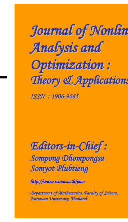

Journal of Nonlinear Analysis and Optimization
Vol. 5, No. 1, (2014), 57-66
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

NONLINEAR ERGODIC THEOREMS FOR WIDELY MORE GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

MAYUMI HOJO[1] AND WATARU TAKAHASHI[*,2,3]

[1] Shibaura Institute of Technology, Tokyo 135-8548, Japan
[2] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan;
Keio Research and Education Center for Natural Sciences, Keio University, Japan
[3]Department of Mathematical and Computing Sciences, Tokyo Institute of Technology,
Tokyo 152-8552, Japan

ABSTRACT. In this paper, using strongly asymptotically invariant nets, we first obtain some properties of widely more generalized hybrid mappings in a Hilbert space. Then, using the idea of mean convergence by Shimizu and Takahashi [24, 25], we prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. This generalizes the Kawasaki and Takahashi nonlinear ergodic theorem.

KEYWORDS : Banach limit; generalized hybrid mapping; Hilbert space; nonexpansive mapping; nonspreading mapping; strongly asymptotically invariant net
AMS Subject Classification:47H10

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. For a mapping $T : C \longrightarrow C$, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \longrightarrow C$ is called nonexpansive [28] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \longrightarrow C$ is called nonspreading [19], hybrid [29] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2,$$

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively; see also [12] and [33]. These three mappings are independent and they are deduced from a firmly nonexpansive mapping in a Hilbert space; see [29]. A mapping $F : C \longrightarrow H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

[*]The second author is partially supported by Grant-in-Aid for Scientific Research No.23540188 from Japan Society for the Promotion of Science.

[*] *Corresponding author.*
Email address : mayumi-h@shibaura-it.ac.jp(M. Hojo); wataru@is.titech.ac.jp(W. Takahashi) .

Article history : Received 29 October 2013, Accepted 25 March 2014.

for all $x, y \in C$; see, for instance, Goebel and Kirk [7]. The class of nonspreading mappings was first defined in a strictly convex, smooth and reflexive Banach space. The resolvents of a maximal monotone operator are nonspreading mappings; see [19] for more details. These three classes of nonlinear mappings are important in the study of the geometry of infinite dimensional spaces. Indeed, by using the fact that the resolvents of a maximal monotone operator are nonspreading mappings, Takahashi, Yao and Kohsaka [34] solved an open problem which is related to Ray's theorem [23] in the geometry of Banach spaces. Motivated by these mappings, Kocourek, Takahashi and Yao [17] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings and hybrid mappings. A mapping $T : C \longrightarrow C$ is said to be *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such a mapping an (α, β)-generalized hybrid mapping. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [18] and Iemoto and Takahashi [12]. Moreover, they proved the following nonlinear ergodic theorem which generalizes Baillon's theorem [2].

Theorem 1.1 ([17]). *Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H, let T be a generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Very recently Kawasaki and Takahashi [15] introduced a class of nonlinear mappings in a Hilbert space which covers contractive mappings [3] and generalized hybrid mappings. A mapping $T : C \longrightarrow C$ is called *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$; see also Kawasaki and Takahashi [14]. A mapping $T : C \longrightarrow C$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [15], they extended the nonlinear ergodic theorem of [17] to widely more generalized hybrid mappings.

In this paper, using strongly asymptotically invariant nets, we first obtain some properties of widely more generalized hybrid mappings in a Hilbert space. Then, using the idea of mean convergence by Shimizu and Takahashi [24, 25], we prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. This generalizes the Kawasaki and Takahashi nonlinear ergodic theorem.

2. PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known [28] that for any $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \leq 2\langle y - x, y \rangle, \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.3}$$

for any $x, y, z, w \in H$. Let C be a nonempty subset of H. It is well-known that if C is closed and convex and $T : C \longrightarrow C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Ito and Takahashi [13]. For a simpler proof of such a result in a Hilbert space, see, for example, [16]. Let D be a nonempty, closed and convex subset of H and $x \in H$. Then we know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the metric projection of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for any $x \in H$ and $u \in D$; see [28] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [31].

Lemma 2.1. *Let D be a nonempty, closed and convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$ and $x_n \longrightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties. For $f \in \ell^\infty$, define $\ell_1 : \ell^\infty \longrightarrow \ell^\infty$ as follows:

$$\ell_1 f(k) = f(1 + k), \quad \forall k \in \mathbb{N} \cup \{0\}.$$

A net $\{\mu_\alpha\}$ of means on ℓ^∞ is said to be *strongly asymptotically invariant* if

$$\|\mu_\alpha - \ell_1^* \mu_\alpha\| \longrightarrow 0,$$

where ℓ_1^* is the adjoint operator of ℓ_1. See [6] for more details. The following definition which was introduced by Takahashi [26] is crucial in the fixed point theory. Let h be a bounded function of $\mathbb{N} \cup \{0\}$ into H. Then, for any mean μ on ℓ^∞, there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z\rangle = (\mu)_k \langle h(k), z\rangle, \quad \forall z \in H.$$

Such a h_μ is contained in $\overline{\text{co}}\{h(k)\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A. In particular, let T be a mapping of a subset C of a Hilbert space H into itself such that $\{T^k x\}$ is bounded for some $x \in C$. Putting $h(k) = T^k x$ for all $k \in \mathbb{N} \cup \{0\}$, we have that there exists $z_0 \in H$ such tat

$$\mu_k \langle T^k x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such z_0 by $S_\mu x$.

From Kawasaki and Takahashi [15], we also know the following fixed point theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 2.1 ([15]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0\end{aligned}$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\alpha + \beta + \zeta + \eta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) *and* (2).

3. NONLINEAR ERGODIC THEOREMS

In this section, using the technique developed by Takahashi [26], we prove a mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following three lemmas. The following lemma was proved by Kawasaki and Takahashi [15].

Lemma 3.1. *Let H be a real Hilbert space, let C be a closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\zeta + \eta \ge 0$ *and* $\alpha + \beta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\varepsilon + \eta \ge 0$ *and* $\alpha + \gamma > 0$.

Then T is quasi-nonexpansive.

The following two lemmas are crucial in the proof of our main theorem.

Lemma 3.2. *Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \ge 0$ *and* $\alpha + \beta + \zeta + \eta \ge 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$, $\zeta + \eta \ge 0$ *and* $\alpha + \gamma + \varepsilon + \eta \ge 0$.

Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on ℓ^∞. For any $x \in C$, define $S_{\mu_\nu} x$ as follows:

$$\langle S_{\mu_\nu} x, y\rangle = (\mu_\nu)_k \langle T^k x, y\rangle, \quad \forall y \in H.$$

Then $\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0$*. In addition, if* C *is bounded, then*

$$\lim_\nu \sup_{x\in C} \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0.$$

Proof. Let $x \in C$. Since $F(T)$ is nonempty and $T : C \longrightarrow C$ is quasi-nonexpansive from Lemma 3.1, we obtain that

$$\|T^{n+1}x - y\| \le \|T^n x - y\|$$

for any $n \in \mathbb{N}\cup\{0\}$ and $y \in F(T)$. Then $\{T^n x\}$ is bounded. Furthermore, we have that for any $x \in C$ and $y \in F(T)$

$$\begin{aligned}
\|S_{\mu_\nu}x - y\|^2 &= \langle S_{\mu_\nu}x - y, S_{\mu_\nu}x - y\rangle \\
&= (\mu_\nu)_k \langle T^k x - y, S_{\mu_\nu}x - y\rangle \\
&\le \|\mu_\nu\| \sup_k |\langle T^k x - y, S_{\mu_\nu}x - y\rangle| \\
&\le \sup_k \|T^k x - y\| \cdot \|S_{\mu_\nu}x - y\| \\
&\le \sup_k \|x - y\| \cdot \|S_{\mu_\nu}x - y\| \\
&= \|x - y\| \cdot \|S_{\mu_\nu}x - y\|
\end{aligned}$$

and hence

$$\|S_{\mu_\nu}x - y\| \le \|x - y\|. \tag{3.1}$$

Therefore, $\{S_{\mu_\nu}x\}$ is also bounded. Since T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself, we obtain that

$$\begin{aligned}
&\alpha\|Tz - T^{k+1}x\|^2 + \beta\|z - T^{k+1}x\|^2 + \gamma\|Tz - T^k x\|^2 + \delta\|z - T^k x\|^2 \\
&\quad +\varepsilon\|z - Tz\|^2 + \zeta\|T^k x - T^{k+1}x\|^2 + \eta\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \le 0
\end{aligned}$$

for all $k \in \mathbb{N}\cup\{0\}$ and $z \in C$. By (2.3) we obtain that

$$\begin{aligned}
&\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 - 2\langle z - Tz, T^k x - T^{k+1}x\rangle \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 + \|z - T^k x\|^2 + \|Tz - T^{k+1}x\|^2 \\
&\qquad -\|z - T^{k+1}x\|^2 - \|Tz - T^k x\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&(\alpha+\eta)\|Tz - T^{k+1}x\|^2 + (\beta-\eta)\|z - T^{k+1}x\|^2 + (\gamma-\eta)\|Tz - T^k x\|^2 \\
&\quad +(\delta+\eta)\|z - T^k x\|^2 + (\varepsilon+\eta)\|z - Tz\|^2 + (\zeta+\eta)\|T^k x - T^{k+1}x\|^2 \le 0.
\end{aligned}$$

From

$$\begin{aligned}
&(\gamma-\eta)\|Tz - T^k x\|^2 = (\gamma+\alpha)\|Tz - T^k x\|^2 - (\alpha+\eta)\|Tz - T^k x\|^2 \\
&\quad = (\alpha+\gamma)(\|z - Tz\|^2 + \|z - T^k x\|^2 - 2\langle z - Tz, z - T^k x\rangle) \\
&\qquad -(\alpha+\eta)\|Tz - T^k x\|^2,
\end{aligned}$$

we have that

$$\begin{aligned}
&(\alpha+\eta)\|Tz - T^{k+1}x\|^2 + (\beta-\eta)\|z - T^{k+1}x\|^2 \\
&\quad +(\alpha+\gamma)(\|z - Tz\|^2 + \|z - T^k x\|^2 - 2\langle z - Tz, z - T^k x\rangle) \\
&\quad -(\alpha+\eta)\|Tz - T^k x\|^2 + (\delta+\eta)\|z - T^k x\|^2 \\
&\quad +(\varepsilon+\eta)\|z - Tz\|^2 + (\zeta+\eta)\|T^k x - T^{k+1}x\|^2 \le 0
\end{aligned}$$

and hence

$$\begin{aligned}(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)+(\beta-\eta)\|z-T^{k+1}x\|^2\\ -2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\delta+\eta)\|z-T^kx\|^2\\ +(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

By $\alpha+\beta+\gamma+\delta\ge 0$, we have that

$$-(\beta-\eta)=-(\beta+\delta)+\delta+\eta\le\alpha+\gamma+\delta+\eta.$$

From this inequality and $\zeta+\eta\ge 0$ we obtain that

$$\begin{aligned}(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\ +(\beta-\eta)(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\ -2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

for any $k\in\mathbb{N}\cup\{0\}$ and $z\in C$. We apply μ_ν to both sides of this inequality. We have that

$$\begin{aligned}(\alpha+\eta)(\mu_\nu)_k(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\ +(\beta-\eta)(\mu_\nu)_k(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\ -2(\alpha+\gamma)(\mu_\nu)_k\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

and hence

$$\begin{aligned}-|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|Tz-T^kx\|^2\\ -|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|z-T^kx\|^2\\ -2(\alpha+\gamma)\langle z-Tz,z-S_{\mu_\nu}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned} \tag{3.2}$$

Replacing z by $S_{\mu_\nu}x$ in (3.2), we have that

$$\begin{aligned}-2(\alpha+\gamma)\langle S_{\mu_\nu}x-TS_{\mu_\nu}x,S_{\mu_\nu}x-S_{\mu_\nu}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\\ \le|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|TS_{\mu_\nu}x-T^kx\|^2\\ +|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|S_{\mu_\nu}x-T^kx\|^2\end{aligned}$$

and hence

$$\begin{aligned}(\alpha+\gamma+\varepsilon+\eta)\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\\ \le|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|TS_{\mu_\nu}x-T^kx\|^2\\ +|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|S_{\mu_\nu}x-T^kx\|^2.\end{aligned}$$

Since $\{TS_{\mu_\nu}x\}$, $\{S_{\mu_\nu}x\}$ and $\{T^nx\}$ are bounded and $\|\mu_\nu-\ell_1^*\mu_\nu\|\longrightarrow 0$, we have that

$$(\alpha+\gamma+\varepsilon+\eta)\limsup_{n\longrightarrow\infty}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\le 0.$$

Since $\alpha+\gamma+\varepsilon+\eta>0$, we have that $\lim_{n\longrightarrow\infty}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|=0$. In addition, if C is bounded, then

$$\limsup_{n\longrightarrow\infty}\sup_{x\in C}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|\le 0$$

and hence $\lim_{n\longrightarrow\infty}\sup_{x\in C}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|=0$.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta\ge 0$, $\alpha+\beta+\zeta+\eta>0$, $\alpha+\gamma>0$ and $\varepsilon+\eta\ge 0$. This completes the proof. □

Lemma 3.3. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

If $x_\nu \rightharpoonup z$ *and* $x_\nu - Tx_\nu \longrightarrow 0$*, then* $z \in F(T)$.

Proof. We give the proof for the case of (2). Since $T : C \longrightarrow C$ is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we have that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned} \tag{3.3}$$

for any $x, y \in C$. Replacing x by x_ν in (3.3), we have that

$$\begin{aligned}\alpha\|Tx_\nu - Ty\|^2 + \beta\|x_\nu - Ty\|^2 + \gamma\|Tx_\nu - y\|^2 + \delta\|x_\nu - y\|^2 \\ + \varepsilon\|x_\nu - Tx_\nu\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned} \tag{3.4}$$

From this inequality, we have that

$$\begin{aligned}&\alpha(\|Tx_\nu - x_\nu\|^2 + \|x_\nu - Ty\|^2 + 2\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle) + \beta\|x_\nu - Ty\|^2 \\ &\quad + \gamma(\|Tx_\nu - x_\nu\|^2 + \|x_\nu - y\|^2 + 2\langle Tx_\nu - x_\nu, x_\nu - y\rangle) + \delta\|x_\nu - y\|^2 \\ &\qquad + \varepsilon\|x_\nu - Tx_\nu\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

From $\|x_\nu - Ty\|^2 = \|x_\nu - y\|^2 + \|y - Ty\|^2 + 2\langle x_\nu - y, y - Ty\rangle$, we also have

$$\begin{aligned}&(\alpha + \beta + \gamma + \delta)\|x_\nu - y\|^2 \\ &\quad + (\alpha + \beta + \zeta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle x_\nu - y, y - Ty\rangle \\ &\qquad + (\alpha + \gamma + \varepsilon)\|Tx_\nu - x_\nu\|^2 + 2\alpha\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle \\ &\qquad\quad + 2\gamma\langle Tx_\nu - x_\nu, x_\nu - y\rangle + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

From $\alpha + \beta + \gamma + \delta \geq 0$ we obtain that

$$\begin{aligned}&(\alpha + \beta + \zeta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle x_\nu - y, y - Ty\rangle \\ &\quad + (\alpha + \gamma + \varepsilon)\|Tx_\nu - x_\nu\|^2 + 2\alpha\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle \\ &\qquad + 2\gamma\langle Tx_\nu - x_\nu, x_\nu - y\rangle + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

Since $x_\nu \rightharpoonup z$ and $x_\nu - Tx_\nu \longrightarrow 0$, we have that

$$(\alpha + \beta + \zeta + \eta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle z - y, y - Ty\rangle \leq 0.$$

Putting $y = z$, we have that

$$(\alpha + \beta + \zeta + \eta)\|z - Tz\|^2 \leq 0.$$

Since $\alpha + \beta + \zeta + \eta > 0$, we have that $z \in F(T)$.

Similarly, by replacing the variables x and y in (3.3), we can obtain the desired result for the case when $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$. This completes the proof. □

Now we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that T satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$, $\varepsilon+\eta \geq 0$ *and* $\alpha+\beta+\zeta+\eta>0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta>0$.

Let $\{\mu_\nu\}$ *be a srongly asymptotically invariant net of means on* ℓ^∞ *and let* P *be the metric projection of* H *onto* $F(T)$. *Then for any* $x \in C$, *the net* $\{S_{\mu_\nu}x\}$ *converges weakly to a fixed point* p *of* T, *where* $p = \lim_{n\to\infty} PT^n x$.

Proof. Let $x \in C$. As in the proof of Lemma 3.2, we have that $\{T^n x\}$ is bounded and $\{S_{\mu_\nu}x\}$ is bounded. Therefore, there exist a subnet $\{S_{\mu_{\nu_\omega}}x\}$ of $\{S_{\mu_\nu}x\}$ and $p \in H$ such that $\{S_{\mu_{\nu_\omega}}x\}$ converges weakly to p. Using $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$, $\varepsilon+\eta \geq 0$ and $\alpha+\beta+\zeta+\eta>0$, we have from Lemma 3.2 that

$$\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0. \tag{3.5}$$

We have from Lemma 3.3 that $p \in F(T)$. Since $F(T)$ is closed and convex from Lemma 3.1, the metric projection P from H onto $F(T)$ is well-defined. By Lemma 2.1, there exists $q \in F(T)$ such that $\{PT^n x : n \in \mathbb{N}\}$ converges strongly to q. To complete the proof, we show that $q=p$. Note that the metric projection P satisfies

$$\langle z-Pz, Pz-u\rangle \geq 0$$

for any $z \in H$ and for any $u \in F(T)$; see [27]. Therefore, we have that

$$\langle T^k x - PT^k x, PT^k x - y\rangle \geq 0$$

for any $k \in \mathbb{N}\cup\{0\}$ and $y \in F(T)$. From the properties of the metric projection P and $PT^{n-1}x \in F(T)$, we obtain that

$$\begin{aligned}\|T^k x - PT^k x\| &\leq \|T^k x - PT^{k-1}x\| \\ &\leq \|T^{k-1}x - PT^{k-1}x\|.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\langle T^k x - PT^k x, y-q\rangle &\leq \langle T^k x - PT^k x, PT^k x - q\rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x-Px\| \cdot \|PT^k x - q\|.\end{aligned}$$

We apply μ_ν to both sides of this inequality. Then we obtain that

$$(\mu_\nu)_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\|(\mu_\nu)_k\|PT^k x - q\|. \tag{3.6}$$

Replacing ν by ν_ω in (3.6), we have that

$$(\mu_{\nu_\omega})_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\|(\mu_{\nu_\omega})_k\|PT^k x - q\|.$$

Since $\{\mu_{\nu_\omega}\}$ has a subnet converging a Banach limit λ in the weak* topology, we obtain that

$$\lambda_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\|\lambda_k\|PT^k x - q\|.$$

Since $\{S_{\mu_{\nu_\omega}}x\}$ converges weakly to p, $\{PT^n x\}$ converges strongly to q and λ is a Banach limit, we obtain that

$$\langle p-q, y-q\rangle \leq 0.$$

Putting $y=p$, we obtain $\|p-q\|^2 \leq 0$ and hence $q=p$. This completes the proof.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ and $\alpha+\gamma+\varepsilon+\eta>0$. □

Using Theorem 3.1, we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Kawasaki and Takahashi [14].

Theorem 3.2. *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

Then for any $x \in C$ the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to a fixed point p of T and $p = \lim_{n\to\infty} PT^n x$, where P is the metric projection of H onto $F(T)$.

Proof. For any $f = (x_0, x_1, x_2, \dots) \in \ell^\infty$, define

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on ℓ^∞; see [27, p.78]. Furthermore, we have that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 3.1. □

References

1. K. Aoyama, Y. Kimura, W. Takahasi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., 67 (2007), 2350–2360.
2. J -B. Baillon, Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert, C. R. Acad. Sci. Sér. A–B, 280 (1975), 1511–1514.
3. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., 3 (1922), 133–181.
4. F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, Math. Z, 100 (1967), 201–225.
5. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl., 20 (1967), 197–228.
6. M. M. Day, Amenable semigroup, Illinois J. Math., 1 (1957), 509–544.
7. K. Goebel and W. A. Kirk, Topics in metric fixed point theory,Cambridge University Press, Cambridge, 1990.
8. M. Hojo and W. Takahashi, Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces, Sci. Math. Jpn., 73 (2011), 31–40.
9. M. Hojo, T. Suzuki and W. Takahashi, Fixed point theorems and convergence theorems for generalized hybrid non-self mappings in Hilbert spaces, J. Nonlinear Convex Anal., 14 (2013), 363–376.
10. T. Ibaraki and W. Takahashi, Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications, Taiwanese J. Math., 11 (2007), 929–944.
11. T. Ibaraki and W. Takahashi, Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces, J. Nonlinear Convex Anal., 10 (2009), 21–32.
12. S. Iemoto and W. Takahashi,Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Anal., 71 (2009), 2082–2089.
13. S. Itoh and W. Takahashi, The common fixed point theory of singlevalued mappings and multivalued mappings, Pacific J. Math., 79 (1978), 493–508.
14. T. Kawasaki and W. Takahashi, Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces, J. Nonlinear Convex Anal., 13 (2012), 529–540.
15. T. Kawasaki and W. Takahashi, Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal., 14 (2013), 71–87.

16. Y. Kimura, W. Takahashi and J.-C. Yao, Strong convergence of an iterative scheme by a new type of projection method for a family of quasinonexpansive mappings, J. Optim. Theory Appl., 149 (2011), 239–253.
17. P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, Taiwanese J. Math., 14 (2010), 2497–2511.
18. F. Kohsaka and W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces, SIAM J. Optim., 19 (2008), 824–835.
19. F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. (Basel), 91 (2008), 166–177.
20. Y. Kurokawa and W. Takahashi, Weak and strong convergence theorems for nonlspreading mappings in Hilbert spaces, Nonlinear Anal., 73 (2010), 1562–1568.
21. L.-J. Lin and W. Takahashi, Attractive point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces, Taiwanese J. Math., 16 (2012), 1763–1779.
22. W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953), 506–510.
23. W. O. Ray, The fixed point property and unbounded sets in Hilbert space, Trans. Amer. Math. Soc., 2 (1980), 5531–537.
24. T. Shimizu and W. Takahashi, Strong convergence theorem for asymptotically nonexpansive mappings, Nonlinear Anal., 26 (1996), 265–272.
25. T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl., 211 (1997), 71–83.
26. W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, Proc. Amer. Math. Soc., 81 (1981), 253–256.
27. W. Takahashi, Nonlinear Functional Analysis. Fixed Points Theory and its Applications, Yokohama Publishers,Yokohama, 2000.
28. W. Takahashi, Introduction to Nonlinear and Convex Analysis, Yokohama Publishers,Yokohama, 2009.
29. W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, J. Nonlinear Convex Anal., 11 (2010), 79–88.
30. W. Takahashi and Y. Takeuchi, Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space, J. Nonlinear Convex Anal., 12 (2011), 399–406.
31. W. Takahashi and M. Toyoda,Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl., 118 (2003), 417–428.
32. W. Takahashi, N.-C. Wong and J.-C. Yao, Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal., 13 (2012), 745–757.
33. W. Takahashi and J.-C. Yao, Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces, Taiwanese J. Math., 2 (2011), 457–472.
34. W. Takahashi, J.-C. Yao and F. Kohsaka, The fixed point property and unbounded sets in Banach spaces, Taiwanese J. Math., 14 (2010), 733–742.