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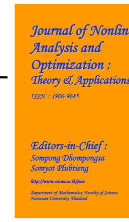

HYBRID FIXED POINT THEOREMS WITH PPF DEPENDENCE IN BANACH ALGEBRAS WITH APPLICATIONS

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ABSTRACT. In this paper, a couple of hybrid fixed point theorems with PPF dependence are proved in a Banach algebra and they are then applied to some nonlinear hybrid functional differential equations of delay and neutral type for proving the existence of PPF dependent solutions under some mixed Lipschitz and compactness type conditions.

KEYWORDS : Fixed point theorem; PPF dependence; Banach space; Functional differential equations.
AMS Subject Classification: 47H10; 34K10

1. INTRODUCTION

In recent papers [2, 7], the authors proved some fundamental fixed point theorems for nonlinear operators in Banach spaces satisfying the condition of linear contraction, wherein the domain and range of the operators are not same. The fixed point theorems of this kind are called PPF dependent fixed point theorems and are useful for proving the existence (and uniqueness) of solutions of nonlinear functional differential and integral equations which may depend upon the past, present and future. The properties of a special minimal or Razumikhin class of functions are employed in the development of existence theory of PPF solutions for certain nonlinear equations in abstract spaces. A study along this line is further continued in Dhage [5, 6], Agarwal *et. al.* [1], Kutbi and Sintunavarat [10] and Sintunavarat and Kumam [12] and proved some PPF dependent fixed point theorems in Banach spaces. In this paper, we prove some hybrid fixed point theorems with PPF dependence in Banach algebras and discuss some of their applications to nonlinear functional hybrid differential equations for proving the existence of PPF dependent solutions.

Given a Banach space E with norm $\|\cdot\|_E$ and given a closed interval $I = [a, b]$ in $\mathbb{R}$, the set of real numbers, let $E_0 = C(I, E)$ be the Banach space of continuous

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E-valued continuous functions defined on I. We equip the space E_0 with the supremum norm $\|\cdot\|_{E_0}$ defined as

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E. \tag{1.1}$$

Let $c \in I$ arbitrarily fixed. The ***minimal*** or ***Razumikhin class*** or $\mathcal{D}$-***class*** functions (cf. [2, 7]) is defined as

$$\mathcal{R}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}. \tag{1.2}$$

A Razumikhin class of functions $\mathcal{R}_c$ is said to be algebraically closed w.r.t. difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, $\mathcal{R}_c$ is topologically closed if it is closed in the topology of of E_0 generated by the norm $\|\cdot\|_{E_0}$. Similarly, other notions such as compactness and connectedness for $\mathcal{R}_c$ may be defined.

Let $T : E_0 \longrightarrow E$. A point $\phi^* \in E_0$ is called a PPF dependent fixed point of T if $T\phi^* = \phi^*(c)$ for some $c \in I$ and any statement that guarantees the existence of PPF dependent fixed point is called a fixed point theorem with PPF dependence for the mapping T .

As mentioned in Bernfield *et al.* [2], the **Razumikhin class** of functions plays a significant role in proving the existence of PPF-fixed points with different domain and range of the operators. Very recently, generalizing a fixed point theorem of Bernfield *et al.* [2], the present author in Dhage [5] proved a first fixed point theorems with PPF dependence in the setting of nonlinear contraction of the operators in Banach spaces.

Definition 1.1. A nonlinear operator $T : E_0 \longrightarrow E$ is called a nonlinear $(\mathcal{B}, \mathcal{W})$-contraction if there exists an upper continuous function from the right $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\|T\phi - T\xi\|_E \le \psi(\|\phi - \xi\|_{E_0}) \tag{1.3}$$

for all $\phi, \xi \in E_0$, where $\psi(r) < r, r > 0$. T is called $\mathcal{B}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is continuous from right and satisfies (1.3). Finally, T is called $\mathcal{M}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that satisfies satisfies (1.3), where $\lim\limits_{n\longrightarrow\infty} \psi^n(t) = 0$, $t > 0$. We say T is nonlinear contraction if it is either a nonlinear $(\mathcal{D}, \mathcal{W})$ or $\mathcal{B}$ or $\mathcal{M}$-contraction on E_0 into E.

Note that every contraction is a nonlinear $(\mathcal{D}, \mathcal{W})$-contraction and every nonlinear $\mathcal{B}$-contraction is $\mathcal{M}$-contraction. However, the converse of the above statements may not be true. The details of different types of contractions appear in the monographs of Krasnoselskii [9], Browder [4], Boyd and Wong [3], Granas and Dugundji [8] and Mathowski [11]. The following fixed point theorem is a slight generalization of a fixed point theorem proved in Dhage [5] with PPF dependence.

Theorem 1.1. *Suppose that $T : E_0 \longrightarrow E$ is a nonlinear contraction. Then the following statements hold in E_0.*

(a) *If $\mathcal{R}_c$ is algebraically closed with respect to difference, then every sequence $\{\phi_n\}$ of successive iterates of T at each point $\phi_0 \in E_0$ converges to a PPF dependent fixed point of T.*
(b) *If $\mathcal{R}_c$ is topologically closed, then ϕ^* is the only fixed point of T in $\mathcal{R}_c$.*

In this paper, we prove two hybrid fixed point theorems with PPF dependence in a Banach algebra and apply them to hybrid differential equations of functional differential equations of delay and neutral type for proving the existence of solutions with PPF dependence.

2. PPF DEPENDENT HYBRID FIXED POINT THEORY

Throughout subsequent part of this paper, unless otherwise specified, let E denote a Banach algebra with norm $\|\cdot\|_E$. Then $E_0 = C(I,E)$ becomes a Banach algebra with respect to the norm (1.1) and the multiplication " $\cdot$ " defined by

$$(x\cdot y)(t) = x(t)\cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E_0$. When there is no confusion, we simply write xy instead of $x \cdot y$.

While working on fixed point theorems in abstract algebras, the present author introduced a class of $\mathcal{D}$-functions to define the growth of the operators in question. We mention that $\mathcal{D}$-functions are in line with the the growth functions mentioned in definition 1.1 and are useful in practical applications to nonlinear differential equations. Here also we employ same notations and terminologies in what follows.

Definition 2.1. A mapping $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathcal{D}$-function** if it is continuous and nondecreasing function satisfying $\psi(0) = 0$. A mapping $Q : E_0 \longrightarrow E$ is called **strong $\mathcal{D}$-Lipschitz** if there is a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$\|Q\phi - Q\xi\|_E \leq \psi(\|\phi(c) - \xi(c)\|_E) \tag{2.1}$$

for all $\phi, \xi \in E$. The function ψ is called a $\mathcal{D}$-function of Q on E. If $\psi(r) = k\,r$, $k > 0$, then Q is called **strong Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **strong contraction** on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called **strong nonlinear $\mathcal{D}$-contraction** and the function ψ is called $\mathcal{D}$-function of Q on X.

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
&\psi(r) = k\,r, \quad \text{for some constant } k > 0,\\
&\psi(r) = \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
&\psi(r) = \tan^{-1} r,\\
&\psi(r) = \log(1 + r),\\
&\psi(r) = e^r - 1.
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations.

Remark 2.2. If $\phi, \psi\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$.

Another notion that we need in the sequel is the following definition.

Definition 2.3. An operator Q on a Banach space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Our main hybrid fixed point theorem with PPF dependence is the following result in a Banach algebra E.

Theorem 2.1. *Let $A, C : E_0 \longrightarrow E$ and $B : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_A,*
(b) *C is strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_C,*

(c) *B is continuous and compact, and*

(d) $M\psi_A(r) + \psi_C(r) < r$ *if* $r > 0$, *where* $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation

$$A\phi\, B\phi(c) + C\phi = \phi(c) \tag{2.2}$$

has a PPF dependent solution.

Proof. Let $\xi \in E_0$ be fixed and let $c \in [a, b]$ be given. Define an operator $T_{\xi(c)} : E_0 \longrightarrow E$ by

$$T_{\xi(c)}(\phi) = A\phi\, B\xi(c) + C\phi. \tag{2.3}$$

Clearly, $T_{\xi(c)}$ is a strong nonlinear $\mathcal{B}$-contraction on E_0. To see this, let $\phi_1, \phi_2 \in E_0$. Then,

$$\begin{aligned}\|T_{\xi(c)}(\phi_1) - T_{\xi(c)}(\phi_2)\|_E &\le \|A\phi_1 - A\phi_2\|_E\, \|B\xi(c)\|_E + \|C\phi_1 - C\phi_2\|_E \\ &\le \|B(E)\|_E\, \psi_A(\|\phi_1(c) - \phi_2(c)\|_E) + \psi_C(\|\phi_1(c) - \phi_2(c)\|_E) \\ &\le M\, \psi_A(\|\phi_1(c) - \phi_2(c)\|_E) + \psi_C(\|\phi_1(c) - \phi_2(c)\|_E).\end{aligned}$$

This shows that $T_{\xi(c)}$ is a strong nonlinear $\mathcal{D}$-contraction and hence nonlinear $\mathcal{D}$-contraction on E_0. By Theorem 1.1, there is a unique PPF dependent fixed point $\phi^* \in E_0$ such that

$$T_{\xi(c)}(\phi^*) = \phi^*(c) \quad \text{or} \quad A\phi^*\, B\xi(c) + C\phi^* = \phi^*(c).$$

Next, we define a mapping $Q : E \longrightarrow E$ by

$$Q\xi(c) = \phi^*(c) = A\phi^*\, B\xi(c) + C\phi^*. \tag{2.4}$$

It then follows that

$$\begin{aligned}\|Q\xi_1(c) - Q\xi_2(c)\|_E &= \|A\phi_1^*\, B\xi_1(c) - A\phi_2^*\, B\xi_2(c)\|_E \\ &\qquad + \|C\phi_1^* - C\phi_2^*\|_E \\ &\le \|A\phi_1^* - A\phi_2^*\|_E\, \|B\xi_1\|_E + \|A\phi_2\|_E\, \|B\xi_1(c) - B\xi_2(c)\|_E \\ &\qquad + \|C\phi_1^* - C\phi_2^*\|_E \\ &\le M\, \psi_A(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + k\, \|B\xi_1(c) - B\xi_2(c)\|_E \\ &\qquad + \psi_C(\|\phi_1^*(c) - \phi_2^*(c)\|_E) \\ &\le M\, \psi_A(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + \psi_C(\|\phi_1^*(c) - \phi_2^*(c)\|_E) \\ &\qquad + k\, \|B\xi_1(c) - B\xi_2(c)\|_E \end{aligned} \tag{2.5}$$

where k is a bound of A on E_0.

Since B is compact, if $\{B\xi_n(c)\}$ is any sequence in E, then $\{B\xi_n(c)\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{B\xi_n(c)\}$ is convergent. Hence, $\{B\xi_n(c)\}$ is a Cauchy sequence. From inequality (2.5), we obtain

$$\begin{aligned}\|Q\xi_m(c) - Q\xi_n(c)\|_E &\le M\, \psi(\|\phi_m^*(c) - \phi_n^*(c)\|_E) + \psi_C(\|\phi_m^*(c) - \phi_n^*(c)\|_E) \\ &\qquad + k\, \|B\xi_m(c) - B\xi_n(c)\|_E.\end{aligned}$$

Taking the limit superior in above inequality yields

$$\begin{aligned}&\limsup_{m,n \longrightarrow \infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E \\ &\qquad \le M \limsup_{m,n \longrightarrow \infty} \psi_A(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E)\end{aligned}$$

$$+ \limsup_{m,n\longrightarrow\infty} \psi_C(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E)$$
$$+ k \limsup_{m,n\longrightarrow\infty} \|B\xi_m(c) - B\xi_n(c)\|_E$$
$$\leq M\,\psi_A\left(\limsup_{m,n\longrightarrow\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right)$$
$$+ \psi_C\left(\limsup_{m,n\longrightarrow\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right).$$

Hence,

$$\lim_{m,n\longrightarrow\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E = 0.$$

As a result, $\{Q\xi_n(c)\}$ is a Cauchy sequence. Since E is complete, $\{Q\xi_n(c)\}$ has a convergent subsequence. Now a direct application of Schauder fixed point principle yields that there is a point $\xi \in E_0$ such that $Q\xi^*(c) = \xi^*(c)$. Consequently $A\xi^*\, B\xi^*(c) + C\xi^* = \xi^*(c)$. This completes the proof. □

Theorem 2.2. *Let $A : E_0 \longrightarrow E$ and $B, C : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_A,*
(b) *B is continuous and compact,*
(c) *C is continuous and compact, and*
(d) *$M\psi_A(r) < r$ if $r > 0$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation

$$A\phi\, B\phi(c) + C\phi(c) = \phi(c) \tag{2.6}$$

has a PPF dependent solution.

Proof. The proof is similar to Theorem 2.2 with appropriate modifications. □

Remark 2.4. If we consider Theorems 2.1 and 2.1 in a closed, convex and bounded subset of the Banach space E, then condition of the boundedness of the operator A is not required because in that case the boundedness of A follows immediately from the strong Lipschitz condition.

Remark 2.5. If we take $\psi_{\mathcal{A}}(r) = \frac{L_1 r}{K+r}$ and $\psi_{\mathcal{C}}(r) = L_2\, r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\frac{L_1 M}{K+r} + L_2 < 1$ for each real number $r > 0$. Similarly, if $\psi_{\mathcal{A}}(r) = L_1\, r$, and $\psi_{\mathcal{C}}(r) = \frac{L_2 r}{K+r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $M\, L_1 + \frac{L_2 M}{K+r} < 1$ for each real number $r > 0$.

In view of above remark, we obtain the following results as special cases of Theorems 2.1 and 2.2 as corollaries.

Corollary 2.3. *Let $A, C : E_0 \longrightarrow E$ and $B : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong Lipschitz with the Lipschitz constant L_1, and*
(b) *C is strong Lipschitz with the Lipschitz with the Lipschitz constant L_2,*
(c) *B is continuous and compact, and*
(d) *$ML_1 + L_2 < 1$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation (2.2) has a PPF dependent solution.

Corollary 2.4. *Let $A : E_0 \longrightarrow E$ and $B, C : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong Lipschitz with the Lipschitz constant L_1,*
(b) *B is continuous and compact,*
(c) *C is continuous and compact, and*
(d) *$ML_1 < 1$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation (2.6) has a PPF dependent solution.

3. APPLICATIONS

In this section, we apply the abstract result of the previous section to functional differential equations for proving the existence of solutions under a weaker Lipschitz condition. Given a closed interval $I_0 = [-r, 0]$ in $\mathbb{R}$ for some real number $r > 0$, let $\mathcal{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\phi\|_{\mathcal{C}} = \sup_{\theta \in I_0} |\phi(\theta)|. \tag{3.1}$$

It is clear that $\mathcal{C}$ is a Banach space with this norm called the history space of the problem under consideration.

Given the closed and bounded interval $J = [-r, T]$ in $\mathbb{R}$, let $C(J, \mathbb{R})$ denote the Banach space of continuous and real-valued functions defined on J with the usual supremum norm $\|\cdot\|$. Given a function $x \in C(J, \mathbb{R})$, for each $t \in I = [0, T]$, define a function $t \longrightarrow x_t \in \mathcal{C}$ by

$$x_t(\theta) = x(t + \theta),\ \theta \in I_0, \tag{3.2}$$

where the argument θ represents the delay in the argument of solutions.

Now we are equipped with the necessary details to study the nonlinear problems of functional differential equations.

3.1. **FUNCTIONAL DIFFERENTIAL EQUATION OF DELAY TYPE.** Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid differential equation of functional differential equations of delay type (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right] &= g(t, x_t), \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.3}$$

for all $t \in I$, where $f : I \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $g : I \times \mathcal{C} \longrightarrow \mathbb{R}$ are continuous.

By a solution x of the HDE (3.3) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{x - k(t, x)}{f(t, x)}$ is continuous in I for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (3.3) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

We consider the following hypotheses in what follows.

(H_1) There exist real numbers $L > 0$ and $K > 0$ such that

$$|g(t, x) - g(t, y)| \leq \frac{L|x(0) - y(0)|}{K + |x(0) - y(0)}$$

for all $t \in I$ and $x, y \in \mathcal{C}$.

(H_2) The function f is uniformly continuous and there exists a real number $M_f > 0$ such that
$$0 < |f(t,x)| \leq M_f$$
for all $t \in I$ and $x \in \mathbb{R}$.

(H_3) The function k is uniformly continuous and there exists a real number $M_k > 0$ such that
$$|k(t,x)| \leq M_k$$
for all $t \in I$ and $x \in \mathbb{R}$.

Remark 3.1. If $L < K$ in hypothesis (H_1), then it reduces to the usual Lipschitz condition of g, namely,
$$|g(t,x) - g(t,y)| \leq (L/K)|x(0) - y(0)|,$$
for all $t \in I$ and $x, y \in \mathcal{C}$.

Theorem 3.1. *Assume that the hypotheses (H_1) through (H_3) hold. Furthermore, if $LT \max\{M_f, 1\} \leq K$, then the HDE (3.3) has a solution defined on J.*

Proof. Set $E = C(J, \mathbb{R})$. Then E is a Banach algebra with respect to the usual supremum norm $\|\cdot\|_E$ and the multiplication " $\cdot$ " defined by
$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$
for all $t \in I$, whenever $x, y \in E$.

Define a set of functions
$$\widehat{E} = \{\hat{x} = (x_t)_{t\in I} : x_t \in \mathcal{C}, x \in C(I, \mathbb{R}) \text{ and } x_0 = \phi\}. \tag{3.4}$$
Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by
$$\|\hat{x}\|_{\widehat{E}} = \sup_{t\in I} \|x_t\|_{\mathcal{C}}. \tag{3.5}$$

Clearly, $\hat{x} \in C(I_0, \mathbb{R}) = \mathcal{C}$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. Then, $\{(x_t^n)_{t\in I}\}$ is a Cauchy sequence in $\mathcal{C}$ for each $t \in I$. This further implies that $\{x_t^n(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^n(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is Banach space.

Now the HDE (3.3) is equivalent to the nonlinear hybrid integral equation (in short HIE)
$$x(t) = \begin{cases} k(t, x(t)) + \big[f(t, x(t))\big] \left(\dfrac{\phi(0)}{f(0, \phi(0))} + \displaystyle\int_0^t g(s, x_s)\, ds \right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.6}$$

Consider three operators $A : \widehat{E} \longrightarrow \mathbb{R}$, $B : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ and $C : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by
$$A\hat{x} = A(x_t)_{t\in I} = \begin{cases} \dfrac{\phi(0)}{f(0, \phi(0))} + \displaystyle\int_0^t g(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.7}$$
$$Bx(t) = \begin{cases} f(t, x(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.8}$$

and

$$Cx(t) = \begin{cases} k(t, x(t)), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.9}$$

Then the HIE (3.6) is equivalent to the operator equation

$$A\hat{x}\, B\hat{x}(0) + C\hat{x}(0) = \hat{x}(0). \tag{3.10}$$

We shall show that the operators A, B and C satisfy all the conditions of Theorem 2.2. First we show that A is a bounded operator on $\widehat{E}$ into E. Now for any $\hat{x} \in \widehat{E}$, one has

$$\begin{aligned} \|A\hat{x}\|_E &\leq \|A0\|_E + \|A(x_t)_{t\in I} - A0\|_E \\ &\leq \|A0\|_E + \left| \int_0^t g(s, x_s)\, ds - \int_0^t g(s, 0)\, ds \right| \\ &\leq \|A0\|_E + \int_0^t \frac{L|x_s(0) - 0|}{K + |x_t(0) - 0|}\, ds \\ &\leq \|A0\|_E + \int_0^t \frac{L\|\hat{x}(0)\|_E}{K + \|\hat{x}(0)\|_E}\, ds \\ &\leq \|A0\|_E + LT \end{aligned}$$

which shows that A is a bounded operator on $\widehat{E}$ with bound $\|A0\|_E + LT$.

Next, we prove that A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$. Then,

$$\begin{aligned} \|A\hat{x} - A\hat{y}\|_E &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\ &= \left| \int_0^t g(s, x_s)\, ds - \int_0^t g(s, y_s)\, ds \right| \\ &\leq \int_0^t \frac{L|x_s(0) - y_s(0)|}{K + |x_s(0) - y_s(0)|}\, ds \\ &\leq \int_0^t \frac{L\|\hat{x}(0) - \hat{y}(0)\|_E}{K + \|\hat{x}(0) - \hat{y}(0)\|_E}\, ds \\ &= \psi_A(\|\hat{x}(0) - \hat{y}(0)\|_E) \end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_A(r) = \dfrac{LTr}{K + r}$. Hence, A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_A.

Next, we show that B is compact and continuous operator on $C(J, \mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J, \mathbb{R})$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Then by continuity of f,,

$$\lim_{n\longrightarrow\infty} Bx_n(t) = \lim_{n\longrightarrow\infty} f(s, x_n(s)) = f(s, x(s)) = Bx(t)$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim\limits_{n\longrightarrow\infty} Bx_n(t) = 1 = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \longrightarrow Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \leq |f(s, x(s))| \leq M_f$$

for all $t \in J$, and $|Bx(t)| \leq 1$ for all $t \in I_0$. From this it follows that

$$|Bx(t)| \leq \max\{M_f\, ,\, 1\} = M^*$$

for all $t \in J$, whence B is uniformly bounded on E.

To show equi-continuity, let $t, \tau \in I$. Then, from the uniform continuity of f it follows that

$$|Bx(t) - Bx(\tau)| \leq |f(t, x(t)) - f(\tau, x(\tau))| < \epsilon$$

for all $x \in C(J, \mathbb{R})$. If $\tau \in I_0$ and $t \in I$, then $\tau \longrightarrow 0$ and $t \longrightarrow 0$ whenever, $|\tau - t| \longrightarrow 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \leq |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \longrightarrow 0 \text{ as } t \longrightarrow \tau$$

for all $x \in C(J, \mathbb{R})$. From this, it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzella-Ascoli theorem yields that B is a compact operator on E into itself. Similarly, it can be shown that the operators C is also a compact and continuous operator on E into itself.

Finally,

$$M\psi_A(r) = \frac{LT \max\{M_f, 1\}\, r}{K + r} < r$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Moreover, here the Razumikhin class $\mathcal{R}_0$, $0 \in [-r, T]$ is $C([0, T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.2 yields that integral equation (3.6) has a solution on J with PPF dependence. This further implies that the HDE (3.3) has a PPF dependent solution on J. This completes the proof. □

3.2. **FUNCTIONAL DIFFERENTIAL EQUATION OF NEUTRAL TYPE.** Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid functional differential equation of neutral type (in short HDE)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t, x_t)}{f(t, x_t)}\right] &= g(t, x(t)), \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.11}$$

for all $t \in I$, where $f : I \times \mathcal{C} \longrightarrow \mathbb{R} \setminus \{0\}$, $k : I \times \mathcal{C} \longrightarrow \mathbb{R}$ and $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

By a solution x of the FDE (3.11) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{y - k(t, y)}{f(t, y)}$ is continuous in I for all $y \in \mathcal{C}$, and
(ii) x satisfies the equations in (3.11) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

We consider the following hypotheses in what follows.

(H_4) There exist real numbers $L_1 > 0$ and $K_1 > 0$ such that

$$|f(t, x) - f(t, y)| \leq \frac{L_1 |x(0) - y(0)|}{K_1 + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

(H_5) There exists a real number $M_g > 0$ such that

$$|g(t, x)| \leq M_g$$

for all $t \in I$ and $x \in \mathbb{R}$.

(H_6) There exist real numbers $L_2 > 0$ and $K_2 > 0$ such that

$$|k(t,x) - k(t,y)| \leq \frac{L_2|x(0) - y(0)|}{K_2 + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

Theorem 3.2. *Assume that the hypotheses (H_4) through (H_6) hold. Furthermore, if*

$$L_1\left[\|\phi\|_{\mathcal{C}} + M_g T\right] + L_2 \leq \min\{K_1, K_2\},$$

then the HDE (3.11) has a solution defined on J.

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a Banach algebra with respect to the norm and the multiplication as defined in the proof of Theorem 3.1. Define a set of functions $\widehat{E}$ by (3.4) which is equipped with the norm $\|\hat{x}\|_{\widehat{E}}$ defined by (3.5) Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathcal{C}$. It can be shown as in Theorem 3.1 that $\widehat{E}$ is Banach space.

Now the HDE (3.10) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} k(t,x_t) + \left[f(t,x_t)\right]\left(\phi(0) + \displaystyle\int_0^t g(s,x(s))\,ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.12}$$

Consider three operators $A, B : \widehat{E} \longrightarrow \mathbb{R}$, $B : C(J,\mathbb{R}) \longrightarrow \mathbb{R}$ and $C : C(J,\mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$A\hat{x} = A(x_t)_{t\in I} = \begin{cases} f(t,x_t), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.13}$$

$$Bx(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t g(s,x(s))\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.14}$$

and

$$C\hat{x} = C(x_t)_{t\in I} = \begin{cases} k(t,x_t), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.15}$$

Then the HIE (3.11) is equivalent to the operator equation

$$A\hat{x}\,B\hat{x}(0) + C\hat{x} = \hat{x}(0). \tag{3.16}$$

We shall show that the operators A, B and C satisfy all the condition of Theorem 2.1. First we show that A is bounded on $\widehat{E}$.

$$\begin{aligned} |A\hat{x}| &\leq |A0| + |A(x_t)_{t\in I} - A0| \\ &\leq |f(t,0)| + |f(t,x_t) - f(s,0)| \\ &\leq F_0 + \frac{L|x_t(0) - 0|}{K + |x_t(0) - 0|} \\ &\leq F_0 + \frac{L\|\hat{x}(0)\|_E}{K + \|\hat{x}(0)\|_E} \\ &= F_0 + L, \end{aligned}$$

for all $\hat{x} \in \widehat{E}$, where $F_0 = \sup_{t\in I}|f(t,0)|$. Hence, A is bounded on $\widehat{E}$ with bound $F_0 + L$.

Next, we show that a strong $\mathcal{B}$-Lipschitz on $\widehat{E}$. Then,

$$\|A\hat{x} - A\hat{y}\|_E = |A(x_t)_{t\in I} - A(y_t)_{t\in I}|$$

$$\begin{aligned}
&= |f(t,x_t) - f(t,y_t)| \\
&\leq \frac{L_1|x_t(0) - y_t(0)|}{K_1 + |x_t(0) - y_t(0)|} \\
&\leq \frac{L_1\|\hat{x}(0) - \hat{y}(0)\|_E}{K_1 + \|\hat{x}(0) - \hat{y}(0)\|_E} \\
&= \psi_A(\|\hat{x}(0) - \hat{y}(0)\|_E)
\end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_A(r) = \dfrac{L_1 r}{K_1 + r}$. Hence, A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_A. Similarly, it can be shown that C is also a A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function $\psi_C(r) = \dfrac{L_2 r}{K_2 + r}$.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} Bx_n(t) &= \phi(0) + \lim_{n\longrightarrow\infty}\int_0^t g(s, x_n(s))\,ds \\
&= \phi(0) + \int_0^t \lim_{n\longrightarrow\infty} g(s, x_n(s))\,ds \\
&= Bx(t)
\end{aligned}$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim\limits_{n\longrightarrow\infty} Bx_n(t) = \phi(t) = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \longrightarrow Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \leq |\phi(0)| + \int_0^t |g(s, x(s))|\,ds \leq \|\phi\|_{\mathcal{C}} + M_g T$$

for all $t \in J$ which shows that $B(E)$ is uniformly bounded set in E. To show equi-continuity, let $t, \tau \in I$. Then,

$$|Bx(t) - Bx(\tau)| \leq \left|\int_\tau^t |g(s, x(s))|\,ds\right| \leq M_g\,|t - \tau|.$$

If $\tau \in I_0$ and $t \in I$, then $\tau \longrightarrow 0$ and $t \longrightarrow 0$ whenever, $|\tau - t| \longrightarrow 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \leq |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \leq M_g\,|t - \tau|.$$

From the above inequalities it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzelá-Ascoli theorem yields that B is a compact operator on E into itself. Finally,

$$\begin{aligned}
M\psi_A(r) + \psi_C(r) &= \frac{L_1\left[\|\phi\|_{\mathcal{C}} + M_g T\right]r}{K_1 + r} + \frac{L_2 r}{K_2 + r} \\
&\leq \frac{\left[L_1\left(\|\phi\|_{\mathcal{C}} + M_g T\right) + L_2\right]r}{\min\{K_1, K_2\} + r} \\
&< r
\end{aligned}$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Again, here the Razumikhin class $\mathcal{R}_0$, $0 \in [-r, T]$ is $C([0, T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.1 yields that the integral equation (3.12) has a solution on J with PPF dependence. This further implies that the HDE (3.11) has a PPF dependent solution defined on J. This completes the proof. □

Remark 3.2. Finally, we conclude this paper with the remark that the functional differential equations considered here are of simple nature, however, other complex nonlinear functional differential equations involving the arguments of past, present and future can also be considered and studied for the existence theorems on similar lines with appropriate modifications.

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