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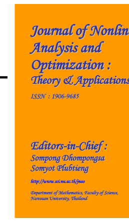

ON GENERALIZED LIPSCHITZIAN MAPPING AND EXPANSIVE LIPSCHITZ CONSTANT

MOHAMMAD IMDAD*, AHMED H. SOLIMAN[1,2], MOHAMED A. BARAKAT[2]

*Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
[2]Department of Mathematics, Faculty of Science, King Khaled University, Abha 9004, Saudi Arabia

ABSTRACT. In this paper, we introduce a new uniformly generalized Lipschitzian type condition for a one-parameter semigroup of self-mappings and utilize the same to show that a uniformly generalized Lipschitzian semigroup of nonlinear self-mappings of a nonempty closed convex subset C of real Banach space X admits a common fixed point provided the semigroup has a bounded orbit and k is appropriately larger than one. Finally, we prove that a semigroup of self mappings $T = \{T(t) : t \in G\}$ defined on a nonempty weakly compact convex subset $\overline{C}$ of a Banach space X with a weak uniform normal structure satisfying $\liminf_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = \lim_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = k < WCS(X)\mu_0$ admits a common fixed point where $\mu_0 = \inf\{\mu \geq 1 : \mu(1-\delta_X(1/\mu)) \geq (1/2)\}$, and $WCS(X)$ is the weak convergent sequence coefficient of X while $\Big| \, ||T(t)|| \, \Big|$ is the exact Lipschitz constant of $T(t)$. Our such result is an extension of the corresponding results due to L.C. Ceng, H. K. Xu and J.C. Yao [5] and L. C. Zeng [29].

1. INTRODUCTION

Let X be a real Banach space equipped with uniform normal structure and C be a nonempty closed convex subset of X. A mapping $T : C \longrightarrow C$ is said to be Lipschitzian if, for each integer $n \geq 1$, there exists a constant $k_n > 0$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \text{ for all } x, y \in C.$$

A Lipschitzian mapping is said to be a k-uniformly Lipschitzian mapping if $k_n \equiv k$ for all $n \geq 1$.

A Banach space X is said to have weak normal structure if every weakly compact convex subset C of X with more than one point contains a nondiametral point, that is, $x_0 \in C$ for which

$$\sup\{||x_0 - y|| : y \in C\} < diam(C).$$

* *Corresponding author.*
Email address : mhimdad@yahoo.co.in (Mohammad Imdad), Ahmed H. Soliman (ahsolimanm@gmail.com), Mohamed A. Barakat (Barakat14285@yahoo.com).
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Every Banach space equipped with normal structure also owns the weak normal structure, but the converse is not true. For reflexive Banach space these properties are equivalent.

Bynum [3] defined weak convergent sequence coefficient of X as the number

$$WCS(X) = \inf\{A(\{x_n\})/\inf\{\limsup_{n\longrightarrow\infty} ||x_n - y|| : y \in \overline{co}\{x_n\}\}\},$$

where the first infimum is taken over all weakly convergent sequences in X while $\overline{co}(A)$ denotes the closure of the convex hull of the subset $A \subset X$, and $A(\{x_n\})$ is the asymptotic diameter of $\{x_n\}$ (i.e., the number $\lim_{n\longrightarrow\infty}(\sup\{\|x_i - x_j\| : i, j \geq n\})$. It is readily seen that $1 \leq WCS(X) \leq 2$. Following [6], we say that the Banach space X has a weak uniform normal structure provided $WCS(X) > 1$.

In 1973, Goebel and Kirk [11] posed the question whether or not the constant $\gamma > 1$ satisfying the equation

$$(1 - \delta_X(1/\gamma))\gamma = 1, \tag{1.1}$$

is the largest number for which any k-uniformly Lipschitzian mapping T with $k < \gamma$ has a fixed point where δ_X denotes the modulus of convexity of X.

In 1975, Lifschits [19] proved that a k-uniformly Lipschitzian mapping defined on a Hilbert space with $k < \sqrt{2}$ has a fixed point.

Casini and Maluta [4] and Ishihara and Takahashi [14] proved that a uniformly $k-$Lipschitzian semigroup of self-mappings defined on a Banach space X has a common fixed point provided $k < \sqrt{N(X)}$ where $N(X)$ denotes the uniform normal structure coefficient.

In 1992 Jimenez-Melado [15] defined the GGLD property for a Banach space X as follows:
X is said to have GGLD provided $D[(x_n)] > 1$ where $\{x_n\}$ is any weakly null sequence such that $\lim_{n\longrightarrow\infty} ||x_n|| = 1$, and $D[(x_n)] = \limsup_{m\longrightarrow\infty} \limsup_{n\longrightarrow\infty} ||x_n - x_m||$; he defined also the coefficient $\beta(X)$ of X by $\beta(X) = \inf\{D[(x_n)] : x_n \longrightarrow 0 \text{ weakly and } ||x_n|| \longrightarrow 1\}$.

Thereafter, k-uniformly Lipschitzian mappings have extensively been investigated by many authors. Moreover, some of results proved for uniformly Lipschitzian mapping have been extended to uniformly Lipschitzian semigroups, and even to Lipschitzian semigroups (e.g. [25]-[35]).

Particularly, in 1993, Tan and Xu [25] answered the earlier mentioned question of Goebel and Kirk [11] in negative by proving the following:

Theorem 1.1. ([25], Theorem 3.5) Let C be a nonempty closed convex subset of a real uniformly convex Banach space X while $\tau = \{T_s : s \in G\}$ be a k-uniformly Lipschitzian semigroup of mappings on C with $k < \alpha$ where $\alpha > 1$ is the unique solution of the equation

$$\frac{\alpha^2}{N(X)}\delta_X^{-1}(1 - \frac{1}{\alpha}) = 1 \tag{1.2}$$

wherein $N(X) > 1$ is the normal structure coefficient of X. If there exists an $x_0 \in C$ such that the orbit $\{T_s x_0 : s \in G\}$ is bounded, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

It is easy to prove that $\gamma < \alpha$, where γ and α are the solution of equations (1.1) and (1.2), respectively. Consequently, the constant γ satisfying equation (1.1) is not the biggest number for which every k-uniformly Lipschitzian mapping T with $k < \gamma$ has a fixed point. Indeed, the best possible number γ is still unknown even in the setting of Hilbert spaces. It is therefore an interesting question to find another constant α^* which is strictly bigger than α and for which every k-uniformly Lipschitzian mapping T with $k < \alpha^*$ has a fixed point.

In 1995, Dominguez Benavides et al. [6] applied a new method to construct a sequence which converges to a fixed point of non-expansive mappings.

Theorem 1.2 (cf. [7]). Suppose X is a Banach space which is not Schur. Then:
(i) X has the GGLD property if and only if $\limsup_{n\longrightarrow\infty} ||x_n - x_\infty|| < A(\{x_n\})$, where $\{x_n\}$ is any weakly (not strongly) convergent sequence in X with limit x_∞.
(ii) $\beta(X) = WCS(X)$.

Some years later, Zeng and Yang [32] proved a fixed point result for Lipschitzian semigroups of mappings as follows:

Theorem 1.3. ([32], Theorem 3.1) Let C be a nonempty bounded subset of a uniformly convex Banach Space X, and let $\tau = \{T_s : s \in G\}$ be a k-uniformly Lipschitzian semigroup on C with

$$\liminf_s ||T_s|| < \sqrt{\gamma_0 N(X)},$$

where

$$\gamma_0 = \inf\{\gamma : \gamma(1 - \delta_X(1/\gamma \geq 1/2)),$$

and $||T_s||$ is the exact Lipschitzian constant of T_s. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:
(P1) $x \in E$ implies $w_w(x) \subset E$; where $w_w(x)$ is the weak w-limit set of τ at x, i.e.,

$$w_w(x) = \{y \in X : y = weak - lim_{t_\alpha} T_{t_\alpha} x \text{ for some subnet } \{t_\alpha\} \subset G\}.$$

(P2) τ is asymptotically regular on E; i.e., $\lim_t ||T_{t+s}x - T_t x|| = 0$, $\forall\ s \in G, x \in E$.
Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Also, Kuczumow [18] proved the following theorem:

Theorem 1.4 (cf. [18]) Let C be a nonempty convex weakly compact separable subset of a Banach space X while $T = \{T(t) : t \in G\}$ be an asymptotically regular semigroup of mappings defined on C such that $\liminf_{G\ni t\longrightarrow\infty} \Big|\ ||T(t)||\ \Big| = k < \sqrt{WCS(X)}$. Then there exists $z \in C$ such that $T(t)z = z\ \ \forall\ \ t \in G$.

Recently, Ceng, Xu and Yao [5] studied the existence of fixed points of uniformly Lipschitzian semigroup $\tau = \{T_s : s \in G\}$ of mappings in the setting of Banach space X under conditions weaker than uniform convexity. More precisely, their improvements can be adjudged twofold:
(1) firstly, authors replaced the uniform convexity of X (in Theorem 1.1) by relatively weaker condition of uniform normal structure of X;
(2) secondly, they removed the asymptotic regularity of the semigroup $\tau = \{T_s : s \in G\}$ on the Banach space X(in Theorem 1.3).
Also, Zeng [29] proved the following fixed point result which extends previously known results due to [7, 8, 13, 18]:

Theorem 1.5 (cf. [29]) Let C be a nonempty convex weakly compact subset of a Banach space X equipped with weak uniform normal structure while $T = \{T(t) : t \in G\}$ be asymptotically regular semigroup of mappings on C such that $\liminf_{G\ni t\longrightarrow\infty} \Big|\ ||T(t)||\ \Big| = k < WCS(X)$.
If each $T(t)$ is weakly continuous, then $F(t)$ is nonempty.

The purpose of this paper, is to extend the result due to Ceng, Xu and Yao [5] by using the new definition of uniformly generalized Lipschitzian type mappings for one-parameter semigroups of self-mappings.

Clearly, it remains a natural question whether Theorem 1.5 is true for the estimate $\liminf_{G\ni t\longrightarrow\infty}\Big|\ ||T(t)||\ \Big| = k < WCS(X)\mu_0$, where $\mu_0 \geq 1$.

The other purpose of this paper is to answer the above question as well.

2. PRELIMINARIES

In what follows, we recall some relevant definitions and results in respect of uniformly generalized Lipschitzian mappings in Banach spaces.

In 2001, Jung and Thakur [16] introduced and studied the following class of mappings.

Definition 2.1(see [16]). A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G1-Lipschitzian)" if

$$||T^n x - T^n y|| \leq a_n||x-y|| + b_n(||x-T^n x|| + ||y-T^n y||) + c_n(||x-T^n y|| + ||y-T^n x||)$$

for each $x, y \in X$ and $n \geq 1$, where a_n, b_n and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n > n_0$. Here it may be pointed out that this class of generalized Lipschitzian mappings is relatively larger than the classes of nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings. The earlier mentioned facts can be realized by choosing constants a_n, b_n and c_n suitably.

On other hand, in 2009, Soliman [24] defined another class of generalized Lipschitzian mappings as follows.

Definition 2.2 (see [24]). A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G2-Lipschitzian)" if for each integer $n \geq 1$, there exists a constant $k_n > 0$ (depending on n) such that

$$||T^n x - T^n y|| \leq k_n \max\Big\{||x-y||, \frac{1}{2}||x-T^n x||, \frac{1}{2}||y-T^n y||\Big\}$$

for every $x, y \in X$. If $k_n = k$ for all $n \geq 1$, then T is called uniformly G2-Lipschitzian.

Now, we will define another class of generalized Lipschitzian mappings as follows.

Definition 2.3. A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G3-Lipschitzian)" if for each integer $n \geq 1$ there exists a constant $k_n > 0$ (depending on n) such that

$$||T^n x - T^n y|| \leq k_n \max\Big\{||x-y||, \frac{1}{2\rho}||x-T^n x||, \frac{1}{2\rho}||y-T^n y||, \frac{1}{2\rho}||x-T^n y||, \frac{1}{2\rho}||y-T^n x||\Big\}$$

for every $x, y \in X$, If $k_n \equiv k$ for all $n \geq 1$, then T is called uniformly G3-Lipschitzian, where $\rho > k,\ \rho > 1$.

Definition 2.4. Let C be a closed convex subset of a Banach space X. Then the collection $\tau = \{T_s : s \in G\}$ of mappings of C into itself is said to be Lipschitzian semigroup on C if the following conditions are satisfied:

(i) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$;
(ii) for each $x \in C$, the mapping $t \longrightarrow T_t x$ from G into C is continuous;
(iii)for each $t \in G$, $T_t : C \longrightarrow C$ is continuous on C;
(iv) for each $t \in G$, there exists a constant $k_t > 0$ such that

$$||T_t x - T_t y|| \leq k_t||x-y|| \quad \text{for all } x, y \in C.$$

In particular, if $k_t \equiv k$ then $\tau = \{T_s : s \in G\}$ is called k-uniformly Lipschitzian semigroup on C.

Definition 2.5. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G1-Lipschitzian semigroup if

$$||T(t)x-T(t)y|| \leq a(t)||x-y||+b(t)(||x-T(t)x||+||y-T(t)y||)+c(t)(||x-T(t)y||+||y-T(t)x||)$$

for each $x, y \in X$, where $a(t), b(t)$ and $c(t)$ are nonnegative constants $b(t) + c(t) < 1$, $\sup\{a(t) : t \in G\} = a < \infty$, $\sup\{b(t) : t \in G\} = b < \infty$, and $\sup\{c(t) : t \in G\} = c < \infty$ with $b + c < 1$.

The simplest uniformly G1-Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \longrightarrow X$ whenever $\sup\{a(t) : t \in G\} = a < \infty$, $\sup\{b(t) : t \in G\} = b < \infty$, and $\sup\{c(t) : t \in G\} = c < \infty$ with $b + c < 1$.

Ahmed H. Soliman [24] introduced the following definition.

Definition 2.6. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G2-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty,$$

where

$$||T(t)x - T(t)y|| \leq k(t) \max\Big\{||x-y||, \frac{1}{2}||x-T(t)x||, \frac{1}{2}||y-T(t)y||\Big\}$$

for each $x, y \in X$ and $\max\Big\{||x-y||, \frac{1}{2}||x-T(t)x||, \frac{1}{2}||y-T(t)y||\Big\} \neq 0$.

Finally, we will introduce the following definition,

Definition 2.7. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G3-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty,$$

where

$$||T(t)x - T(t)y|| \leq k(t)M(x, y)$$

for each $x, y \in X$ and $M(x, y) = \max\Big\{||x-y||, \frac{1}{2\rho}||x-T(t)x||, \frac{1}{2\rho}||y-T(t)y||, \frac{1}{2\rho}||x-T(t)y||, \frac{1}{2\rho}||y-T(t)x||\Big\} \neq 0$.

Remark 2.8. The class of uniformly G3-Lipschitzian semigroups is relatively larger than the other classes namely: uniformly G1-Lipschitzian semigroups, uniformly G2-Lipschitzian semigroups, and also uniformly k-Lipschitzian semigroups.

Recall that X is strictly convex if its unit sphere does not contain any line segments, that is, X is strictly convex if and only if the following implication holds:

$$x, y \in X, \quad ||x|| = ||y|| = 1 \text{ and } ||(x+y)/2|| = 1 \quad \Rightarrow x = y.$$

In order to measure the degree of convexity of X, we define its modulus of convexity $\delta_X : [0, 2] \longrightarrow [0, 1]$ by

$$\delta_X(\varepsilon) = \inf\{1 - ||(x+y)/2|| : ||x|| \leq 1, ||y|| \leq 1 \text{ and } ||x-y|| \geq \varepsilon\}.$$

The characteristic of convexity of X is the number $\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$. It is easy to see [10] that X is uniformly convex iff $\varepsilon_0(X) = 0$; uniformly nonsquare iff $\varepsilon_0(X) < 2$; and strictly convex iff $\delta(2) = 1$. Moreover, if $\varepsilon_0(X) < 1$; then X has a normal structure, that is, each bounded convex subset H of X containing more than one points admits a point x_0 such that $\sup\{||x_0 - x|| : x \in H\} < diam(H)$.

The following properties of modulus of convexity of X are quite well-known (see [12]):
(a) δ_X is increasing on [0,2] and moreover strictly increasing on $[\varepsilon_0, 2]$;
(b) δ_X is continuous on [0,2)(but not necessarily at $\varepsilon = 2$);
(c) $\delta_X(2) = 1$ iff X is strictly convex;
(d) $\delta_X(0) = 0$ and $\lim_{\varepsilon \longrightarrow 2^-} \delta_X(\varepsilon) = 1 - \varepsilon_0/2$

(e) $[||a-x|| \leq r, ||a-y|| \leq r$ and $||x-y|| \geq \varepsilon] \Rightarrow ||a-(x+y)/2|| \leq r(1-\delta_X(\varepsilon/r))$.

Recall that the normal structure coefficient $N(X)$ of X is the number (see [3])

$$\inf\Big\{\frac{diamK}{r_K(K)}\Big\},$$

where the infimum is taken over all bounded closed convex subsets K of X with more than one member, and $r_K(K)$ and $diam(K)$ are Chebyshev radius of K relative to it self and the diameter of K, respectively, i.e., $r_K(K) = \inf_{x\in K}\sup_{y\in K}||x-y||$ and $diamK = \sup_{x,y\in K}||x-y||$. A Banach space X is said to have uniform normal structure if $N(X) > 1$. It is known that a Banach space with uniform normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure (see, e.g., [35]). It is also been computed that $N(H) = \sqrt{2}$ for a Hilbert spaces H. The computations of the normal structure coefficient $N(X)$ for general Banach spaces look however complicated. No exact values of $N(X)$ are known except for some special cases (e.g., Hilbert and L^p spaces). In general, we have the following lower bound for $N(X)$ (see [3, 21, 1])

$$N(X) \geq \frac{1}{1-\delta_X(1)}.$$

Other lower bounds for $N(X)$ in terms of some Banach space parameters or constants can be found in [17, 22].

Tan and Xu [25] have also proved that if X is uniformly convex and $\gamma > 1$ is the unique solution of the equation (1.1), then $N(X) > \gamma$. Note that for a Hilbert space H, we have $N(H) = \sqrt{2}$ and $\gamma = \sqrt{5}/2$.

Suppose X is uniformly convex Banach space. Then it is easily seen that the equation

$$\alpha^2\delta_X^{-1}(1-\frac{1}{\alpha})\widetilde{N}(X) = 1 \tag{2.1}$$

has a unique solution $\alpha > 1$, where $\widetilde{N}(X) = 1/N(X)$. Tan and Xu [25] proved that if $\gamma > 1$ and $\alpha > 1$ are the solution of (1.1) and (2.1), respectively, then $\gamma < \alpha$. Note that $\gamma = \sqrt{5}/2$, and $\alpha = \frac{1}{\sqrt{\sqrt{3}-1}} > \gamma$.

We need the notation of asymptotic centers, due to Edelstein [9]. Let C be a nonempty closed convex subset of a Banach space X and let $\{x_t : t \in G\}$ be a bounded net of elements of X. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t\in G}$ with respect to C are the number

$$r_C\{x_t\} = \inf_{y\in C}\limsup_t ||x_t - y||,$$

and respectively, the (possibly empty) set

$$A_C(\{x_t\}) = \{y \in C : \limsup_t ||x_t - y|| = r_C(\{x_t\}).$$

Lemma 2.1. (cf. [25]) If C is a nonempty closed convex subset of a reflexive Banach space X, then for every bounded net $\{x_t\}_{t\in G}$ of elements of X, $A_C(\{x_t\})$ is a nonempty bounded closed convex subset of C. In particular, if X is a uniformly convex Banach space, then $A_C(\{x_t\})$ consists of a single point.

The following lemma can be proved in exactly the same way as in Lim [20] for sequences and the proof is thus omitted here.

Lemma 2.2.(cf. [25]) Suppose X is a Banach space with uniformly normal structure. Then for every bounded net $\{x_t\}_{t\in G}$ of elements of X there exists $y \in \overline{co}(\{x_t : t \in G\})$ such that

$$\limsup_t ||x_t - y|| \leq \widetilde{N}(X)D(\{x_t\}),$$

where $\widetilde{N}(X) = 1/N(X)$, and $\overline{co}(E)$ is the closure of the convex hull of a set $E \subset X$ and $D(\{x_t\}) = \lim_t(\sup\{||x_i - x_j|| : t \leq i, j \in G\})$ is the asymptotic diameter of $\{x_t\}$.

Lemma 2.3 (cf. [34]). Let X be a Banach space, C be a nonempty weakly compact separable subset of X, and $T = \{T(t) : t \in G\}$ be a semigroup of mappings of C into it self with $k = \liminf_{G \ni t \longrightarrow \infty} \Big| ||T(t)|| \Big| < +\infty$. Then there exists a positive sequence $\{t_n\} \subset G$ such that for each $x \in C$, the sequence $\{T(t_n)x\}$ converges weakly.

3. ON GENERALIZED LIPSCHITZIAN MAPPING

The following lemma plays an important role in proving our results.

Lemma 3.1. If $\{T_s x_0; s \in G\}$ is bounded for some $x_0 \in C$ and $\tau = \{T_s; s \in G\}$ is a k-uniformly generalized Lipschitzian semigroup of mappings on C, then $\{T_s x; s \in G\}$ is bounded for each $x \in C$.

We next present the first result of this paper which weakens the uniform convexity assumption in Theorem 1.1.

Theorem 3.2. Suppose C be a nonempty closed convex subset of a real Banach space X with $N(X) > \max(1, \varepsilon_0)$, while $\tau = \{T_s; s \in G\}$ be a uniformly generalized Lipschitzian (in short G3-Lipschitzian) semigroup of mappings on C with $\rho < \alpha_*$ where ε_0 is the characteristic of convexity of X and

$$\alpha_* = \sup\Big\{\alpha : \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} \leq 1 \text{ and } 4\ 1 - \frac{1}{\alpha} \in (0, 1 - \frac{1}{2}\varepsilon_0)\Big\}. \tag{3.1}$$

If $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Proof. Put $\widetilde{N}(X) = N(X)^{-1}$. Observe that the set

$$\Big\{\alpha : \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} \leq 1 \textit{ and } 1 - \frac{1}{\alpha} \in (0, 1 - \frac{1}{2}\varepsilon_0)\Big\} \neq \phi. \tag{3.2}$$

Indeed, by properties (a),(b),(d) of the modulus δ_x of convexity of X, we see that the mapping

$$\delta_x : [\varepsilon_0, 2) \longrightarrow \delta_x([\varepsilon_0, 2)) = [0, 1 - \frac{1}{2}\varepsilon_0)$$

is strictly increasing and continuous, and hence a bijection. Thus, we deduce that

$$\lim_{\alpha \longrightarrow 1^+} \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} = \delta_X^{-1}(0)\widetilde{N}(X) = \varepsilon_0 \widetilde{N}(X) < 1.$$

which amounts to say that there exists $\alpha_0 > 1$ such that $\alpha_0^2 \delta_X^{-1}(1 - \frac{1}{\alpha_0})\widetilde{N}(X) < 1$ and

$$1 - \frac{1}{\alpha_0} \in \delta_x([\varepsilon_0, 2)) = [0, 1 - \frac{1}{2}\varepsilon_0).$$

This verifies our assertion (5).

Since X has a uniform normal structure and X is reflexive, due to the boundedness of $\{T_s x_0 : s \in G\}$ and Lemma 2.1, we conclude that $A_C(\{T_t x_0\} t \in G)$ is nonempty bounded closed and convex subset of C. Then, we can choose $x_1 \in A_C(\{T_t x_0\}_{t \in G})$ such that

$$\limsup_t ||T_t x_0 - x_1|| = \inf_{y \in C} \limsup_t ||T_t x_0 - y||.$$

Since τ satisfies a $k-$uniformly generalized Lipschitzian property, owing to Lemma 3.1 $T_t x_1$ remains bounded.

Consequently we can choose $x_2 \in A_C(\{T_t x_1\}_{t \in G})$ such that

$$\limsup_t ||T_t x_1 - x_2|| = \inf_{y \in C} \limsup_t ||T_t x_1 - y||.$$

Continuing this process, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in C with the following two properties:

(i) for each $n \geq 0$, $\{T_t x_n\}_{t \in G}$ is bounded;

(ii) for each $n \geq 0$, $x_{n+1} \in A_C(\{T_t x_n\}_{t \in G})$; that is x_{n+1} is a point in C such that

$$\lim_t ||T_t x_n - x_{n+1}|| = \inf_{y \in C} \lim_t ||T_t x_n - y||.$$

Write $r_n = r_C(\{T_t x_n\}_{t\in G})$. Then by Lemma 2.2 we have

$$\begin{aligned} r_n &= \limsup_t ||T_t x_n - x_{n+1}|| \\ &\le \widetilde{N}(X) D(\{T_t x_n\}_{t\in G}) \\ &= \widetilde{N}(X) \lim_t (\sup\{||T_i x_n - T_j x_n|| : t \le i, j \in G\}) \\ &\le \widetilde{N}(X) k \lim_t (\sup \max\{||x_n - T_{j-i} x_n||, \frac{1}{2\rho}||x_n - T_i x_n||, \frac{1}{2\rho}||T_j x_n - T_{j-i} x_n||, \\ &\frac{1}{2\rho}||x_n - T_j x_n||, \frac{1}{2\rho}||T_{j-i} x_n - T_i x_n||\} \\ &\le \widetilde{N}(X) k \lim_t (\sup \max\{d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n)\} \\ &\le \widetilde{N}(X).k.d(x_n), \end{aligned}$$

that is,

$$r_n \le \widetilde{N}(X).k.d(x_n) \le \rho.\widetilde{N}(X).d(x_n). \tag{3.3}$$

where

$$d(x_n) = \sup\{||x_n - T_t x_n|| : t \in G\}.$$

We may assume that $d(x_n) > 0$ for all $n \ge 0$ (otherwise x_n is a common fixed point of the semigroup τ and the proof is over). Let $n \ge 0$ be fixed and let $\varepsilon > 0$ be small enough. We can choose $j \in G$ such that

$$||T_j x_{n+1} - x_{n+1}|| > d(x_{n+1}) - \varepsilon$$

and then choose $s_0 \in G$ so large that

$$||T_s x_n - x_{n+1}|| < r_n + \varepsilon \le \rho(r_n + \varepsilon)$$

for all $s \ge s_0$. Now, for $s \ge s_0 + j$,

$$\begin{aligned} ||T_s x_n - T_j x_{n+1}|| &\le k \max\{||T_{s-j} x_n - x_{n+1}||, \frac{1}{2\rho}||T_s x_n - T_{s-j} x_n||, \frac{1}{2\rho}||x_{n+1} - T_j x_{n+1}||, \\ &\frac{1}{2\rho}||T_{s-j} x_n - T_j x_{n+1}||, \frac{1}{2\rho}||x_{n+1} - T_s x_n||\} \\ &\le k \max\{||T_{s-j} x_n - x_{n+1}||, \frac{1}{2\rho}||T_s x_n - T_{s-j} x_n||, \frac{1}{2\rho} \limsup_t ||x_{n+1} - T_{j+t} x_n||, \\ &\frac{1}{2\rho} \limsup_t ||T_{s-j} x_n - T_{j+t} x_n||, \frac{1}{2\rho}||x_{n+1} - T_s x_n||\} \\ &\le k \max\{r_n + \varepsilon, \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon), \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon)\} \end{aligned}$$

so that

$$||T_s x_n - T_j x_{n+1}|| \le k(r_n + \varepsilon) \le \rho(r_n + \varepsilon).$$

Then owing to property (e), it follows that

$$||T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})|| \le \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big)$$

for $s \ge s_0 + j$ and hence

$$r_n \le \limsup_s ||T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})|| \le \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big).$$

Taking the limit as $\varepsilon \longrightarrow 0$, we obtain

$$r_n \le \rho r_n \Big(1 - \delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big)\Big)$$

which implies that

$$\delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big) \le 1 - \frac{1}{\rho} \tag{3.4}$$

or

$$d(x_{n+1}) \leq \rho r_n \delta_X^{-1}(1-\frac{1}{\rho}). \tag{3.5}$$

Indeed, if $d(x_{n+1})/(\rho r_n) \in [0, \varepsilon_0)$, then noticing that $\delta_X : [\varepsilon_0, 2) \longrightarrow [0, 1-\varepsilon_0/2)$ is a bijection and that $1-\frac{1}{\rho}$ lies in $[0, 1-\varepsilon_0/2)$. By assumption $k < \rho < \alpha_*$, we have $\delta_X^{-1}(1-\frac{1}{\rho}) \geq \varepsilon_0$; hence $d(x_{n+1})/(\rho r_n) \leq \delta_X^{-1}(1-\frac{1}{\rho})$ and (3.5) follows. If $d(x_{n+1})/(\rho r_n) \in [\varepsilon_0, 2]$, then it is clear that $d(x_{n+1})/(\rho r_n) \leq \delta_X^{-1}(1-\frac{1}{\rho})$. This also shows that (3.5) is true. Therefore, utilizing (3.3) and (3.5), we obtain

$$d(x_{n+1}) \leq \rho^2 \widetilde{N}(X) \delta_X^{-1}(1-\frac{1}{\rho}) d(x_n). \tag{3.6}$$

Write $A = \rho^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\rho})$. Then $A < 1$. Indeed, from the assumption that $\rho < \alpha_*$ it follows that there exists an $\tilde{\alpha} > \rho$ such that

$$\tilde{\alpha}^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\tilde{\alpha}}) \leq 1 \text{ and } (1-\frac{1}{\tilde{\alpha}}) \in \delta_X((\varepsilon_0, 2)).$$

It then turns out that $\delta_X^{-1}(1-\frac{1}{\rho}) < \delta_X^{-1}(1-\frac{1}{\tilde{\alpha}})$, and

$$A = \rho^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\rho}) < \tilde{\alpha}^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\tilde{\alpha}}) \leq 1.$$

Hence, it is follows from (3.6) that

$$d(x_n) \leq A d(x_{n-1}) \leq ... \leq A^n d(x_0). \tag{3.7}$$

Since

$$\begin{aligned} ||x_{n+1} - x_n|| \quad & \leq \limsup_t ||T_t x_n - x_{n+1}|| + \limsup_t ||T_t x_n - x_n|| \\ & \leq r_n + d(x_n) \leq 2d(x_n). \end{aligned}$$

We get from (3.7) that $\sum_{n=1}^{\infty} ||x_{n+1} - x_n|| < \infty$, and hence $\{x_n\}$ is a norm-Cauchy. Let $z = ||.|| - \lim_n x_n$. Finally, we have for each $s \in G$,

$$\begin{aligned} ||z - T_s z|| \quad & = \lim_{n \longrightarrow \infty} ||x_n - T_s x_n|| \\ & \leq \lim_{n \longrightarrow \infty} d(x_n) = 0 \end{aligned}$$

so that $T_s z = z$ for all $s \in G$. This completes the proof of the theorem.

Corollary 3.4. Suppose that X is a real Banach space with $N(X) > \max(1, \varepsilon_0)$, and C is a nonempty closed convex subset of X, while $\tau = \{T_s; s \in G\}$ is a uniformly generalized Lipschitzian (in short G2-Lipschitzian) semigroup on C with $\rho < \alpha_*$. Also, ε_0 is the characteristic of convexity of X and

$$\alpha_* = \sup\Big\{\alpha : \alpha^2 \delta_X^{-1}(1-\frac{1}{\alpha})N(X)^{-1} \leq 1 \; and \; 1-\frac{1}{\alpha} \in (0, 1-\frac{1}{2}\varepsilon_0)\Big\}.$$

If $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Theorem 3.5. Let C be a nonempty bounded subset of a uniformly convex Banach space X, and $\tau = \{T_s : s \in G\}$ be a k-uniformly generalized Lipschitzian (in short G3-Lipschitzian) semigroup of mappings on C such that

$$\rho < \sqrt{\gamma_0 N(X)}, \text{ where } \gamma_0 = \inf\{\gamma \geq 1 : \gamma(1-\delta_X(1/\gamma)) \geq 1/2. \tag{3.8}$$

Also, there exists a nonempty bounded closed convex subset E of C with the following property $(\mathbb{R})$:
$(\mathbb{R})$ $x \in E$ implies $w_w(x) \subset E$.
Then there exists $z \in E$ such that $T_s z = z$ for all $s \in G$.

Proof. Take an $x_0 \in E$ and, consider for $t \in G$, the bounded net $\{T_s x_0 : t \le s \in G\}$. Owing to Lemma 2.2, we have a $y_t \in \overline{co}\{T_s x_0 : t \le s \in G\}$ such that

$$\limsup_s ||T_s x_0 - y_t|| \le \widetilde{N}(X) D(\{T_s x_0\} t \le s \in G), \tag{3.9}$$

where $\widetilde{N}(X) = 1/N(X)$ and $D(\{T_s x_0\} t \le s \in G)$ denotes the asymptotic diameter of the net $\{z_l\}$ i.e, the number

$$\lim_t (sup\{||z_i - z_j|| : t \le i, j \in G\}).$$

Since X is reflexive, $\{y_t\}$ admits a subnet $\{y_{t_\beta}\}$ converging weakly to some $x_1 \in X$. From (3.9) and the weak lower semicontinuity of the functional $\limsup_t ||T_t x_0 - y||$, it follows that

$$\limsup_t ||T_t x_0 - x_1|| \le \widetilde{N}(X) D(\{T_t x_0\} t \in G). \tag{3.10}$$

It is also seen that $x_1 \in \bigcap_{t\in G} \overline{co}\{T_s x_0 : t \le s \in G\}$ and

$$||z - x_1|| \le \limsup_t ||z - T_t x_0|| \ \text{ for all } \ z \in X. \tag{3.11}$$

Owing to Property ($\mathbb{R}$) and the fact that $\bigcap_{t\in G} \overline{co}\{T_s x_0 : t \le s \in G\} = \overline{co}\{w_w(x_0)\}$ which is easy to prove by using the Separation Theorem (see [2]). As, we know that x_1 lies in E, we can repeat the above process and obtain a sequence $\{x_n\}_{n=0}^{\infty}$ in E with the properties: (for all nonnegative integers $n \ge 0$),

$$\limsup_t ||T_t x_n - x_{n+1}|| \le \widetilde{N}(X) D(\{T_t x_n\} t \in G) \tag{3.12}$$

and

$$||z - x_{n+1}|| \le \limsup_t ||z - T_t x_n|| \ \text{ for all } \ z \in X. \tag{3.13}$$

Write $r_n = \limsup_t ||T_t x_n - x_{n+1}||$ and $d(x_n) = \sup\{||x_n - T_t x_n|| : t \in G\}$. Thus in view of (3.12), we have

$$\begin{aligned}
r_n &= \limsup_t ||T_t x_n - x_{n+1}|| \\
&\le \widetilde{N}(X) D(\{T_t x_n\}_{t\in G}) \\
&= \widetilde{N}(X) \lim_t (\sup\{||T_i x_n - T_j x_n|| : t \le i, j \in G\}) \\
&\le \widetilde{N}(X) k \lim_t (\sup\max\{||x_n - T_{j-i} x_n||, \frac{1}{2\rho}||x_n - T_i x_n||, \frac{1}{2\rho}||T_j x_n - T_{j-i} x_n||, \frac{1}{2\rho}||x_n - T_j x_n|| \\
&, \frac{1}{2\rho}||T_{j-i} x_n - T_i x_n||\} \\
&\le \widetilde{N}(X) k \lim_t (\sup\max\{d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n)\} \\
&\le \widetilde{N}(X).k.d(x_n),
\end{aligned}$$

so that

$$r_n \le \widetilde{N}(X).k.d(x_n) \le \rho.\widetilde{N}(X).d(x_n). \tag{3.14}$$

We may assume that $d(x_n) > 0$ for all $n \ge 0$. Let $n \ge 0$ be fixed and let $\varepsilon > 0$ be small enough. First choose $j \in G$ such that

$$||T_j x_{n+1} - x_{n+1}|| > d(x_{n+1}) - \varepsilon$$

and then choose $s_0 \in G$ so large that

$$||T_s x_n - x_{n+1}|| < r_n + \varepsilon \le \rho(r_n + \varepsilon)$$

for all $s \ge s_0$. Now, for $s \ge s_0 + j$,

$$||T_s x_n - T_j x_{n+1}|| \qquad \le k \max\{||T_{s-j} x_n - x_{n+1}||, \frac{1}{2\rho}||T_s x_n - T_{s-j} x_n||, \frac{1}{2\rho}||x_{n+1} - T_j x_{n+1}||,$$

$$\frac{1}{2\rho}||T_{s-j}x_n - T_jx_{n+1}||, \frac{1}{2\rho}||x_{n+1} - T_sx_n||\}$$
$$\leq k\max\{||T_{s-j}x_n - x_{n+1}||, \frac{1}{2\rho}||T_sx_n - T_{s-j}x_n||, \frac{1}{2\rho}\limsup_t ||x_{n+1} - T_{j+t}x_n||,$$
$$\frac{1}{2\rho}\limsup_t ||T_{s-j}x_n - T_{j+t}x_n||, \frac{1}{2\rho}||x_{n+1} - T_sx_n||\}$$
$$\leq k\max\{r_n + \varepsilon, \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon), \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon)\}.$$

so that

$$||T_sx_n - T_jx_{n+1}|| \leq k(r_n + \varepsilon) \leq \rho(r_n + \varepsilon)$$

Then, it follows from property (e) that (for $s \geq s_0 + j$),

$$||T_sx_n - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \leq \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big).$$

Hence from (3.13) (taking $z := (x_{n+1} + T_jx_{n+1})/2$), we obtain

$$\begin{aligned}\frac{1}{2}(d(x_{n+1} - \varepsilon)) &< ||\frac{1}{2}(T_jx_{n+1} - x_{n+1})|| \\ &\leq ||T_jx_{n+1} - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \\ &\leq \limsup_t ||T_tx_n - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \\ &\leq \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big). \end{aligned} \tag{3.15}$$

Taking the limit as $\varepsilon \longrightarrow 0$, we have

$$\frac{1}{2}d(x_{n+1}) \leq \rho r_n\Big(1 - \delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big)\Big). \tag{3.16}$$

On the other hand, using (3.13)we easily find (for each $j \in G$),

$$||T_jx_{n+1} - x_{n+1})|| \leq \limsup_t ||T_{j+t}x_n - x_{n+1}|| = r_n \leq \rho r_n.$$

It turns out that

$$d(x_{n+1}) \leq \rho r_n \tag{3.17}$$

Combining (3.16) and (3.17) and using the definition of γ_0 in (3.8), we infer that $(\rho r_n)/d(x_{n+1}) \geq \gamma_0$. It turns out from (3.14) that

$$d(x_{n+1}) \leq \frac{\rho}{\gamma_0}r_n \leq \frac{\rho^2}{\gamma_0 N(X)}d(x_n).$$

Consequently, we obtain

$$d(x_n) \leq Ad(x_{n-1} \leq A^n d(x_0)),$$

where $A = \rho^2[\gamma_0 N(X)]^{-1} < 1$ by assumption. Noticing that

$$\begin{aligned}||x_{n+1} - x_n|| &\leq \limsup_t ||T_tx_n - x_{n+1}|| + \limsup_t ||T_tx_n - x_n|| \\ &\leq r_n + d(x_n) \\ &\leq (1 + k\widetilde{N}(X))d(x_n) \\ &\leq (1 + k\widetilde{N}(X))A^n d(x_0),\end{aligned}$$

so that the series $\sum_{n=1}^{\infty} ||x_{n+1} - x_n||$ is convergent. This implies that $\{x_n\}$ is strongly convergent. Let $z = ||.|| - \lim_n x_n$. Then, we have (for each $s \in G$)

$$\begin{aligned}||z - T_sz|| &= \lim_{n\longrightarrow\infty} ||x_n - T_sx_n|| \\ &\leq \lim_{n\longrightarrow\infty} d(x_n) = 0\end{aligned}$$

so that $T_sz = z$ for all $s \in G$ and this concludes the proof.

Corollary 3.6. Let C be a nonempty bounded subset of a uniformly convex Banach space X and $\tau = \{T_s : s \in G\}$ be a k-uniformly generalized Lipschitzian (in short G2-Lipschitzian) semigroup of mappings on C such that

$$\rho < \sqrt{\gamma_0 N(X)}, \text{ where } \gamma_0 = \inf\{\gamma \geq 1 : \gamma(1 - \delta_X(1/\gamma)) \geq 1/2.$$

Suppose that there exists a nonempty bounded closed convex subset E of C with the following property $(\mathbb{R})$:
$(\mathbb{R})$ $x \in E$ implies $w_w(x) \subset E$.
Then there exists $z \in E$ such that $T_s z = z$ for all $s \in G$.

4. ON EXPANSIVE LIPSCHITZ CONSTANT

We prove our final result as follows.

Theorem 4.1. Let C be a nonempty convex weakly compact subset of a Banach space X equipped with weak uniform normal structure and $T = \{T(t) : t \in G\}$ be an asymptotically regular $k-$uniformly Lipschitzian semigroup of mappings on C such that

$$\liminf_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = \lim_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = k < WCS(X)\mu_0,$$

where $\mu_0 = \inf\{\mu \geq 1 : \mu(1 - \delta_X(1/\mu)) \geq (1/2)\}$.
If each $T(t)$ is weakly continuous, then $F(t)$ is nonempty.

Proof. Firstly, let us choose a sequence of positive real numbers $\{t_n\} \subset G$ which increase monotonically to $+\infty$ such that

$$\liminf_{t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = \lim_{n \longrightarrow \infty} \Big| \, ||T(t_n)|| \, \Big| = k < WSC(X).$$

Since one can construct (cf. [23]) a nonempty convex closed separable subset C_0 of C which is invariant under $T(t_n)$ (i.e., $(T(t_n)C_0 \subset C_0 \quad \forall \ n = 0, 1, 2, ...)$), we may assume for a while that C itself is separable. In view of lemma 2.1, by passing to a subsequence it is possible to assume that for each $x \in C$ the sequence $\{T(t_n)x\}$ is weakly convergent. Now we define a sequence $\{x_n\}_{n=1}^{\infty}$ in C as

$$x_0 \in C \quad \text{arbitrary}, \qquad x_{m+1} = w - \lim_{n \longrightarrow \infty} T(t_n)x_m, \ m \geq 0.$$

Now, it is easy to show that $x_{m+1} = w - \lim_{j \longrightarrow \infty} T(t_j + s)x_m \quad \forall \ s \in G, \ m \geq 0$. Define

$$R_m = \limsup_{j \longrightarrow \infty} ||T(t_j)x_m - x_{m+1}|| \ \forall \ \ m \geq 0.$$

Now, we prove that $R_m \leq WCS(X)^{-1}D[(T(t_n)x_m)] \quad \forall \ m \geq 0$. Indeed, let $R_m \neq 0$, and let $\{t_{n_i}\}$ be a subsequence of $\{t_n\}$ such that

$$R_m = \limsup_{n \longrightarrow \infty} ||T(t_n)x_m - x_{m+1}|| = \lim_{j \longrightarrow \infty} ||T(t_{n_j})x_m - x_{m+1}||,$$

Also, define a sequence $\{y_j\}$ as

$$y_j = (T(t_{n_j})x_m - x_{m+1})/R_m \qquad \forall \ \ j \geq 1$$

then $||y_j|| \longrightarrow 1$ and $y_j \longrightarrow 0$.
Owing to Lemma 2.1, we have

$$\begin{aligned} WCS(X) = \beta(X) \quad & \leq D[(y_j)] = \limsup_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} || \frac{T(t_{n_j})x_m - x_{m+1}}{R_m} - \frac{T(t_{n_i})x_m - x_{m+1}}{R_m} || \\ & \leq \frac{1}{R_m} \limsup_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} ||T(t_{n_j})x_m - T(t_{n_i})x_m|| \\ & \leq \frac{1}{R_m} D[(T(t_{n_j})x_m)] \leq \frac{1}{R_m} D[(T(t_n)x_m)] \end{aligned}$$

so that

$$R_m \leq WCS(X)^{-1}D[(T(t_n)x_m)]. \tag{4.1}$$

Let $n \geq 0$ be fixed and $\varepsilon > 0$ be small enough. Choose $t_s, t_a \in \{t_n\}$ such that

$$||T(t_n)x_{m+1} - T(t_a)x_{m+1}|| > D[(T(t_n)x_{m+1})] - \varepsilon,$$

so that for every $s > n$,

$$\begin{aligned} ||T(t_s)x_m - w - \lim_{n\longrightarrow\infty} T(t_n)x_{m+1}|| &= w - \lim_{n\longrightarrow\infty} ||T(t_n)T(t_s - t_n)x_m - T(t_n)x_{m+1}|| \\ &\leq w - \lim_{n\longrightarrow\infty} \Big| \, ||T(t_n)|| \, \Big| \, ||T(t_s - t_n)x_m - x_{m+1}|| \\ &\leq (k+\epsilon)(R_m + \varepsilon). \end{aligned}$$

Also; for all $s > a$,

$$\begin{aligned} ||T(t_s)x_m - w - \lim_{a\longrightarrow\infty} T(t_a)x_{m+1}|| &\leq w - \lim_{a\longrightarrow\infty} \Big| \, ||T(t_a)|| \, \Big| \, ||T(t_s - t_a)x_m - x_{m+1}|| \\ &\leq (k+\epsilon)(R_m + \varepsilon). \end{aligned}$$

Then; in view of the property (e):

$$\begin{aligned} w - \lim_{n,a\longrightarrow\infty} ||T(t_s)x_m &- \frac{1}{2}(T(t_a)x_{m+1} + T(t_n)x_{m+1})|| \\ &\leq (k+\epsilon)(R_m - \varepsilon)\Big[1 - \delta_X(\frac{D[(T(t_n)x_{m+1})] - \varepsilon}{k(R_m - \varepsilon)})\Big]. \end{aligned} \quad (4.2)$$

Since, each $T(t)$ is weakly continuous,

$$\begin{aligned} \tfrac{1}{2}(D[(T(t_n)x_{m+1})] - &\varepsilon) < ||\frac{1}{2}(T(t_n)x_{m+1} - T(t_a)x_{m+1})|| \\ &\leq ||T(t_n)x_{m+1} - \frac{1}{2}(T(t_n)x_{m+1} + T(t_a)x_{m+1})|| \\ &\leq \lim_{l\longrightarrow\infty} ||T(t_n + t_l)x_m - \frac{1}{2}(T(t_n)x_{m+1} + T(t_a)x_{m+1})|| \end{aligned} \quad (4.3)$$

so that in view of (4.2) and (4.3), we have

$$\tfrac{1}{2}(D[(T(t_n)x_{m+1})] - \varepsilon) \leq (k+\epsilon)(R_m - \varepsilon)\Big[1 - \delta_X(\tfrac{D[(T(t_n)x_{m+1})] - \varepsilon}{(k+\epsilon)(R_m - \varepsilon)})\Big].$$

Taking the limit as $\varepsilon \longrightarrow \infty$, we have

$$\tfrac{1}{2}D[(T(t_n)x_{m+1})] \leq kR_m\Big[1 - \delta_X(\tfrac{D[(T(t_n)x_{m+1})]}{kR_m})\Big]. \quad (4.4)$$

Since, each $T(t)$ is weakly continuous, therefore $T(t_i)x_m = w - \lim_{l\longrightarrow\infty} T(t_i + t_l)x_{m-1}$ for all $i \geq 1$, so that

$$\begin{aligned} D[(T(t_n)x_{m+1})] &= \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} ||T(t_n)x_{m+1} - T(t_l)x_{m+1}|| \\ &\leq \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} \lim_{i\longrightarrow\infty} ||T(t_n + t_i)x_m - T(t_l)x_{m+1}|| \\ &\leq k \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} \lim_{i\longrightarrow\infty} ||T(t_n + t_i - t_l)x_m - x_{m+1}|| \\ &\leq kR_m. \end{aligned} \quad (4.5)$$

Combining (4.4) and (4.5) and using the definition of μ_0, we get $(kR_m)/D[(T(t_n)x_{m+1})] \geq \mu_0$. Owing to (4.1), we have

$$\begin{aligned} D[(T(t_n)x_{m+1})] &\leq \frac{k}{\mu_0}R_m \\ &\leq \frac{k}{\mu_0 WCS(X)}D[(T(t_n)x_m)] \\ &\leq A.D[(T(t_n)x_m)] \leq ... \leq A^m.D[(T(t_n)x_0)] \end{aligned}$$

where $A = \frac{k}{\mu_0 WCS(X)} < 1$ by assumption. It follows from the weak lower semicontinuity of the norm $||.||$ of X that,

$$\begin{aligned} ||x_{m+1} - x_m|| &\leq \limsup_{j\longrightarrow\infty} \Big[||x_m - T(t_j)x_m|| + ||T(t_j)x_m - x_{m+1}||\Big] \\ &\leq \limsup_{j\longrightarrow\infty} \liminf_{i\longrightarrow\infty} ||T(t_i)x_{m-1} - T(t_j)x_m|| + R_m \end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{j\longrightarrow\infty}\liminf_{i\longrightarrow\infty}||T(t_j)T(t_i-t_j)x_{m-1}-T(t_j)x_m||+R_m\\
&\leq \limsup_{j\longrightarrow\infty}\limsup_{i\longrightarrow\infty}||T(t_j)T(t_i-t_j)x_{m-1}-T(t_j)x_m||+R_m\\
&\leq \limsup||T(t_j)||\limsup_{j\longrightarrow\infty}\limsup_{i\longrightarrow\infty}||T(t_i-t_j)x_{m-1}-x_m||+R_m\\
&\leq kR_{m-1}+R_m\\
&\leq k\frac{D[(T(t_n)x_{m-1})]}{WCS(X)}+\frac{D[(T(t_n)x_m)]}{WCS(X)}\\
&\leq \frac{kA^{m-1}+A^m}{WCS(X)}D[(T(t_n)x_0)].
\end{aligned}
$$

Hence, $\{x_m\}$ is a Cauchy sequence. Let $x_\infty=\lim_{m\longrightarrow\infty}x_m$. Then, $x_\infty\in C$, and

$$
\begin{aligned}
||T(t_j)x_\infty-x_\infty|| &= ||T(t_j)x_\infty-\lim_{m\longrightarrow\infty}x_m||\\
&\leq \lim_{m\longrightarrow\infty}\limsup_{k\longrightarrow\infty}||T(t_j)x_m-T(t_k)x_{m-1}||\\
&\leq ||T(t_j)||\lim_{m\longrightarrow\infty}\limsup_{k\longrightarrow\infty}||x_m-T(t_k-t_j)x_{m-1}||\\
&\leq ||T(t_j)||\lim_{m\longrightarrow\infty}R_{m-1}=0.
\end{aligned}
$$

Obviously, $T(t_j)x_\infty=x_\infty$ for all $j\geq 1$. Now we prove that $T(s)x_\infty=x_\infty$ for all $s\in G$,

$$
\begin{aligned}
||T(s)x_\infty-x_\infty|| &= \lim_{m\longrightarrow\infty}||T(s)x_m-x_m||\\
&\leq \lim_{m\longrightarrow\infty}\lim_{j\longrightarrow\infty}||T(s+t_j)x_{m-1}-x_m||\\
&\leq \lim_{m\longrightarrow\infty}\lim_{k\longrightarrow\infty}||T(t_k)x_{m-1}-x_m||\\
&\leq \lim_{m\longrightarrow\infty}R_{m-1}=0
\end{aligned}
$$

so that $T(s)x_\infty=x_\infty$. This completes the proof.

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