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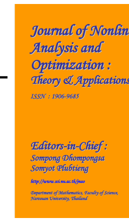

FIXED POINT THEOREMS FOR SOME GENERALIZED NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, at first we introduce C_α condition, which is weaker than α-nonexpansivity and present some fixed point theorems for mappings satisfying this condition, in $CAT(0)$ spaces. Our results extend and improve some results in [6]. In the sequal, we introduce fundamentaly nonexpansive mapping which generalizes the Suzuki's generalized nonexpansive mapping and consequently we give some fixed point results for this kind of mappings.

KEYWORDS: $CAT(0)$ spaces; α-nonexpansive mappings; fixed point; Condition C; Opial property.
AMS Subject Classification: 47H09, 47H10.

1. INTRODUCTION

Fixed point theory for nonexpansive and related mappings has played a fundamental role in many aspects of functional analysis for many years. In this paper, we apply generalized nonexpansive definitions which are strong enough to generate a fixed point but do not force the map to be continuous in spite of this fact that in most of the fixed point theorems in this field either continuity is explicitly assumed or, the nonexpansive definitions themselves imply continuity. In 2008, Suzuki [13] introduced condition C as below:
Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy condition (C) (or Suzuki's generalized nonexpansive) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \quad implies \quad ||Tx - Ty|| \leq ||x - y||,$$

for all $x, y \in C$.

Proposition 1.1. *Every nonexpansive mapping satisfies condition (C), but the inverse is not true (see [13] example1).*

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As can be seen, this condition does not force the map to be continuous. Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(\acute{t})) = |t - \acute{t}|$ for all $t, \acute{t} \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y for each $x, y \in X$. Write $c(\alpha 0 + (1-\alpha)l) = \alpha x \oplus (1-\alpha)y$ for $\alpha \in (0,1)$. The space X is said to be of hyperbolic type [8] if it satisfies

$$d(p, \alpha x \oplus (1-\alpha)y) \leq \alpha d(p,x) + (1-\alpha)d(p,y) \quad \forall p \in X. \tag{1.1}$$

Let $v_1, v_2, ..., v_n \subset X$ and $\lambda_1, \lambda_2, ..., \lambda_n \subset (0,1)$ with $\sum_{i=1}^{n} \lambda_i = 1$. We write, by induction,

$$\bigoplus_{i=1}^{n} \lambda_i \nu_i := (1-\lambda_n)(\frac{\lambda_1}{1-\lambda_n}\nu_1 \oplus \frac{\lambda_2}{1-\lambda_n}\nu_2 \oplus ... \oplus \frac{\lambda_{n-1}}{1-\lambda_n}\nu_{n-1})\lambda_n \nu_n. \tag{1.2}$$

The definition of $\oplus$ in (3.3) is an ordered one in the sense that it depends on the order of points $v_1, ..., v_n$. Under (3.2) we can see that

$$d(\bigoplus_{i=1}^{n} \lambda_i \nu_i, x) \leq \sum_{i=1}^{n} \lambda_i d(\nu_i, x) \tag{1.3}$$

for each $x \in X$. A subset $Y \subseteq X$ is said to convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X,d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$. A geodesic metric space is said to be a $CAT(0)$ space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$: $d(x,y) \leq d_{E^2}(\bar{x}, \bar{y})$.

Lemma 1.2 ([1], see Proposition 2.2). *Let X be a $CAT(0)$ space. Then for each $p, q, r, s \in X$ and $\alpha \in [0,1]$,*

$$d(\alpha p \oplus (1-\alpha)q, \alpha r \oplus (1-\alpha)s) \leq \alpha d(p,r) + (1-\alpha)d(q,s).$$

In particular, (3.2) holds in $CAT(0)$ spaces. Let X be a complete $CAT(0)$ space and (x_n) be a bounded sequence in X. For $x \in X$ set:

$$r(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is the set:

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

It is known that in a $CAT(0)$ space, $A((x_n))$ consists of exactly one point [4], and distance function in $CAT(0)$ spaces, is convex (see page 159 of [1]). Also every $CAT(0)$ space has the *Opial* property, i.e. if (x_n) is a sequence in K and $\Delta - \lim x_n = x$, then for each $y(\neq x) \in K$ we have

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y)$$

Definition 1.3. (see [11], Definition 3.1) A sequence (x_n) in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of (u_n) for every sequence (u_n) of (x_n). In this case, we write $\Delta - \lim_n x_n = x$ and call x the $\Delta - \lim$ of (x_n).

We also need the following theorem which is presented in [12] (see Corollary 2.8).

Theorem 1.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition (C) then $F(T)$ (the set of fixed points of T) is nonempty, closed and convex.*

2. GENERALIZED α-NONEXPANSIVE MAPPINGS

Recently, in 2010, the authors in [6] proved some fixed point theorems for α-nonexpansive mappings introduced by Goebel and Pineda [9] as follows :
A mapping T on a nonempty closed convex subset C of a Banach space X is said to be α-nonexpansive if for given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$ we have

$$\sum_{i=1}^{n} \alpha_i ||T^i x - T^i y|| \leq ||x - y||, \quad \forall x, y \in C.$$

The above definition generalizes the nonexpansive one. Now, we are going to generalize α-nonexpansivity by Suzuki's method:

Definition 2.1. Let C be a nonempty closed convex subset of a Banach space X. For a given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$, $p \in \{1, 2, ..., n\}$, a mapping $T : C \longrightarrow C$ is said to satisfy condition C_α if

$$\frac{1}{2}||x - \sum_{i=1}^{p} \alpha_i T^i x|| \leq ||x - y|| \quad \textit{implies} \quad \sum_{i=1}^{p} \alpha_i ||T^i x - T^i y|| \leq ||x - y||, \qquad (2.1)$$

for all $x, y \in C$.

In the case $p = n$, it is easy to show every α-nonexpansive mapping satisfies condition C_α, but the converse is not necessarily true.

Example 2.2. Define a mapping T on $[0, \infty]$ by $Tx = [\frac{x}{3}]$. Then for $\alpha = (\frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{2}{5})$ and $x = 3k, y = 3k - p$ for $0 < p < 1$ (for example let $x = 729$ and $y = 728.5$, therefore $Tx = 243, T^2 x = 81, T^3 x = 27, T^4 x = 9, T^5 x = 3$ and $Ty = 242, T^2 y = 80, T^3 y = 26, T^4 y = 8, T^5 y = 2$) we have

$$\sum_{i=1}^{5} \alpha_i d(T^i x, T^i y) \nleq d(x, y)$$

thus T is not α-nonexpansive, but T satisfies condition C_α.

For technical reason we always assume that the first coefficient α_1 is nonzero. If T satisfies condition C_α then

$$\frac{1}{2}||x - \sum_{i=1}^{p} \alpha_i T^i x|| \leq ||x - y||$$

implies

$$\sum_{i=1}^{p} \alpha_i ||T^i x - T^i y|| \leq ||x - y||,$$

on the other hand

$$||\sum_{i=1}^{p} \alpha_i T^i x - \sum_{i=1}^{p} \alpha_i T^i y|| \leq \sum_{i=1}^{p} \alpha_i ||T^i x - T^i y||.$$

So if we set $T_{\alpha_p} x = \sum_{i=1}^{p} \alpha_i T^i x$ for all $x \in C$ then it follows that the mapping T_{α_p} satisfies condition C. However, we can't imply that if T_{α_p} satisfies condition C then T satisfies condition C_α because it is much weaker.

3. Fixed point theorems

In this section, we prove some fixed point theorems for mapping satisfying condition C_α in a $CAT(0)$ space. First, we mentioned the definition of condition C_α in $CAT(0)$ spaces as follow:

Definition 3.1. Let C be a nonempty bounded, closed and convex subset of a $CAT(0)$ space X. For a given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \alpha_i = 1$, $p \in \{1, 2, ..., n\}$, a mapping $T : C \longrightarrow C$ is said to satisfy condition C_α if

$$\frac{1}{2} d(x, \bigoplus_{i=1}^{p} \alpha_i T^i x) \leq d(x, y) \quad implies \quad \sum_{i=1}^{p} \alpha_i d(T^i x, T^i y) \leq d(x, y), \tag{3.1}$$

for all $x, y \in C$.

Theorem 3.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition C_α and for all $n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be such that $\alpha_i \geq 0, i = 2, ..., n, \alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $\sum_{i=1}^{n} \alpha_i = 1$, then $F(T) = F(T_{\alpha_p})$ for all $p \in \{1, ..., n\}$.*

Proof. It is clear that $F(T) \subset F(T_{\alpha_p})$. Next, we show that $F(T_{\alpha_p}) \subset F(T)$. Since T satisfies condition C_α, for $x \in F(T_{\alpha_p})$ and for all $k \in \{1, 2, ..., m\}$ we have

$$0 = \frac{1}{2} d(x, \bigoplus_{i=1}^{p} \alpha_i T^i x) \leq d(x, T^k x),$$

let $x \neq Tx$, then for all $m \in \{1, 2, ..., n\}$ we can write

$$\begin{aligned}
d(T^m x, Tx) \leq\ & \tfrac{1}{\alpha_1} d(T^{m-1} x, x) \\
\leq\ & \tfrac{1}{\alpha_1} (d(T^{m-1} x, Tx) + d(Tx, x)) \\
\leq\ & \tfrac{1}{\alpha_1^2} d(T^{m-2} x, x) + \tfrac{1}{\alpha_1} d(Tx, x) \\
\leq\ & \tfrac{1}{\alpha_1^2} (d(T^{m-2} x, Tx) + d(Tx, x)) + \tfrac{1}{\alpha_1} d(Tx, x) \\
& . \\
& . \\
& . \\
\leq\ & (\tfrac{1}{\alpha_1^{m-1}} + ... + \tfrac{1}{\alpha_1^2} + \tfrac{1}{\alpha_1}) d(Tx, x).
\end{aligned}$$

So one can write

$$\begin{aligned} d(x,Tx) &= d(T_{\alpha_p}x,Tx) \\ &= d(\bigoplus_{i=1}^{p}\alpha_i T^i x,Tx) \\ &\leq \alpha_2 d(T^2x,Tx)+\alpha_3 d(T^3x,Tx)+...+\alpha_p d(T^px,Tx) \\ &\leq \frac{\alpha_2}{\alpha_1}d(Tx,x)+(\frac{\alpha_3}{\alpha_1^2}+\frac{\alpha_3}{\alpha_1})d(Tx,x)+...+(\frac{\alpha_p}{\alpha_1^{p-1}}+...+\frac{\alpha_p}{\alpha_1^2}+\frac{\alpha_p}{\alpha_1})d(Tx,x) \\ &= (\frac{\alpha_2+\alpha_3+...+\alpha_p}{\alpha_1}+\frac{\alpha_3+...+\alpha_p}{\alpha_1^2}+...+\frac{\alpha_p}{\alpha_1^{p-1}})d(Tx,x) \\ &\leq (\frac{1-\alpha_1}{\alpha_1}+\frac{1-\alpha_1}{\alpha_1^2}+...+\frac{1-\alpha_1}{\alpha_1^{p-1}})d(x,Tx) \\ &= \frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}}d(x,Tx). \end{aligned}$$

Since $\alpha_1 > \frac{1}{\sqrt[n-1]{2}} \geq \frac{1}{\sqrt[p-1]{2}}$ this implies that $\frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} < 1$ which lead to a contradiction, therefore $x = Tx$ and this complete the proof. □

Corollary 3.2. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T: K \to K$ satisfies condition C_α and for all $n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be such that $\alpha_i \geq 0$ for $i = 2, ..., n, \alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $\sum_{i=1}^{n}\alpha_i = 1$ then $F(T)$ is nonempty closed and convex.*

Proof. Since T_{α_p} satisfies condition C, it follows by Theorem 1.1 and Theorem 3.1 that $F(T)$ is nonempty closed and convex. □

Therefore the existence problem of a fixed point of mapping $T: K \longrightarrow K$ satisfying condition C_α can be directly obtained by the existence of a fixed point of mapping T_α which satisfies condition C. Next, we show that the approximate fixed point sequences for these two mappings are the same.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be as in Theorem 3.1. Let K be a bounded closed convex subset of a complete $CAT(0)$ space X and $T: K \to K$ satisfies condition C_α. Suppose (x_m) be a bounded sequence in K and $\alpha_1 d(T^n x_m, T^{n+1}x_m) \leq d(T^n x_m, T^{n-1}x_m)$. Then $d(x_m, Tx_m) \longrightarrow 0$ if and only if $d(x_m, T_{\alpha_p}x_m) \longrightarrow 0$ as $m \longrightarrow \infty$.*

Proof. Let $d(x_m, Tx_m) \longrightarrow 0$. Since $\alpha_1 d(T^n x_m, T^{n+1}x_m) \leq d(T^n x_m, T^{n-1}x_m)$ one can write

$$\begin{aligned} d(T^k x_m, x_m) &\leq d(T^k x_m, T^{k-1}x_m)+...+d(T^2x_m,Tx_m)+d(Tx_m,x_m) \\ &\leq (\frac{1}{\alpha_1^{k-1}}+...+\frac{1}{\alpha_1}+1)d(Tx_m,x_m). \end{aligned}$$

So $d(T^k x_m, x_m) \longrightarrow 0$ as $m \longrightarrow 0$ for all $k \in \{1, 2, ..., n\}$. Thus by the above equation

$$\begin{aligned} d(T_{\alpha_p}x_m, x_m) &= d(\bigoplus_{i=1}^{p}\alpha_i T^i x_m, x_m) \\ &\leq \sum_{i=1}^{p}\alpha_i d(T^i x_m, x_m) \longrightarrow 0, \quad as \quad m \longrightarrow \infty. \end{aligned}$$

Again we can write

$$\begin{aligned} d(T^k x_m, Tx_m) &\leq d(T^k x_m, T^{k-1}x_m)+...+d(T^2x_m,Tx_m) \\ &\leq (\frac{1}{\alpha_1^{k-1}}+...+\frac{1}{\alpha_1})d(Tx_m,x_m). \end{aligned}$$

Now, conversely, assume that $d(x_m, T_{\alpha_p}x_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Since

$$\begin{aligned} d(x_m, Tx_m) \leq & \; d(x_m, T_{\alpha_p}x_m) + d(T_{\alpha_p}x_m, Tx_m) \\ = & \; d(x_m, T_{\alpha_p}x_m) + d(\bigoplus_{i=1}^{p} \alpha_i T^i x_m, Tx_m) \\ \leq & \; d(x_m, T_{\alpha_p}x_m) + \alpha_2 d(T^2x_m, Tx_m) + ... + \alpha_p d(T^p x_m, Tx_m) \\ \leq & \; d(x_m, T_{\alpha_p}x_m) + \tfrac{\alpha_2}{\alpha_1} d(Tx_m, x_m) + ... + (\tfrac{\alpha_p}{\alpha_1^{p-1}} + ... + \tfrac{\alpha_p}{\alpha_1}) d(Tx_m, x_m) \\ = & \; d(x_m, T_{\alpha_p}x_m) + (\tfrac{\alpha_2 + ... + \alpha_p}{\alpha_1} + ... + \tfrac{\alpha_p}{\alpha_1^{p-1}}) d(Tx_m, x_m) \\ \leq & \; d(x_m, T_{\alpha_p}x_m) + (\tfrac{1-\alpha_1}{\alpha_1} + \tfrac{1-\alpha_1}{\alpha_1^2} + ... + \tfrac{1-\alpha_1}{\alpha_1^{p-1}}) d(x_m, Tx_m) \\ = & \; d(x_m, T_{\alpha_p}x_m) + \tfrac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} d(x_m, Tx_m), \end{aligned}$$

and $\beta_p = \frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} < 1$, hence

$$(1 - \beta_p) d(x_m, Tx_m) \leq d(x_m, T_{\alpha_p}x_m).$$

Which implies that $d(x_m, Tx_m) \longrightarrow 0$ as $m \longrightarrow \infty$. □

Remark 3.3. Note that if K is a bounded closed convex subset of a strictly convex Banach space and $T : K \longrightarrow K$ satisfies condition C,then $F(T)$ is closed and convex [13]. Hence if we use this, instead of Theorem 1.1, then we can write all the above results in the setting where Chakkrid Klin-eam and Suthep Suantai [6] worked in and generalize all their mentioned results.

4. FUNDAMENTALLY NONEXPANSIVE MAPPINGS

In this section, we want to generalize Suzuki's generalized nonexpansive mappings in another manner as follow:

Definition 4.1. Let X be a $CAT(0)$ space and K be a bounded closed convex subset of X. A mapping $T : K \longrightarrow K$ is said to be fundamentally nonexpansive if

$$d(T^2x, Ty) \leq d(Tx, y),$$

for all $x, y \in K$.

Proposition 4.2. *Every mapping which satisfies condition (C) is fundamentally nonexpansive, but the inverse is not true.*

Proof. By taking $\acute{x} = Tx, \acute{y} = y$, we see that every nonexpansive mapping is fundamentally nonexpansive. So by Lemma 3.4 part (iii) in [13] the desired result is obtained. □

Example 4.3. Define a mapping T on $[0, 2]$ by

$$T(x) = \begin{cases} 0 & if x \neq 2, \\ 1 & if x = 2. \end{cases}$$

By taking $x = 2, y = 1.5$ we have

$$\frac{1}{2} d(T(2), 2) \leq d(2, 1.5)$$

but

$$d(T(2), T(1.5)) \nleq d(2, 1.5).$$

Therefore T is fundamentally nonexpansive, but T is not nonexpansive or even satisfies condition (C).

Theorem 4.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive and $F(T) \neq \emptyset$, then $F(T)$ is Δ-closed and convex set.*

Proof. Suppose (x_n) is a sequence in $F(T)$ which Δ-converges to some $y \in K$. We want to show $y \in F(T)$. In order to prove this, one can write

$$d(x_n, Ty) = d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y)$$

therefore

$$\limsup_n d(x_n, Ty) \leq \limsup_n d(x_n, y).$$

By the uniqueness of asymptotic center, we obtain $Ty = y$.

$F(T)$ is convex: let $x, z \in F(T)$, then we have:

$$d(x, Ty) = d(T^2x, Ty) \leq d(Tx, y) = d(x, y),$$

and

$$d(z, Ty) = d(T^2z, Ty) \leq d(Tz, y) = d(z, y).$$

For $y \in [x, z]$, we have $d(x, y) + d(y, z) = d(x, z)$

$$d(x, z) \leq d(x, Ty) + d(Ty, z) \leq d(x, y) + d(y, z) = d(x, z).$$

Therefore $d(x, Ty) = d(x, y)$ and $d(Ty, z) = d(y, z)$, because if $d(x, Ty) < d(x, y)$ or $d(Ty, z) < d(y, z)$, then we obtain the contradiction $d(x, z) < d(x, z)$, therefore $Ty \in [x, z]$ and $Ty = y$, which means $[x, z] \subset F(T)$. □

Lemma 4.4. [7] *Let (z_n) and (w_n) be bounded sequences in K and $\lambda \in (0, 1)$. Suppose that $z_{n+1} = \lambda w_n + (1 - \lambda)z_n$ and $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$ for all $n \in N$. Then $\limsup_n d(w_n, z_n) = 0$.*

Lemma 4.5. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive, then always there exists an approximate fixed point sequence for T.*

Proof. Define a sequence (x_n) in K by $x_1 \in K$ and

$$x_{n+1} = \alpha Tx_n \oplus (1 - \alpha)x_n$$

for $n \in \mathbb{N}$, where α is a real number belonging to $[0, 1]$. Then we have

$$d(Tx_{n+1}, Tx_n) = \alpha d(T^2x_n, Tx_n) \leq \alpha d(Tx_n, x_n) = d(x_{n+1}, x_n).$$

for $n \in \mathbb{N}$, hence

$$d(Tx_{n+1}, Tx_n) \leq d(x_{n+1}, x_n).$$

So by Lemma 4.4,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

holds. □

Lemma 4.6. [5] *Let (x_n) be a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Theorem 4.2. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive, then $F(T)$ is nonempty.*

Proof. By Lemma 4.6, the asymptotic center of any bounded sequence is in K, particularly, the asymptotic center of approximate fixed point sequence for T is in K. Let $A((x_n)) = \{y\}$, we want to show that y is a fixed point of T. In order to prove this, one can write

$$d(x_n, Ty) = d(T^2 x_n, Ty) \leq d(Tx_n, y) = d(x_n, y)$$

therefore

$$\limsup_n d(x_n, Ty) \leq \limsup_n d(x_n, y).$$

By the uniqueness of the asymptotic center $Ty = y$. □

Corollary 4.7. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ is fundamentally nonexpansive, then $F(T)$ is nonempty, Δ-closed and convex.*

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