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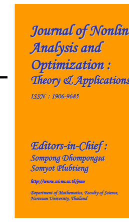

TWO GENERAL FIXED POINT RESULTS ON WEAK PARTIAL METRIC SPACE

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ABSTRACT. In this work, we obtain two fixed point results on weak partial metric space. Our results are extend and generalize some previous results.

KEYWORDS : Fixed point; Partial metric; Weak partial metric.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

The concept of partial metric p on a nonempty set X was introduced by Matthews [8]. One of the most interesting properties of a partial metric is that $p(x, x)$ may not be zero for $x \in X$. Also, each partial metric p on a nonempty set X generates a T_0 topology on X. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then many authors gave some generalizations of this result on this space (See [1, 3, 7, 9, 10, 11, 12]). Recently, Chi, Karapınar and Thanh [4] obtained a fixed point theorem using a new type contractive condition, which is quite different from usual contractive conditions.

On the other hand, Heckman defined the concept of weak partial metric space and viewed some topological properties of it. Then Altun and Durmaz [2] proved the fundamental fixed point theorem on this space. Also, Durmaz et al [5], obtained some generalization of the result of [2]. In this work, we continue to study on fixed point theory in weak partial metric space. For this, we use Chi, Karapınar and Thanh type contractive condition.

2. PRELIMINARIES

In this section, we recall partial metric and weak partial metric space and some properties of them.

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Definition 2.1 ([8]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) such that for all $x, y, z \in X$:

(i) $x = y \iff p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom),
(ii) $p(x,x) \leq p(x,y)$ (small self-distance axiom),
(iii) $p(x,y) = p(y,x)$ (symmetry),
(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Example 2.2. A mapping $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by

$$p(x,y) = \begin{cases} |x-y| & , \quad x, y \in [0,1) \\ \max\{x,y\} & , \quad \text{otherwise} \end{cases}$$

is a partial metric on $\mathbb{R}^+$.

Example 2.3. Let $p : \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ be defined by

$$p(x,y) = \begin{cases} 0 & , \quad x = y \geq 0 \\ 2^{-|x|} & , \quad x \neq 0 \text{ and } y = 0 \\ 2^{-|y|} & , \quad x = 0 \text{ and } y \neq 0 \\ 2^{-\min\{|x|,|y|\}} & , \quad otherwise \end{cases}$$

is a partial metric on $\mathbb{N} \cup \{0\}$.

Example 2.4. Let $P(\mathbb{N})$ be the set all subsets of $\mathbb{N}$. If

$$p(x,y) = 1 - \sum_{n \in x \cap y} 2^{-n}$$

for all $x, y \in P(\mathbb{N})$, then p is a partial metric on $P(\mathbb{N})$.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

and

$$\begin{aligned} p^w(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned}$$

are ordinary metrics on X. It is easy to see that p^s and p^w are equivalent metrics on X. For example, let $X = \mathbb{R}^+$ and $p(x,y) = \max\{x,y\}$, then $p^s(x,y) = |x-y| = p^w(x,y)$.

Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$. Since τ_p may not be Hausdorff, then if there exists the limit of a sequence may not be unique, too.

Remark 2.5. A sequence $\{x_n\}$ in a PMS (X,p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. Indeed, let $\{x_n\}$ converges to $x \in X$, with respect to τ_p, then there for all $\varepsilon > 0$, exists a positive integer n_0 such that $x_n \in B_p(x,\varepsilon)$ for $n \geq n_0$. Therefore, considering the small self distance property we have $p(x,x) \leq p(x_n,x) < p(x,x) + \varepsilon$ for $n \geq n_0$ and so letting limit $n \to \infty$, we have $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. The converse may be shown similarly.

Example 2.6. Let $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$. Define a sequence in X by $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to any point of X.

Definition 2.7. (X, p) is a partial metric space. Then

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.

(ii) (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is well known that, every convergent sequence on an ordinary metric space is Cauchy, but this is not true on partial metric space. For example, let $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$. Define a sequence $\{x_n\}$ by $\{x_n\} = \{0, 1, 0, 1, \cdots\}$, then it converges to any point of $[1, \infty)$, but it is not a Cauchy sequence. Also, we know that an ordinary metric is continuous and so sequentially continuous, but this is not true as shown in Example 2.2 for a partial metric.

The following lemma have an important role in the proof of our main result.

Lemma 2.8. *Assume that $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

According to [8], a sequence $\{x_n\}$ in X converges, with respect to τ_{p^s}, to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

By omitting the small self distance axiom, Heckmann [6] introduced the concept of weak partial metric space (for short WPMS), which is generalized version of Matthews' partial metric space. That is, the function $p : X \times X \to \mathbb{R}^+$ is called weak partial metric on X if it satisfies T_0 seperation axiom, symmetry and modified triangular inequality. Heckmann also shows that, if p is weak partial metric on X, then for all $x, y \in X$ we have the following weak small self-distance property

$$p(x, y) \geq \frac{p(x, x) + p(y, y)}{2}. \tag{2.1}$$

Weak small self-distance property shows that WPMS are not far from small self-distance axiom. It is clear that PMS is a WPMS, but the converse may not be true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \dfrac{x + y}{2}$ for all $x, y \in \mathbb{R}^+$. For another example, for $x, y \in \mathbb{R}$ the function $p(x, y) = \dfrac{e^x + e^y}{2}$ is a non partial metric but weak partial metric on $\mathbb{R}$.

The concepts of convergence of a sequence, Cauchy sequence and completeness in WPMS are defined as in PMS. Following Heckmann, in [2, 5] gave some fundamental fixed point results on weak partial metric space such that:

Theorem 2.1. *([2])Let (X, p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$\begin{aligned} p(Fx, Fy) \quad \leq \quad & ap(x, y) + bp(x, Fx) + cp(y, Fy) + \\ & dp(x, Fy) + ep(y, Fx) \end{aligned}$$

for all $x, y \in X$, where $a, b, c, d, e \geq 0$ and, if $d \geq e$, then $a + b + c + 2d < 1$, if $d < e$, then $a + b + c + 2e < 1$. Then F has a unique fixed point.

Theorem 2.2. *([5])Let (X, p) be a complete WPMS, $\alpha \in [0, 1)$ and $T : X \to X$ a mapping. Suppose that for each $x, y \in X$ the following condition holds:*

$$p(Tx, Ty) \leq \max\{\alpha p(x, y), \min\{p(x, x), p(y, y)\}\}$$

Then:

(1) *the set* $X_p = \{x \in X : p(x,x) = \inf\{p(y,y) : y \in X\}\}$ *is nonempty,*

(2) *there is a unique* $u \in X_p$ *such that* $u = Tu$,

(3) *for each* $x \in X_p$ *the sequence* $\{T^n x\}$ *converges with respect to the metric* p^w *to* u.

3. THE MAIN RESULT

Theorem 3.1. *Let* (X,p) *be a complete weak partial metric space and* $T : X \to X$ *be a mapping such that for all* $x, y \in X$

$$p(Tx,Ty) \leq \max \left\{ \begin{array}{l} ap(x,y), bp(x,Tx), cp(y,Ty), \\ d\{p(x,Ty)+p(y,Tx)\}, \\ \min\{p(x,x),p(y,y)\} \end{array} \right\} \tag{3.1}$$

for some $a,b,c \in [0,1)$ *and* $d \in [0,\frac{1}{2})$. *Then*

(a) $X_p = \{x \in X : p(x,x) = \inf\{p(y,y) : y \in X\}\}$ is nonempty,

(b) There is a unique $u \in X_p$ such that $u = Tu$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the sequence defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. First, we will prove that X_p is nonempty. For this, by taking $x = x_{n-1}$ and $y = x_n$ in (3.1) and then

$$\begin{aligned} p(Tx_{n-1},Tx_n) &\leq \max \left\{ \begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ d\{p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})\}, \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ d\{p(x_{n-1},x_n)+p(x_n,x_{n+1})\}, \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ 2dp(x_{n-1},x_n), 2dp(x_n,x_{n+1}), \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array} \right\}. \end{aligned}$$

We suppose that $\alpha = \max\{a,b,c,2d\}$, then

$$p(x_n,x_{n+1}) \leq \max \left\{ \begin{array}{c} \alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array} \right\} \tag{3.2}$$

So we consider this in two cases:

Case I:

$$\max\{\alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} = \alpha p(x_n,x_{n+1})$$

then we obtain

$$p(x_n,x_{n+1}) \leq \alpha p(x_n,x_{n+1})$$

since $\alpha \in [0,1)$, we say that $p(x_n,x_{n+1}) = 0$ and then $x_n = Tx_n$. Since $p(x_n,x_n) \leq 2p(x_n,x_{n+1})$, we obtain $p(x_n,x_n) = 0$. This implies that X_p is nonempty.

Case II:

$$\max\{\alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} \neq \alpha p(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$, then from (3.2), we obtain

$$\begin{aligned} p(x_n,x_{n+1}) &\leq \max\{\alpha p(x_{n-1},x_n), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} \\ &\leq \max\left\{\alpha p(x_{n-1},x_n), \frac{p(x_{n-1},x_{n-1})+p(x_n,x_n)}{2}\right\} \end{aligned} \tag{3.3}$$

$$\leq \quad p(x_{n-1}, x_n).$$

Hence $\{p(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. It follows that, there exist $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r.$$

If $r = 0$, then $p(x_n, x_n) \leq 2p(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} p(x_n, x_n) = 0$. Now, we consider the case $r > 0$. To do this, we set

$$r_n = \max\{\alpha p(x_{n-1}, x_n), \min\{p(x_{n-1}, x_{n-1}), p(x_n, x_n)\}\}$$

for all $n \in \mathbb{N}$. From (3.3) and $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$ we have $\lim_{n\to\infty} r_n = r$.

We shall prove that $r_n = \alpha p(x_n, x_{n-1})$ for finite n. If $r_n = \alpha p(x_n, x_{n-1})$ for infinitely many n then there exists a sequence $\{n_k\}$ of positive integers such that

$$r_{n_k} = \alpha p(x_{n_k}, x_{n_k-1}).$$

Letting $n_k \to \infty$ we obtain $r = \alpha r$. This is a contradiction with $\alpha \in [0, 1)$ and $r > 0$. Hence $r_n = \alpha p(x_n, x_{n-1})$ for finite n. Combining this fact with the definition of r_n , we can deduce that

$$\lim_{n\to\infty} p(x_n, x_n) = r.$$

Now for each $n = 1, 2, ...$ by $(P4)$ in the definition of weak partial metric space , we have

$$\begin{aligned}\min\{p(x_n, x_n), p(x_{n+2}, x_{n+2})\} &\leq \frac{p(x_n, x_n) + p(x_{n+2}, x_{n+2})}{2} \\ &\leq p(x_n, x_{n+2}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}).\end{aligned}$$

It follows from the above inequalities and

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_n) = r$$

that

$$\lim_{n\to\infty} p(x_n, x_{n+2}) = r.$$

By induction we infer that

$$\lim_{n\to\infty} p(x_n, x_{n+s}) = r$$

for every positive integers s that is equivalent to saying that $\lim_{n,m\to\infty} p(x_n, x_m) = r$. Hence $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete there exist $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$ that is

$$r = p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Let us prove

$$p(u, Tu) \leq p(u, u).$$

For each n, we have

$$\begin{aligned}\min\{p(u, u), p(Tu, Tu)\} &\leq \frac{p(u, u) + p(Tu, Tu)}{2} \\ &\leq p(u, Tu) \\ &\leq p(u, x_n) + p(x_n, Tu) - p(x_n, x_n). \qquad (3.4)\end{aligned}$$

Now we need some computations for $p(x_n, Tu)$. So from (3.1) we have

$$p(Tu, x_n) \quad = \quad p(Tu, Tx_{n-1})$$

$$\begin{aligned}
&\leq \max\left\{\begin{array}{l} ap(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},Tx_{n-1}), \\ d\left\{p(u,Tx_{n-1})+p(x_{n-1},Tu)\right\}, \\ \min\left\{p(u,u),p(x_{n-1},x_{n-1})\right\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), \\ 2dp(u,x_n), 2dp(x_{n-1},Tu), p(u,x_{n-1}) \end{array}\right\} \\
&\leq \max \begin{array}{l} p(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), \\ 2dp(u,x_n), 2dp(x_{n-1},Tu) \end{array} \\
&\leq \max\left\{\begin{array}{l} p(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), 2dp(u,x_n), \\ 2d\left[p(x_{n-1},u)+p(u,Tu)-p(u,u)\right] \end{array}\right\} \\
&\leq \max\left\{p(u,u), bp(u,Tu), 2dp(u,Tu)\right\} \\
&\leq \max\left\{p(u,u), \alpha p(u,Tu)\right\} \qquad (3.5)
\end{aligned}$$

and we get $\{p(Tu,x_n)\}$ is bounded sequence. Thus it has a convergent subsequence $\{p(Tu,x_{n_k})\}$. Taking the limits from (3.5) as $n_k \to \infty$ and we get

$$\lim_{n_k\to\infty} p(Tu,x_{n_k}) \leq \max\left\{p(u,u), \alpha p(u,Tu)\right\}.$$

Also letting $n_k \to \infty$ in (3.4) and combining with the above fact , we have

$$\begin{aligned}
p(u,Tu) &\leq p(u,u) + \max\left\{p(u,u), \alpha p(u,Tu)\right\} - p(u,u) \\
&\leq \max\left\{p(u,u), \alpha p(u,Tu)\right\} \\
&\leq p(u,u).
\end{aligned}$$

Set

$$\rho_p = \inf\left\{p(y,y) : y \in X\right\}$$

For each $k = 1, 2, ...$ we can fix $x^k \in X$ such that

$$p(x^k,x^k) \leq \rho_p + \frac{1}{k}.$$

By what we have proved for each $k = 1, 2, ...$ we can seek u^k such that $T^n x^k \to u^k$ as $n \to \infty$ and

$$p(Tu^k,u^k) \leq p(u^k,u^k) = r_{u^k}.$$

We shall show that

$$\lim_{n,m\to\infty} p(u^n,u^m) = \rho_p.$$

Given $\epsilon > 0$ and put $n_0 := \left[\frac{3}{\varepsilon(1-\alpha)}\right] + 1$. If $k \geq n_0$ then using (3.1) we have

$$\begin{aligned}
\rho_p &\leq p(Tu^k,Tu^k) \\
&\leq \max\left\{\begin{array}{l} ap(u^k,u^k), bp(u^k,Tu^k), cp(u^k,Tu^k), \\ d\left\{p(u^k,Tu^k)+p(u^k,Tu^k)\right\}, \\ \min\left\{p(u^k,u^k),p(u^k,u^k)\right\} \end{array}\right\} \\
&\leq \max\left\{\alpha p(u^k,Tu^k), p(u^k,u^k)\right\} \\
&\leq p(u^k,u^k)
\end{aligned}$$

and so

$$\begin{aligned}
\rho_p &\leq p(Tu^k,Tu^k) \leq p(u^k,u^k) = r_{u^k} \leq p(x^k,x^k) \\
&\leq \rho_p + \frac{1}{k} \leq \rho_p + \frac{1}{n_0} < \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right].
\end{aligned}$$

This implies that

$$U_k \quad : \quad = p(x^k,x^k) - p(Tx^k,Tx^k)$$

$$< \quad \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right] - \rho_p = \left[\frac{\varepsilon(1-\alpha)}{3}\right].$$

Also if $k \geq n_0$ then

$$p(u^k, u^k) = r_{u_k} \leq p(x^k, x^k) < \rho_p + \frac{1}{k} < \rho_p + \frac{1}{n_0}$$

implies that

$$p(u^k, u^k) < \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right].$$

Now, for each $m, n > n_0$, it follows from $p(u^k, Tu^k) \leq p(u^k, u^k)$ for all $k = 1, 2, ...$ that

$$\begin{aligned}
p(u^n, u^m) &\leq p(u^m, Tu^m) + p(Tu^n, u^n) + p(Tu^m, Tu^n) \\
&\quad -p(Tu^m, Tu^m) - p(Tu^n, Tu^n) \\
&\leq p(u^m, u^m) + p(u^n, u^n) + p(Tu^m, Tu^n) \\
&\quad -p(Tu^m, Tu^m) - p(Tu^n, Tu^n) \\
&= U_m + U_n + p(Tu^m, Tu^n) \\
&\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + p(Tu^m, Tu^n). \qquad (3.6)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p(Tu^m, Tu^n) &\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\{p(u^m, Tu^n) + p(u^n, Tu^m)\}, \\ \min\{p(u^m, u^m), p(u^n, u^n)\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\left\{\begin{array}{l} p(u^m, u^n) + p(u^n, Tu^n) - p(u^n, u^n) + \\ p(u^n, u^m) + p(u^m, Tu^m) - p(u^m, u^m) \end{array}\right\}, \\ \min\{p(u^m, u^m), p(u^n, u^n)\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\left\{\begin{array}{l} p(u^m, u^n) + p(u^n, u^n) - p(u^n, u^n) + \\ p(u^n, u^m) + p(u^m, u^m) - p(u^m, u^m) \end{array}\right\}, \\ \min\{p(u^m, u^m), p(u^n, u^n)\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ 2dp(u^m, u^n), \min\{p(u^m, u^m), p(u^n, u^n)\} \end{array}\right\} \\
&\leq \max\{\alpha p(u^m, u^n), p(u^m, u^m), p(u^n, u^n)\}.
\end{aligned}$$

By combining the above inequality with (3.6) we get

$$\begin{aligned}
p(u^n, u^m) &\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + p(Tu^m, Tu^n) \\
&\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + \max\{\alpha p(u^m, u^n), p(u^m, u^m), p(u^n, u^n)\}.
\end{aligned}$$

This implies that

$$p(u^n, u^m) \leq \max\left\{\begin{array}{l} \alpha p(u^m, u^n) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ p(u^m, u^m) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right], p(u^n, u^n) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\}.$$

Thus

$$\rho_p \quad \leq \quad p(u^n, u^m)$$

$$\begin{aligned}
&\leq \max\left\{\begin{array}{l} \frac{2}{3}\varepsilon, p(u^m,u^m)+2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ p(u^n,u^n)+2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} \frac{2}{3}\varepsilon, \rho_p+\left[\frac{\varepsilon(1-\alpha)}{3}\right]+2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ \rho_p+\left[\frac{\varepsilon(1-\alpha)}{3}\right]+2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\} \\
&\leq \max\left\{\frac{2}{3}\varepsilon, \rho_p+(1-\alpha)\varepsilon\right\} \\
&\leq \max\left\{\frac{2}{3}\varepsilon, \rho_p+\varepsilon\right\} \\
&= \rho_p+\varepsilon.
\end{aligned}$$

Therefore $\lim\limits_{n,m\to\infty} p(u^n,u^m)=\rho_p$, hence $\{u^n\}$ is Cauchy sequence. Since (X,p) is complete, there exist $y\in X$ such that $u^n\to y$ as $n\to\infty$ that is

$$p(y,y)=\lim_{n\to\infty}p(u^n,y)=\lim_{n,m\to\infty}p(u^n,u^m)=\rho_p.$$

Hence $y\in X_p$ or $X_p\neq\emptyset$. In this way (a) is proved.

Now if $y\in X_p$ then there exist $u\in X$ such that

$$p(u,Tu)\leq p(u,u)=r_y$$

where

$$u=\lim_{n\to\infty}T^n y.$$

We have

$$\rho_p\leq p(Tu,Tu) \text{ and } \rho_p\leq p(u,u)=p(u,Tu)$$

and

$$\rho_p\leq \frac{p(Tu,Tu)+p(u,u)}{2} \leq p(u,Tu)=p(u,u)=r_y\leq p(y,y)=\rho_p$$

so

$$p(u,u)=p(Tu,u)=p(Tu,Tu)$$

or $u=Tu$. To finish the proof we have to show that if $u,v\in X_p$ are both fixed point of T then $u=v$. Indeed it follows from $Tu=u$,$Tv=v$ and $p(u,u)=p(v,v)=\rho_p$ that

$$\begin{aligned}
p(u,v) &= p(Tu,Tv) \\
&\leq \max\left\{\begin{array}{l} ap(u,v), bp(u,Tu), cp(v,Tv), \\ d\{p(u,Tv)+p(v,Tu)\}, \\ \min\{p(u,u),p(v,v)\} \end{array}\right\} \\
&\leq \max\{\alpha p(u,v), p(u,u), p(v,v)\}.
\end{aligned}$$

This implies that $(1-\alpha)p(u,v)\leq 0$ or $p(u,v)\leq p(u,u)=p(v,v)=\rho_p$. If

$$(1-\alpha)p(u,v)\leq 0$$

then $p(u,v)=0$, that is $u=v$. If

$$p(u,v)\leq p(u,u)=p(v,v)=\rho_p$$

then $p(u,v)=p(u,u)=p(v,v)$ that is $u=v$.

□

Corollary 3.1. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \max \left\{ \begin{array}{l} ap(x, y), bp(x, Tx), cp(y, Ty), \\ d\{p(x, Ty) + p(y, Tx)\} \end{array} \right\}$$

for some $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then X_p is nonempty and there is a unique $u \in X_p$ such that $u = Tu$.

Corollary 3.2. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \max\{ap(x, y), \min\{p(x, x), p(y, y)\}\}$$

for some $a \in [0, 1)$. Then X_p is nonempty and there is a unique $u \in X_p$ such that $u = Tu$.

Corollary 3.3. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq ap(x, y) + bp(x, Tx) + cp(y, Ty) + d\{p(x, Ty) + p(y, Tx)\}$$

for some $a + b + c + d < 1$ and $a, b, c, d \geq 0$. Then T has a unique fixed point.

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References

1. I. Altun, A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, Fixed Point Theory and Applications (2011), Article ID 508730, 10 pp.
2. I. Altun, G. Durmaz, Weak partial metric spaces and some fixed point results, Applied General Topology 13 (2012), 179-191.
3. I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology and its Applications 157 (2010), 2778-2785.
4. K.P. Chi, E. Karapınar, T.D. Thanh, A generalized contraction principle in partial metric space, Mathematical and Computer Modelling 55 (2012), 16731681.
5. G. Durmaz, Ö. Acar, I. Altun, Some fixed point results on weak partial metric spaces, Filomat 27 (2013), 317-326.
6. R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7 (1999), 71-83.
7. D. Ilic, V. Pavlovic, V. Rakocevic, Some new extensions of Banach's contraction principle to partial metric space, Appl. Math. Lett. 24 (2011), 1326-1330.
8. S. G. Matthews, Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. 728 (1994), 183-197.
9. H. K. Nashine, Fixed point results for mapping satisfying (ψ, φ)-weakly contractive condition in ordered partial metric spaces, Math. Slovaca 63 (2013), 871-882.
10. S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istid. Math. Univ. Trieste 36 (2004), 17-26.
11. S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory and Applications (2010), Article ID 493298, 6 pp.
12. O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. General Topology 6 (2005), 229-240.