

## BEST PROXIMITY THEOREMS FOR REICH TYPE CYCLIC ORBITAL AND REICH TYPE MEIR-KEELER CONTRACTION MAPS

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**ABSTRACT.** In this manuscript, Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction are defined and also some related best proximity point theorems are obtained. These theorems generalize some results of Kirk-Srinivasan-Veeramani and Karpagam-Agrawal.

**KEYWORDS :** Cyclic contraction; Best proximity points; Reich type cyclic orbital contraction; Reich type Meir Keeler cyclic contraction spaces.

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### 1. INTRODUCTION AND PRELIMINARIES

Banach Contraction mapping principle is a fundamental result in fixed point theory. It says that every contraction in a complete metric space has a unique fixed point. In this theorem, the contraction is necessarily conditions. It is natural to consider the following question. Is it possible to prove the existence and uniqueness of a fixed point of mappings that are not continuous. One of the successive answers of this question was given by Kirk-Srinivasan-Veeramani [11] by defining cyclic contraction. We should also notice that Banach considered the fixed point of self-mapping, but the authors [11] investigated the existence and uniqueness a best proximity point of non-self mappings.

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For the sake of completeness, we recollect some definitions and basic results on the topic in the literature. Suppose that  $(X, d)$  is a metric spaces. Let  $A$  and  $B$  be non-empty subsets of  $X$ . We say that  $T : A \cup B \rightarrow A \cup B$  is a *cyclic map* if  $T(A) \subset B$  and  $T(B) \subset A$ . Furthermore, a point  $x \in A \cup B$  is called a best proximity point if  $d(x, Tx) = d(A, B)$  where  $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ .

A number of authors have attracted attention to the concept of cyclic mapping and best proximity points, and published interesting results on these topics, see e.g. [1, 2, 4, 5, 6, 7, 8, 10, 13, 12, 14, 18, 19, 16].

The initial result, in this direction, was given by Kirk-Srinivasan-Veeramani [11] in 2003.

**Theorem 1.1.** ([11, Theorem 1.1]) *Let  $A$  and  $B$  be non-empty closed subsets of a complete metric space  $(X, d)$  and  $T : A \cup B \rightarrow A \cup B$  be a map satisfying  $T(A) \subset B$  and  $T(B) \subset A$  and there exists  $k \in (0, 1)$  such that  $d(Tx, Ty) \leq kd(x, y)$  for all  $x \in A$  and  $y \in B$ . Then,  $T$  has a unique fixed point in  $A \cap B$ .*

In this paper, we first introduce the notions of Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction. Later, we investigate the existence and uniqueness of a best proximity point Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction. Our results generalize, improve and extend some results in the literatures, such as, [10, 11].

## 2. MAIN RESULTS

We start to this section with following definition.

**Definition 2.1.** (See [10]) Let  $A$  and  $B$  be non-empty subsets of a metric space  $(X, d)$ . A cyclic map  $T : A \cup B \rightarrow A \cup B$  is said to be a cyclic orbital contraction if for some  $x \in A$  there exists a  $k_x \in (0, 1)$  such that

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \quad (2.1)$$

Definition 2.1 is generalized in the following way:

**Definition 2.2.** Let  $A$  and  $B$  be non-empty subsets of a metric space  $(X, d)$ . A cyclic map  $T : A \cup B \rightarrow A \cup B$  said to be a Reich type cyclic orbital contraction if for some  $x \in A$  there exists a  $k_x \in (0, \frac{1}{3})$  such that

$$d(T^{2n}x, Ty) \leq k_x [d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)], \quad n \in \mathbb{N}, \quad y \in A. \quad (2.2)$$

Karpagam and Agrawal proved the following interesting theorem.

**Theorem 2.1.** ([10, Theorem 2.2]) *Let  $A$  and  $B$  be non-empty closed subsets of a complete metric space  $(X, d)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic orbital contraction. Then  $A \cap B$  is non empty and  $T$  has a unique fixed point.*

The following theorem is a generalization of Theorem 2.1.

**Theorem 2.2.** *Suppose  $A$  and  $B$  is non-empty closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \cup B \rightarrow A \cup B$  be a Reich type cyclic orbital contraction. Then  $A \cap B$  is non empty and  $T$  has a unique fixed point.*

*Proof.* Take  $x \in A$ . By (2.2), we have

$$d(T^2x, Tx) \leq k_x [d(x, Tx) + d(T^2x, Tx) + d(Tx, x)] \quad (2.3)$$

which yields that

$$d(T^2x, Tx) \leq t_x d(Tx, x), \quad (2.4)$$

where  $t_x = \frac{2k_x}{1-k_x}$ . Since  $k_x \in (0, \frac{1}{3})$ , then  $t_x \in (0, 1)$ . Analogously, by using substitution  $Tx = u$  together with (2.2), we have

$$d(T^3x, T^2x) = d(T^2(Tx), T(Tx)) \leq k_u[d(u, Tu) + d(T^2u, Tu) + d(Tu, u)] \quad (2.5)$$

and thus

$$d(T^3x, T^2x) = d(T^2u, Tu) \leq t_u d(Tu, u) \quad (2.6)$$

where  $t_u = \frac{2k_u}{1-k_u}$ . Since  $k_u \in (0, \frac{1}{3})$ , then  $t_u \in (0, 1)$ . Keeping (2.4) and (2.6) in the mind, we obtain that

$$\begin{aligned} d(T^3x, T^2x) &= d(T^2u, Tu) \\ &\leq t_u d(Tu, u) \\ &\leq t_u d(T(Tx), Tx) \\ &\leq t_u t_x d(Tx, x) \end{aligned}$$

where  $t_u \cdot t_x < 1$ . By using the same argument, we derive that

$$d(T^4x, T^3x) = d(T^3(Tx), T^2(Tx)) \leq t_u^2 t_x d(Tx, x). \quad (2.7)$$

Iteratively, we have

$$d(T^{n+1}x, T^n x) \leq t_u^{n-1} t_x d(Tx, x), \quad (2.8)$$

for any  $n \in \mathbb{N}$ , either  $n$  or  $n + 1$  is even. Consequently, we obtain that

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^n x) \leq \left( \sum_{n=1}^{\infty} t_u^{n-1} \right) t_x d(Tx, x) < \infty.$$

Thus, we conclude that  $\{T^n x\}$  is a Cauchy sequence. As a result, there exists a  $z \in A \cup B$  such that  $T^n x \rightarrow z$ . Note that  $\{T^{2n} x\}$  is a sequence in  $A$  and  $\{T^{2n-1} x\}$  is a sequence in  $B$  in a way that both sequences tend to same limit  $z$ . Due to the fact that  $A$  and  $B$  are closed, we observe  $z \in A \cap B$ . Hence,  $A \cap B \neq \emptyset$ .

We claim that  $Tz = z$ . Observe that

$$\begin{aligned} d(Tz, z) &= \lim_{n \rightarrow \infty} d(Tz, T^{2n} x) \\ &\leq k_x \lim_{n \rightarrow \infty} [d(z, T^{2n-1} x) + d(T^{2n} x, T^{2n-1} x) + d(Tz, z)] \\ &\leq k_x d(Tz, z) \end{aligned}$$

which is equivalent to  $(1 - k_x)d(Tz, z) = 0$ . On account of  $k \in (0, \frac{1}{3})$ , we conclude that  $d(Tz, z) = 0$ , that is,  $Tz = z$ .

We shall prove that  $z$  is the unique fixed point of the operator  $T$ . Suppose, on the contrary, that there exists  $w \in A \cup B$  such that  $z \neq w$  and  $Tw = w$ . Taking account into  $T$  is a cyclic mapping, we derive that  $w \in A \cap B$ . On the other hand, we have

$$\begin{aligned} d(z, w) &= d(z, Tw) \\ &= \lim_{n \rightarrow \infty} d(T^{2n} x, Tw) \\ &\leq \lim_{n \rightarrow \infty} k_x [d(T^{2n-1} x, w) + d(T^{2n} x, T^{2n-1} x) + d(Tw, w)] \\ &= k_x d(z, w) \leq d(z, w) \end{aligned}$$

which concludes that  $(1 - k_x)d(z, w) \leq 0$  where  $k_x \in (0, \frac{1}{3})$ . Thus  $z = w$  and hence  $z$  is the unique fixed point of  $T$ .  $\square$

As an immediate consequence of the above theorem we get the following result.

**Corollary 2.3.** *Suppose  $T$  is a self map on a complete metric space  $(X, d)$ . If for some  $x \in X$ , there exists a  $k_x \in (0, \frac{1}{3})$  such that*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n-1}x, T^{2n}x) + d(Ty, y)], \quad n \in \mathbb{N}, \quad y \in X$$

*then,  $T$  has a unique fixed point.*

**Example 2.4.** Let  $A = B = [0, 1] = X$  with the metric  $d(x, y) = |x - y|$ . Define  $T : X \rightarrow X$  as follows:

$$T(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Fix any  $x \in [\frac{1}{2}, 1]$ . By setting  $k_x = \frac{1}{3}$ , we have  $Tx = \frac{1}{2}$ ,  $T^2x = \frac{1}{2}$ ,  $\dots$ ,  $T^n x = \frac{1}{2}$ ,  $\forall n$  and for every  $y \in [0, 1]$  we have

$$Ty = \begin{cases} 1 & \text{if } y \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

$$d(T^{2n}x, Ty) = \begin{cases} \frac{1}{2} & \text{if } y \in [0, \frac{1}{2}) \\ 0 & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

and thus,  $d(T^{2n-1}x, y) = |\frac{1}{2} - y|$  and

$$d(Ty, y) = \begin{cases} |1 - y| & \text{if } y \in [0, \frac{1}{2}) \\ |\frac{1}{2} - y| & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

Therefore the Reich type cyclic orbital contraction condition

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$$

For each  $n \in \mathbb{N}$  and for each  $y \in [0, 1]$  satisfies for  $k_x = \frac{1}{3}$ . Thus, by the Theorem 2.2,  $T$  has the unique fixed point and it is observe that  $x = \frac{1}{2}$  is the unique fixed point of  $T$ .

**Remark 2.5.** Notice that the statement (2.2) in Definition 2.2 could not be generalized to the following condition:

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n}x, y) + d(Ty, x)]; \quad n \in \mathbb{N}; \quad y \in A \quad (2.9)$$

since both  $T^{2n}x$  and  $y$  lies in  $A$ , the statement (2.9) fails to be cyclic. To avoid such cases, throughout of the manuscript we define and use the notion "opposite parity": We say that  $p, q \in \mathbb{N}$  are opposite parity if either  $T^p x \in A$ ,  $T^q x \in B$  or  $T^p x \in B$ ,  $T^q x \in A$  holds.

### 3. REICH TYPE CYCLIC ORBITAL MEIR-KEELER CONTRACTIONS

We start to this section with the following definition.

**Definition 3.1.** (See [10]) Let  $(X, d)$  be a metric space, and  $A$  and  $B$  be nonempty subsets of  $X$ . Assume that  $T : A \cup B \rightarrow A \cup B$  is a cyclic map such that, for some  $x \in A$ , and for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$d(T^{2n-1}x, y) < d(A, B) + \varepsilon + \delta \quad \text{implies} \quad d(T^{2n}x, Ty) < d(A, B) + \varepsilon, \quad n \in \mathbb{N}, y \in A. \quad (3.1)$$

Then  $T$  is said to be a cyclic orbital Meir-Keeler contraction.

We extend the Definition 3.1 in the following way.

**Definition 3.2.** Suppose that  $(X, d)$  is a metric space, and  $A$  and  $B$  are nonempty subsets of  $X$ . Assume that  $T : A \cup B \rightarrow A \cup B$  is a cyclic map such that, for some  $x \in A$ , and for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$\begin{aligned} R(T^{2n-1}x, y) &< d(A, B) + \varepsilon + \delta \\ \text{implies } d(T^{2n}x, Ty) &< d(A, B) + \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \quad (3.2)$$

where  $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$ . Then  $T$  is said to be a Reich Type cyclic orbital Meir-Keeler contraction.

Now we prove two technical results, which would be usee in the sequel.

**Proposition 3.3.** *Suppose that  $A, B$  are nonempty and closed subsets of a metric space  $X$  and  $T : A \cup B \rightarrow A \cup B$  is a Reich type cyclic orbital Meir-Keeler contraction. If  $x \in A$  satisfies condition (3.2) then  $d(T^{n+1}x, T^n x) \rightarrow d(A, B)$ , as  $n \rightarrow \infty$ .*

*Proof.* Assume that  $T$  is Reich type cyclic orbital Meir-Keeler contraction. Let  $x \in A$  for which the expression (3.2) is satisfied. Since either  $n$  or  $n+1$  is even, then for each  $x \in A$ , we get  $\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \geq d(A, B)$ .

Now, we examine the case

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] = d(A, B).$$

By regarding (3.2), we derive that

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon$$

which is equivalent to

$$d(T^{n+1}x, T^n x) < \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] + \varepsilon.$$

Consequently, we have

$$d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x), \text{ as } \varepsilon \rightarrow 0.$$

Let us consider the other case:

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] > d(A, B).$$

We set  $\varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] - d(A, B) > 0$ . For this  $\varepsilon_1$ , there exists a  $\delta$ , satisfying (3.2). Also  $R(T^n x, T^{n-1}x) < d(A, B) + \varepsilon_1 + \delta$ . Next

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)].$$

Hence, we get that  $d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x)$  for all  $n \in \mathbb{N}$ .

Let  $s_n = d(T^{n+1}x, T^n x)$ . So, the sequence  $\{s_n\}$  is a non-increasing and bounded below by  $d(A, B)$ . As a result,  $\{s_n\}$  converges to some  $s$  with  $s \geq d(A, B)$ .

We shall show that  $s = d(A, B)$ . Suppose, on the contrary, that  $s > d(A, B)$ . So we have  $\varepsilon = s - d(A, B) > 0$ . Consequently, there exists a  $\delta > 0$  which satisfies (3.2). Regarding  $\{d(T^{n+1}x, T^n x)\} \rightarrow s$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\begin{aligned} s &\leq \frac{1}{3}[d(T^{n+1}x, T^n x) + d(T^{n+2}x, T^{n+1}x) + d(T^{n+1}x, T^n x)] \\ &< s + \delta = \varepsilon + d(A, B) + \delta, \end{aligned}$$

for all  $n \geq n_0$ . So we conclude that

$$d(T^{n+2}x, T^{n+1}x) < d(A, B) + \varepsilon = s, \forall n \geq n_0,$$

a contradiction. Therefore, we have  $s = d(A, B)$ .  $\square$

**Proposition 3.4.** *Suppose that  $A$  and  $B$  are nonempty and closed subsets of a metric space  $X$  and  $T : A \cup B \rightarrow A \cup B$  is a Reich type cyclic orbital Meir-Keeler contraction. Suppose also that  $d(A, B) = 0$ . Then, for each  $\varepsilon > 0$ , there exist  $n_1 \in \mathbb{N}$  and a  $\delta > 0$  such that*

$$d(T^p x, T^q x) < \varepsilon + \delta \text{ implies that } d(T^{p+1} x, T^{q+1} x) < \varepsilon \quad (3.3)$$

where  $p$  and  $q$  are opposite parity, with  $p, q \geq n_1$ .

*Proof.* First, we take  $x \in X$  for which (3.2) holds. Owing to the fact that  $T$  is a Reich type cyclic orbital Meir-Keeler contraction, for a given  $\varepsilon > 0$ , there exists a  $\delta > 0$  satisfying (3.2). So, we have

$$\begin{aligned} \frac{1}{3}[d(T^{2n-1} x, y) + d(T^{2n} x, T^{2n-1} x) + d(Ty, y)] &< \varepsilon + \delta \\ \text{implies } d(T^{2n} x, Ty) &< \varepsilon, n \in \mathbb{N}, y \in A. \end{aligned} \quad (3.4)$$

Without loss of generality we may choose

$$\delta < \varepsilon. \quad (3.5)$$

Regarding  $d(A, B) = 0$  and Proposition 3.3, one can choose  $n_1 \in \mathbb{N}$  in a way that

$$d(T^n x, T^{n+1} x) < \frac{\delta}{2}, \text{ for each } n \geq n_1. \quad (3.6)$$

We shall show that  $d(T^p x, T^q x) < \varepsilon + \delta$  implies that  $d(T^{p+1} x, T^{q+1} x) < \varepsilon$ . For this purpose, fix  $n \geq n_1$  and take  $p, q \in \mathbb{N}$  which are opposite parity with  $p, q \geq n_1$ . Assume that  $d(T^p x, T^q x) < \varepsilon + \delta$ . Without loss of generality we may assume  $T^p x \in A$  and  $T^q x \in B$  with  $p = 2n$  and  $q = 2m - 1$ . Otherwise, we revise the indices respectively.

Therefore, we have  $d(T^p x, T^q x) = d(T^{2n} x, T^{2m-1} x) < \varepsilon + \delta$ , for  $m \geq n$ . Then, regarding (3.6) we get

$$\frac{1}{3}[d(T^{2m-1} x, T^{2n} x) + d(T^{2m} x, T^{2m-1} x) + d(T^{2n+1} x, T^{2n} x)] \leq \frac{1}{3}[\varepsilon + \delta + \frac{\delta}{2} + \frac{\delta}{2}] < \varepsilon + \delta. \quad (3.7)$$

Regarding (3.4) under the assumption  $y = T^{2n} x$ , the inequality (3.7) yields that

$$d(T^{2n+1} x, T^{2m} x) = d(T^{p+1} x, T^{q+1} x) < \varepsilon$$

Hence, we derive that for a given  $\varepsilon > 0$ , there exist  $n_1 \in \mathbb{N}$  and a  $\delta > 0$  such that

$$d(T^p x, T^q x) < \varepsilon + \delta \text{ implies that } d(T^{p+1} x, T^{q+1} x) < \varepsilon \quad (3.8)$$

where  $p$  and  $q$  are opposite parity, with  $p, q \geq n_1$ . □

Next we prove a necessary condition for a cyclic orbital Meir-Keeler contraction and we use this to prove the main result.

**Lemma 3.5.** *Suppose that  $X$  is a complete metric space,  $A$  and  $B$  non-empty, closed subsets of  $X$  such that  $d(A, B) = 0$ . Suppose also that  $T : A \cup B \rightarrow A \cup B$  is a Reich Type cyclic orbital Meir-Keeler contraction. Then*

$$d(T^{2n} x, Ty) < R(T^{2n-1} x, y) \text{ if } T^{2n-1} x \neq y, \quad (3.9)$$

where  $R(T^{2n-1} x, y) = \frac{1}{3}[d(T^{2n-1} x, y) + d(T^{2n} x, T^{2n-1} x) + d(Ty, y)]$ .

*Proof.* To reach (3.9), it is sufficient to show that (3.2) is equivalent to the following condition: For each  $\varepsilon > 0$  there exists  $\delta$  such that

$$\begin{aligned} \varepsilon &\leq R(T^{2n-1} x, y) < \varepsilon + \delta \\ \text{implies } d(T^{2n} x, Ty) &< \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \quad (3.10)$$

where  $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$  and recall that  $d(A, B) = 0$ .

It is evident that (3.2) implies (3.10). Now, suppose that (3.10) holds. Now, we fix  $T^{2n-1}x, y \in A \cup B$  and  $\varepsilon > 0$ . If  $R(T^{2n-1}x, y) < \varepsilon$ , since (3.10) we have  $d(T^{2n}x, Ty) \leq R(T^{2n-1}x, y)$  and consequently  $d(T^{2n}x, Ty) < \varepsilon$ . If  $R(T^{2n-1}x, y) \geq \varepsilon$ , then immediately (3.2) holds. Hence, (3.10) and (3.2) are equivalent under the condition  $d(A, B) = 0$ .

We shall show that if (3.10) holds then  $d(T^{2n}x, Ty) \leq R(T^{2n-1}x, y)$ . If  $R(T^{2n-1}x, y) = 0$  then  $T^{2n-1}x = y$ . Thus  $d(T^{2n}x, Ty) \leq R(T^{2n-1}x, y)$ . Suppose  $R(T^{2n-1}x, y) \neq 0$  and fix  $\varepsilon \leq R(T^{2n-1}x, y)$ . Choose a  $\delta > 0$  such that (3.10) holds. Recall that  $R(T^{2n-1}x, y) \leq d(T^{2n}x, Ty)$  which contradicts with (3.10).  $\square$

Now we prove the existence, uniqueness and convergence theorem for a Reich type cyclic orbital Meir-Keeler contraction.

**Theorem 3.1.** *Suppose that  $X$  is a complete metric space,  $A$  and  $B$  non-empty, closed subsets of  $X$  with  $d(A, B) = 0$ . Assume that  $T : A \cup B \rightarrow A \cup B$  is a Reich type cyclic orbital Meir-Keeler contraction. Then, there exists a fixed point, say  $z \in A \cap B$ , such that for each satisfying (3.2), the sequence  $\{T^{2n}x\}$  converges to  $z$ .*

*Proof.* First, we take  $x \in A$ . We shall show that  $\{T^m x\}$  is a Cauchy sequence. Suppose, on the contrary, that the sequence  $\{T^n x\}$  is not Cauchy. Then, there exists an  $\varepsilon > 0$  and a subsequence in  $\{T^{n(i)}\}$  of  $\{T^n x\}$  with

$$d(T^{n(i)}x, T^{n(i+1)}x) > 2\varepsilon. \quad (3.11)$$

For this  $\varepsilon$ , there exists  $\delta > 0$  such that

$$R(T^{2n-1}x, y) < \varepsilon + \delta \text{ implies that } d(T^{2n}x, Ty) < \varepsilon \quad (3.12)$$

where  $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$ . We set  $r = \min\{\varepsilon, \delta\}$  and  $d_m = d(T^m x, T^{m+1}x)$ . Owing to Proposition 3.3, one can choose  $n_0 \in \mathbb{N}$  such that

$$d^m = d(T^m x, T^{m+1}x) < \frac{r}{4}, \quad \text{for } m \geq n_0. \quad (3.13)$$

Let  $n(i) \geq N$ . Suppose that  $d(T^{n(i)}x, T^{n(i+1)-1}x) \leq \varepsilon + \frac{r}{2}$ . Then triangle inequality implies that

$$\begin{aligned} d(T^{n(i)}x, T^{n(i+1)}x) &\leq d(T^{n(i)}x, T^{n(i)-1}x) + d(T^{n(i+1)-1}x, T^{n(i+1)}x) \\ &< \varepsilon + \frac{r}{2} + d_{n(i+1)-1} \\ &< 2\varepsilon \end{aligned} \quad (3.14)$$

which contradicts the assumption (3.11). Therefore, there are values of  $k$  with  $n(i) \leq k \leq n(i+1)$  such that  $d(T^{n(i)}, T^k x) > \varepsilon + \frac{r}{2}$ . opposite parity. We assume that  $d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2}$ . Then

$$d_{n(i)} = d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{4}$$

which contradicts with (3.13). Hence, there are values of  $k$  with  $n(i) \leq k \leq n(i+1)$  such that  $d(T^{n(i)}, T^k x) < \varepsilon + \frac{r}{2}$  where  $k$  and  $n(i)$  are opposite parity. Choose smallest integer  $k$  with  $k \geq n(i)$  such that  $d(T^{n(i)}x, T^k x) \geq \varepsilon + \frac{r}{2}$ . So,

$$d(T^{n(i)}x, T^{k-1}x) < \varepsilon + \frac{r}{2}. \quad (3.15)$$

Hence,

$$\begin{aligned} d(T^{n(i)}x, T^{k-1}x) &\leq d(T^{n(i)}x, T^{k-1}x) + d(T^{k-1}x, T^kx) \\ &< \varepsilon + \frac{r}{2} + \frac{r}{4} \\ &= \varepsilon + \frac{3r}{4}. \end{aligned}$$

Then, there exists an integer  $k$  satisfying  $n(i) \leq k \leq n(i+1)$  such that

$$\varepsilon + \frac{r}{2} \leq d(T^{n(i)}x, T^kx) < \varepsilon + \frac{3r}{4}. \quad (3.16)$$

Owing to the facts

$$\begin{aligned} d(T^{n(i)}x, T^kx) &< \varepsilon + \frac{3r}{4} < \varepsilon + r, \\ d(T^{n(i)}x, T^{n(i)+1}x) &= d_{n(i)} < \frac{r}{4} < \varepsilon + r \text{ and} \\ d(T^k, T^{k+1}x) &= d_k < \frac{r}{4} < \varepsilon + r. \end{aligned}$$

We have

$$\begin{aligned} R(T^{n(i)}x, T^kx) &= \frac{1}{3}[d(T^{n(i)}x, T^kx) + d(T^{n(i)}x, T^{n(i)+1}x) + d(T^{k+1}x, T^kx)] \\ &\leq \frac{1}{3}[\varepsilon + r + \varepsilon + r + \varepsilon + r] = \varepsilon + r, \end{aligned}$$

which implies  $d(T^{n(i)+1}, T^{k+1}x) < \varepsilon$ . But,

$$\begin{aligned} d(T^{n(i)+1}x, T^{k+1}x) &\geq d(T^{n(i)}x, T^kx) - d(T^{n(i)}x, T^{n(i)+1}x) - d(T^kx, T^{k+1}x) \\ &> \varepsilon + \frac{r}{2} - \frac{r}{4} - \frac{r}{4} = \varepsilon, \end{aligned}$$

which contradicts the preceding inequality.

Hence the sequence  $\{T^n x\}$  is a Cauchy. Thus, the sequence  $\{T^n x\}$  converges to some  $z \in A$ . Hence,

$$0 \leq d(T^{2n-1}x, z) \leq d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, z) \quad (3.17)$$

tends to zero as well. Thus,

$$\lim_{n \rightarrow \infty} d(T^{2n-1}x, z) = 0. \quad (3.18)$$

Since  $\{T^{2n-1}x\}$  is a sequence in  $B$ , it converges to  $z \in B$ . Taking into account both  $A$  and  $B$  are closed, we get  $z \in A \cap B$ .

We shall show that  $Tz = z$ .

On account of Lemma 3.5

$$\begin{aligned} d(Tz, z) &= \lim_{n \rightarrow \infty} d(T^{2n}x, Tz) \\ &< \lim_{n \rightarrow \infty} R(T^{2n-1}x, z) \\ &= \lim_{n \rightarrow \infty} \frac{1}{3}[d(T^{2n-1}x, z) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \end{aligned}$$

which implies that

$$d(Tz, z) < \frac{1}{3}d(Tz, z).$$

This is a contradiction. Hence, we have  $Tz = z$ .

Finally, we shall show that  $z$  is a unique fixed point of  $T$ . Suppose, on the contrary, that there exists a point  $w \in A \cap B$  such that  $z \neq w$  and  $Tw = w$ . Owing to Lemma 3.5

$$\begin{aligned}
 d(w, z) &= d(Tw, z) \\
 &= \lim_{n \rightarrow \infty} d(T^{2n}x, Tw) \\
 &< \lim_{n \rightarrow \infty} R(T^{2n-1}x, w) \\
 &\leq \lim_{n \rightarrow \infty} \frac{1}{3} [d(T^{2n-1}x, w) + d(T^{2n}x, T^{2n-1}x) + d(Tw, w)] \\
 &\leq \frac{1}{3} [d(z, w) + d(z, z) + d(Tw, w)] \\
 &= \frac{1}{3} d(z, w)
 \end{aligned}$$

which is a contradiction. Hence,  $z = w$ . □

#### 4. CONCLUDING REMARKS

In this paper, two different types of Reich type cyclic contractions, such as Reich type cyclic orbital contractions and Reich type Meir-Keeler cyclic contractions, are introduced and thereby obtained the existence and uniqueness of best proximity for such mappings. We also gave a procedure to find the best proximity point (a type of iterative sequence that converges to the point) for a Reich type Meir-Keeler cyclic contraction. Our results generalize some results of Kirk-Srinivasan-Veeramani [11] and Karpagam-Agrawal [10].

For further research, we propose the studying the existence of best proximity points of Reich type cyclic orbital contractions and Reich type Meir-Keeler cyclic contractions on  $A \cup B$  in a complete metric space in the case that  $d(A, B) > 0$ . Further research may also include studying the necessary condition for convergence of iterative sequence  $\{T^{2n}x\}$ , for any  $x \in A \cup B$ , that converges to the unique best proximity point for a Reich type cyclic orbital contraction or Reich type Meir-Keeler cyclic contraction on  $A \cup B$  in a complete metric space in the case that  $d(A, B) > 0$ .

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#### REFERENCES

1. Al-Thafai, M.A. and N. Shahzad, Convergence and existence for best proximity points, *Non-linear Analysis*, 70, 3665-3671 (2009).
2. J. Anuradha and P. Veeramani, Proximal pointwise contraction, *Topology and its Applications*, 156, 2942-2948(2009).
3. C.-M. Chen, Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces, *Fixed Point Theory and Applications*, 2012, 2012:41 pp13.
4. A.A. Eldered and P. Veeramani, Convergence and existence for best proximity points, *J. Math. Anal. Appl.*, 323,1001-1006 (2006).
5. M. Jleli, E. Karapinar and B. Samet, Best proximity point results for MK-proximal contractions, *Abstract and Applied Analysis*, 2012 pp14. Article Id: 193085
6. E. Karapinar, G.Petruşel and K.Tas, Best Proximity Point Theorems for KT-Types Cyclic Orbital Contraction Mappings, *Fixed Point Theory*,13( 2012),no:2, 537-546.
7. E. Karapinar, Best Proximity Points Of Cyclic Mappings, *Appl. Math. Lett.*, 25 (2012), No: 11, 1761-1766

8. E. Karapinar, Best proximity points of Kannan type cyclic weak  $\phi$ -contractions in ordered metric spaces, *Analele Stiintifice ale Universitatii Ovidius Constanta*, 20,(2012), no:3,51-64.
9. E. Karapinar, E, Fixed point theory for cyclic weak  $\phi$ -contraction, *Appl. Math. Lett.*, 24 (2011) 822-825.
10. S. Karpagam and Agrawal, Sushama, Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps. *Nonlinear Anal.* 74 (2011), no. 4, 1040–1046.
11. W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed Points for mapping satisfying cyclical contractive conditions, *Fixed Point Theory*, 4,79-89 (2003).
12. C. Mongkolkeha and P. Kumam, Best proximity point Theorems for generalized cyclic contractions in ordered metric spaces, *Journal of Optimization Theory and Applications*, (2012) 155:215–226.
13. C. Mongkolkeha and P. Kumam , Best proximity points for asymptotic proximal pointwise weaker Meir-Keeler-type -contraction mappings, *Journal of the Egyptian Mathematical Society*, **21**(2) (2013) - 87 - 90.
14. C. Mongkolkeha and P. Kumam, Some common best proximity points for proximity commuting mappings, *Optimization Letters* 7 (2013), no. 8, 1825 - 1836.
15. G. Petruşhel, Cyclic representations and periodic points, *Studia Univ. Babeş-Bolyai Math.*, 50(2005),107–112.
16. Sh. Rezapour, M. Derafshpour and N. Shahzad, Best proximity point of cyclic  $\varphi$ -contractions in ordered metric spaces, *Topological Methods in Nonlinear Analysis*, **37**, (2011) 193–202.
17. I. A. Rus, Cyclic representations and fixed points, *Ann. T. Popoviciu. Sem. Funct. Eq. Approx. Convexity* 3 (2005), 171–178.
18. G. Sankara Raju Kosuru and P. Veeramani: A Note on Existence and Convergence of Best proximity Points for Pointwise Cyclic Contractions, *Numer. Funct. Anal. Optim.* **32**(7), 821–830, 2011.
19. G. Sankara Raju Kosuru and P. Veeramani, On existence of best proximity pair theorems for relatively nonexpansive mappings, *J. Nonlinear Convex Anal.* **11** (2010), no. 1, 71–77.