. Tang and Q. Zhang, 2014. *Class* $\mathfrak{A}-KKM(X,Y,Z)$*, general* KKM *type theorems, and their applications in topological vector space*, Abstract Appl. Anal. vol. 2014, Article ID 238191, 10pp.
81. D. Turkoglu, M. Abuloha, and T. Abdeljawad, 2010. *KKM mappings in cone metric spaces and some fixed point theorems*, Nonlinear Anal. **72**: 348–353.
82. M.-G. Yang and N.-J. Huang, 2012. *Coincidence theorems for noncompact* $\mathfrak{RC}$*-maps in abstract convex spaces with applications,* Bull. Korean Math. Soc. **49**(6): 1147–1161.
83. M.-G. Yang, N.-J. Huang, and C. H. Lee, 2011. *Coincidence and maximal element theorems in abstract convex spaces and applications*, Taiwan. J. Math. **15**(1): 13–29.