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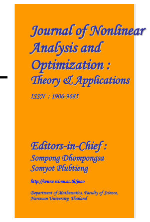

GENERALIZED MINIMAX FRACTIONAL PROGRAMMING PROBLEMS WITH GENERALIZED NONSMOOTH $(F, \alpha, \rho, d, \theta)$-UNIVEX FUNCTIONS

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ABSTRACT. The aim of this paper is to establish the sufficient optimality conditions for a class of nondifferentiable multiobjective generalized minimax fractional programming problems involving $(F, \alpha, \rho, d, \theta)$-univex functions. Subsequently, we apply the optimality condition to formulate a dual model and prove weak, strong and strict converse duality theorems.

KEYWORDS: Generalized minimax fractional programming; $(F, \alpha, \rho, d, \theta)$-univexity; Sufficient optimality conditions; Duality.
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1. INTRODUCTION

Fractional programming is a nonlinear programming method that has knows increasing exposure in the last few decades. Interest of this subject was generated by the fact that various optimization problems from engineering and economics consider the minimization of a ratio between physical and/or economical functions, for example cost/time, cost/volume, cost/profit, or other quantities that measure the efficiency of a system. For example, the productivity of industrial systems, defined as ratio between the realized services in a system within a given period of time and the utilized resources, is used as one of the best indicators of the quality of their operation. See Stancu-Minasian's book [21] which contains the state-of-the art theory and practice developments.

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Consider the following multiobjective generalized fractional programming problem [7]:

$$\text{(GFPP)} \qquad \begin{cases} \min\ E(x) = \big(E_1(x), E_2(x), ..., E_p(x)\big)^T, \\ \text{subject to} \\ g(x) = \big(g_1(x), g_2(x), ..., g_r(x)\big)^T \leq 0, \\ x \in X, \end{cases}$$

where $E_i(x) = \max_{y \in Y} \dfrac{f_i(x, y) + \Phi_i(x)}{h_i(x, y) - \Psi_i(x)}, i = 1, 2, ..., p.$

In addition, X is a closed convex subset of R^n and Y is a compact subset of R^m, $f_i(x, y) : X \times Y \to R$, $h_i(x, y) : X \times Y \to R$, $g : R^n \to R^r$, $\nabla_x f_i(x, y)$ and $-\nabla_x h_i(x, y)$ exist and are continuous with respect to (x, y) for $i = 1, 2, ..., p$, $f_i(x, y)$ and $-h_i(x, y)$ are upper semicontinuous functions with respect to y on Y for $i = 1, 2, ..., p$, g is a locally Lipschitz function on X, $\Phi_i(x), \Psi_i(x) : R^n \to R$ are convex functions on X for $i = 1, 2, ..., p$, $f_i(x, y) + \Phi_i(x) \geq 0, h_i(x, y) - \Psi_i(x) > 0$, $\forall (x, y) \in R^n \times Y, i = 1, 2, ..., p.$

Minimax fractional programming problems have been widely reviewed by many authors and several approaches for sufficient optimality conditions and duality theorems have been studied under different kinds of generalized convexity, see for example [1, 2, 5, 8, 11, 15, 16, 19, 20, 23], and the references therein.

Liang et al. [17] introduced the concept of (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results for the single objective fractional problem. Also, Liang et al. [18] extended their results to multiobjective fractional programs. Ahmad and Husain [1, 2] obtained sufficient optimality conditions and duality theorems for a class of nondifferentiable minimax fractional programming problems under generalized (F, α, ρ, d)-convexity assumptions. Later on, Ahmad [3] extended the work Ahmad and Husain [1, 2] to establish second order duality results for the nondifferentiable minimax fractional programming problem under the assumptions of generalized second order (F, α, ρ, d)-convexity.

On the other hand, Bector et al. [4] defined a new class of function called univex functions in nonlinear programming, which were further generalized by several researcher, and obtained optimality and duality results for a nonlinear multiobjective programming problem. Jayswal [11] focus his study on a nondifferentiable minimax fractional programming problem and established sufficient optimality conditions and duality theorems under the assumption of generalized α-univexity. Gupta et al. [9] obtained duality results for two types of second-order dual models of a nondifferentiable minimax fractional programming problem involving second-order α-univex functions.

Recently, Zheng and Cheng [23] given the concept of generalized (F, ρ, θ)-d-univexity in the setting of Clarke's derivative and derived Kuhn-Tucker type sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional problem with inequality constraints and its three different types of dual problems.

The notion of (V, ρ)-invexity for vector-valued functions was introduced by Kuk et al. [14], which is generalization of the V-invex function given in [13]. Very recently, Tong and Zheng [22] introduced the concept of generalized $(F, \alpha, \rho, \theta)$-$d$-$V$-univex functions involving locally Lipschitz functions and established some alternatives theorems and saddle point necessary optimality conditions for properly efficient solutions of vector optimization problems.

Gao and Rong [7] established Karush-Kuhn-Tucker type necessary conditions for the generalized fractional programming problem (GFPP). Moreover, they also formulated two kinds of dual models for (GFPP) and obtained sufficient optimality conditions and duality theorems under the assumptions of generalized $(F, \alpha, \rho, \theta)$-$V$-convexity.

In this paper, inspired from the work of Ahmad and Husain [1, 2], Gao and Rong [7], Tong and Zheng [22] and Zheng and Cheng [23], we established sufficient optimality conditions and duality theorems for generalized minimax fractional programming problem (GFPP) involving $(F, \alpha, \rho, d, \theta)$-univex functions.

The paper is organized as follow. Some definition and notations are given in Section 2. In Section 3, we derive sufficient optimality conditions for nondifferentiable minimax fractional programming problems under the assumption of generalized $(F, \alpha, \rho, d, \theta)$-univexity. After utilized the optimality condition, a dual problem is formulated and duality results are presented in Section 4. Concluding remarks are presented in Section 5.

2. PRELIMINARIES AND NOTATIONS

Let R^n be the n-dimensional Euclidean space and R^n_+ its non-negative orthant. For $x, y \in R^n$, we let $x \leq y \Leftrightarrow y - x \in R^n_+$; $x < y \Leftrightarrow y - x \in R^n_+ \setminus \{0\}$.

Let $S = \{x \in X : g(x) \leq 0\}$ be the set of all feasible solutions to (GFPP). For each $x \in S$, we define

$$I(x) = \{j : g_j(x) = 0, j = 1, 2, ..., r\},$$

$$Y_i(x) = \left\{ y \in Y : \frac{f_i(x, y) + \Phi_i(x)}{h_i(x, y) - \Psi_i(x)} = \max_{z \in Y} \frac{f_i(x, z) + \Phi_i(x)}{h_i(x, z) - \Psi_i(x)} \right\}, i = 1, 2, ..., p,$$

$$K(x) = \Big\{(s, \hat{t}, \hat{y}) \in N \times R^{s \times p} \times R^{p \times m \times s} : 1 \leq s \leq n + 1, \hat{t} = (t^1, t^2, ..., t^p),$$
$$t^i = (t^i_1, t^i_2, ..., t^i_s)^T \geq 0, \sum_{i=1}^{s} t^i_l = 1, \hat{y} = (y^1, y^2, ..., y^p)^T,$$
$$y^i = (y^i_1, y^i_2, ..., y^i_s), y^i_l \in Y_i(x), l = 1, 2, ..., s, i = 1, 2, ..., p\Big\}.$$

Definition 2.1. A feasible point $\bar{x}$ is said to be an efficient solution of the multiobjective generalized fractional programming problem (GFPP) if there exists no other feasible x such that

$$E_i(x) \leq E_i(\bar{x}), \text{ for all } i \in P = \{1, 2, ..., p\},$$
$$E_k(x) < E_k(\bar{x}), \text{ for at least one } k \neq i.$$

Definition 2.2. [6] The function $f : X \to R$ is said to be locally Lipschitz on X if for each bounded subset B of X, there exists a constant K such that

$$|f(y) - f(x)| \leq K \|y - x\|, \text{ for all points } y \text{ and } x \text{ of } B,$$

where $\|.\|$ denotes the Euclidean norm.

For the function f Lipschitzian on X , Clarke defined the *generalized directional derivative* of f at a point $x \in X$ in the direction $\nu \in R^n$ by

$$f^0(x; \nu) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda\nu) - f(y)}{\lambda}.$$

Also, he defined the subdifferential (or generalized gradient) of the function f at a point x by the unique, nonempty, convex and compact set

$$\partial f(x) = \left\{\xi \in R^n | f^0(x;\nu) \geq \xi^T \nu, \forall \nu \in R^n\right\}.$$

The elements of $\partial f(x)$ are called subgradients.
It then follows that

$$f^0(x;\nu) = \max\left\{\xi^T \nu | \xi \in \partial f(x)\right\}, \text{ for any } x \text{ and } \nu.$$

We remark that when the function f is smooth (continuously differentiable), $\partial f(x)$ is the singleton set $\{\nabla f(x)\}$ and when f is convex, $\partial f(x)$ coincides with the sub-differential of convex functions.

Definition 2.3. A functional $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument, if for $\forall x, \bar{x} \in X$

(i) $F(x, \bar{x}; a_1 + a_2) \leq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2), \quad \forall a_1, a_2 \in R^n,$
(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a), \quad \forall \alpha \in R_+, a \in R^n.$

By (ii), it is clear that $F(x, \bar{x}; 0) = 0$.

To impose the convexity assumptions in the above problem (GFPP), we propose the following definition. Let $f : X \to R$ be a locally Lipschitz and $F : X \times X \times R^n \to R$ be a sublinear functional. Also let $\alpha : X \times X \to R_+ \setminus \{0\}$, $b : X \times X \to R_+$, $\theta : X \times X \to R_+$ such that $x \neq y \Rightarrow \theta(x, y) \neq 0$, $d : R \to R$ with the property that $d(0) = 0$, $\phi : R \to R$ and ρ is a real number.

Definition 2.4. The function f is said to be $(F, \alpha, \rho, d, \theta)$-univex at $y \in X$ with respect to b and ϕ, if the inequality

$$b(x, y)\phi[f(x) - f(y)] \geq F(x, y; \alpha(x, y)\xi) + \rho d^2(\theta(x, y)),$$

holds, for each $x \in X$ and $\xi \in \partial f(y)$.

The function f is said to be $(F, \alpha, \rho, d, \theta)$-univex on X with respect to b and ϕ if it is $(F, \alpha, \rho, d, \theta)$-univex at any point $y \in X$ with respect to the same b and ϕ. In particular, f is said to be strongly $(F, \alpha, \rho, d, \theta)$-univex or (F, α)-univex if $\rho > 0$ or $\rho = 0$, respectively.

It has been revealed in [22] by means of an example that the above class of functions is an extension of F-convex function [10] or η-invex function [12].
Let C be a nonempty subset of X and $d_c(.) : X \to R$ its distance function,

$$d_c(x) = \inf\left\{ \|x - c\| : c \in C\right\}.$$

Throughout the paper, we assume that the sublinear functional F satisfies the following condition D.

Condition D: Let sublinear functional $F : X \times X \times R^n \to R$ satisfy the following relation for some

$$K > 0, K\partial d_x(\bar{x}) \subset \left\{\epsilon \in R^n : F(x, \bar{x}; \epsilon) \leq 0, \forall x \in X\right\}.$$

The following result from [7] is needed in the sequel.

Theorem 2.5. *Let $\bar{x}$ be an efficient solution for (GFPP). If (GFPP) satisfies Calmness Constraints Qualification [6] at $\bar{x}$, in other words, for every $i \in \{1, 2, ..., p\}$, the following problem*

$$(P)_i \qquad \textit{Min } E_i(x)$$

subject to

$$E_k(x) - E_k(\bar{x}) \leq 0, k \neq i, x \in S = \{x \in X : g(x) \leq 0\},$$

satisfies Calmness Constraints Qualification at $\bar{x}$*, then there exist*

$$(s,t,y) \in K(\bar{x}), \lambda \in R^p, \bar{u} \in R^r_+, \bar{e} \in R^p_+,$$

and $K > 0$ *such that*

$$0 \in \sum_{i=1}^{p} \lambda_i \left\{ \sum_{l=1}^{s} t_l^i \left(\nabla_x f_i(\bar{x}, y_l^i) - \bar{e}_i \nabla_x h_i(\bar{x}, y_l^i) \right) + \partial \Phi_i(\bar{x}) + \bar{e}_i \partial \Psi_i(\bar{x}) \right\}$$

$$+ \sum_{j=1}^{r} \bar{u}_j \partial g_j(\bar{x}) + K \partial d_x(\bar{x}), \tag{2.1}$$

$$f_i(\bar{x}, y_l^i) - \bar{e}_i h_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) + \bar{e}_i \Psi_i(\bar{x}) = 0, i = 1, 2, ..., p, l = 1, 2, ..., s, \tag{2.2}$$

$$\sum_{j=1}^{r} \bar{u}_j g_j(\bar{x}) = 0, \tag{2.3}$$

$$\sum_{l=1}^{s} t_l^i = 1, t_l^i \geq 0, i = 1, 2, ..., p, l = 1, 2, ..., s, \tag{2.4}$$

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda_i > 0, i = 1, 2, ..., p. \tag{2.5}$$

Throughout the paper we denote

$$H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(., y_l^i) + \Phi_i(.) - e_i h_i(., y_l^i) + e_i \Psi_i(.) \right\}.$$

3. SUFFICIENT OPTIMALITY CONDITION

In this section, we shall establish a sufficient optimality condition involving generalized convexity assumptions discussed in the previous section.

Theorem 3.1 (Sufficient optimality conditions). *Let* $\bar{x}$ *be a feasible solution to (GFPP). Assume that there exist* $(s,t,y) \in K(\bar{x}), \lambda \in R^p, u \in R^r_+, e \in R^p_+$*, and* $K > 0$ *satisfying the relations (2.1)-(2.5). Assume also that* $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{ f_i(., y_l^i) + \Phi_i(.) - e_i(h_i(., y_l^i) - \Psi_i(.)) \}$ *is* $(F, \alpha_1, \rho_1, d, \theta)$*-univex at* $\bar{x}$ *with respect to* b_0 *and* ϕ_0 *with* $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\sum_{j=1}^{r} u_j g_j(.)$ *is* $(F, \alpha_2, \rho_2, d, \theta)$*-univex at* $\bar{x}$ *with respect to* b_1 *and* ϕ_1 *with* $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$*. Furthermore, assume*

$$\frac{\rho_1}{\alpha_1(x,\bar{x})} + \frac{\rho_2}{\alpha_2(x,\bar{x})} \geq 0. \tag{3.1}$$

Then $\bar{x}$ *is an efficient solution to (GFPP).*

Proof. Suppose to the contrary that $\bar{x}$ is not an efficient solution of (GFPP). Then there exists $x \in S$ such that

$$E_i(x) \leq E_i(\bar{x}) = e_i, \text{ for all } i \in P,$$

$$E_k(x) < E_k(\bar{x}) = e_k, \text{ for at least one } k \neq i.$$

Since $\lambda > 0, \sum_{i=1}^{p} \lambda_i = 1$, we have

$$\sum_{i=1}^{p} \lambda_i \left\{ \max_{y\in Y} \{f_i(x,y) + \Phi_i(x) - e_i(h_i(x,y) - \Psi_i(x))\} \right\} < 0.$$

The above inequality together with (2.2), (2.4) and $y_l^i \in Y_i(\bar{x}), l = 1, 2, ..., s, i = 1, 2, ..., p$, yield

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0$$

$$= \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i h_i(\bar{x}, y_l^i) + e_i \Psi_i(\bar{x}) \right\}.$$

That is,

$$H(x) - H(\bar{x}) < 0.$$

Since $b_0(x, \bar{x}) > 0$ and $V < 0 \Rightarrow \phi_0(V) < 0$, we get

$$b_0(x, \bar{x})\phi_0(H(x) - H(\bar{x})) < 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ at $\bar{x}$, we obtain

$$0 > b_0(x, \bar{x})\phi_0(H(x) - H(\bar{x}))$$

$$\geq F\left(x, \bar{x}; \alpha_1(x, \bar{x}) \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i) \right) + \rho_1 d^2(\theta(x, \bar{x})),$$

$$\forall \xi_l^i \in \nabla_x f_i(\bar{x}, y_l^i) + \partial \Phi_i(\bar{x}), \forall \eta_l^i \in -\nabla_x h_i(\bar{x}, y_l^i) + \partial \Psi_i(\bar{x}),$$

$$l = 1, 2, ..., s, i = 1, 2, ..., p.$$

Since $\alpha_1(x, \bar{x}) > 0$, by the sublinearity of F, we obtain

$$F\left(x, \bar{x}; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i) \right) + \frac{\rho_1 d^2(\theta(x, \bar{x}))}{\alpha_1(x, \bar{x})} < 0,$$

$$\forall \xi_l^i \in \nabla_x f_i(\bar{x}, y_l^i) + \partial \Phi_i(\bar{x}), \forall \eta_l^i \in -\nabla_x h_i(\bar{x}, y_l^i) + \partial \Psi_i(\bar{x}),$$

$$l = 1, 2, ..., s, i = 1, 2, ..., p. \tag{3.2}$$

By the feasibility of x and from (2.3), we have

$$\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x})) \leq 0.$$

Since $b_1(x, \bar{x}) \geq 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, from the above inequality, we get

$$b_1(x, \bar{x})\phi_1 \left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x})) \right) \leq 0.$$

From $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at $\bar{x}$, we obtain

$$0 \geq b_1(x, \bar{x})\phi_1 \left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x})) \right) \geq F\left(x, \bar{x}; \alpha_2(x, \bar{x}) \sum_{j=1}^{r} u_j \gamma_j \right)$$

$$+\rho_2 d^2(\theta(x, \bar{x})), \quad \forall \gamma_j \in \partial g_j(\bar{x}), j = 1, 2, ..., r.$$

Since $\alpha_2(x,\bar{x}) > 0$, by the sublinearity of F, we obtain

$$F\left(x,\bar{x};\sum_{j=1}^{r}u_j\gamma_j\right)+\frac{\rho_2 d^2(\theta(x,\bar{x}))}{\alpha_2(x,\bar{x})}\leq 0,\forall\gamma_j\in\partial g_j(\bar{x}),j=1,2,...,r. \tag{3.3}$$

On adding (3.2), (3.3) and with the sublinear functional F satisfying condition D, we get

$$F\left(x,\bar{x};\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i+e_i\eta_l^i)+\sum_{j=1}^{r}u_j\gamma_j+K\eta\right)$$
$$+\left(\frac{\rho_1}{\alpha_1(x,\bar{x})}+\frac{\rho_2}{\alpha_2(x,\bar{x})}\right)d^2(\theta(x,\bar{x}))<0,$$
$$\forall\xi_l^i\in\nabla_x f_i(\bar{x},y_l^i)+\partial\Phi_i(\bar{x}),\forall\eta_l^i\in-\nabla_x h_i(\bar{x},y_l^i)+\partial\Psi_i(\bar{x}),$$
$$\forall\gamma_j\in\partial g_j(\bar{x}),\forall\eta\in\partial d_x(\bar{x}),l=1,2,...,s,i=1,2,...,p,j=1,2,...,r.$$

By the assumption $\dfrac{\rho_1}{\alpha_1(x,\bar{x})}+\dfrac{\rho_2}{\alpha_2(x,\bar{x})}\geq 0$, we have

$$F\left(x,\bar{x};\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i+e_i\eta_l^i)+\sum_{j=1}^{r}u_j\gamma_j+K\eta\right)<0,$$
$$\forall\xi_l^i\in\nabla_x f_i(\bar{x},y_l^i)+\partial\Phi_i(\bar{x}),\forall\eta_l^i\in-\nabla_x h_i(\bar{x},y_l^i)+\partial\Psi_i(\bar{x}),$$
$$\forall\gamma_j\in\partial g_j(\bar{x}),\forall\eta\in\partial d_x(\bar{x}),l=1,2,...,s,i=1,2,...,p,j=1,2,...,r,$$

which contradicts (2.1). This completes the proof. □

Corollary 3.2. *Let $\bar{x}$ be a feasible solution to (GFPP). Assume that there exist $(s,t,y)\in K(\bar{x}),\lambda\in R^p,u\in R^r_+,e\in R^p_+$, and $K>0$ satisfying the relations (2.1)-(2.5). Assume also that $H(.)=\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i\left\{f_i(.,y_l^i)+\Phi_i(.)\ -e_i(h_i(.,y_l^i)-\Psi_i(.))\right\}$ is strongly $(F,\alpha_1,\rho_1,d,\theta)$-univex at $\bar{x}$ with respect to b_0 and ϕ_0 with $b_0>0,V<0\Rightarrow\phi_0(V)<0$ and $\sum_{j=1}^{r}u_jg_j(.)$ is strongly $(F,\alpha_2,\rho_2,d,\theta)$-univex at $\bar{x}$ with respect to b_1 and ϕ_1 with $b_1\geq 0$, $V\leq 0\Rightarrow\phi_1(V)\leq 0$. Then $\bar{x}$ is an efficient solution to (GFPP).*

Proof. Under the assumptions of this corollary, we know that inequality (3.1) holds. Therefore, $\bar{x}$ is an efficient solution to (GFPP). □

4. DUALITY MODEL

In this section, we consider the following dual for (GFPP) and establish weak, strong and strict converse duality results.

$$\text{(GFMD)}\qquad \max_{(s,t,y)\in K(z)}\ \sup_{(z,\lambda,u,e,K)\in H_1(s,t,y)}\quad e=(e_1,e_2,...,e_p)^T,$$

subject to

$$0\in\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i\left\{\nabla_x f_i(z,y_l^i)+\partial\Phi_i(z)-e_i\left(\nabla_x h_i(z,y_l^i)-\partial\Psi_i(z)\right)\right\}$$

$$+\sum_{j=1}^{r} u_j \partial g_j(z) + K\partial d_x(z), \tag{4.1}$$

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(z, y_l^i) + \Phi_i(z) - e_i \left(h_i(z, y_l^i) - \Psi_i(z) \right) \right\} \geq 0, \tag{4.2}$$

$$\sum_{j=1}^{r} u_j g_j(z) \geq 0, \tag{4.3}$$

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda > 0, e \geq 0, u \geq 0, K > 0,$$

where $H_1(s,t,y) = \{(z, \lambda, u, e, K) \in R^n \times R^p \times R^r \times R^p \times R\}$.

Theorem 4.1 (Weak duality). *Let x and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{ f_i(., y_l^i) + \Phi_i(.)$ $-e_i(h_i(., y_l^i) - \Psi_i(.))\}$ is $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, and*

$$\frac{\rho_1}{\alpha_1(x,z)} + \frac{\rho_2}{\alpha_2(x,z)} \geq 0. \tag{4.4}$$

Then the following can not hold:

$$E_i(x) \leq e_i, \text{ for } i = 1, 2, ..., p,$$

and

$$E_k(x) < e_k, \text{ for at least one } k \in \{1, 2, ..., p\}.$$

Proof. Suppose to the contrary that $E(x) < e$, then we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0.$$

The above inequality together with (4.2) and $y_l^i \in Y_i(x)$ for $l = 1, 2, ..., s, i = 1, 2, ..., p$, yield

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0$$

$$\leq \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(z, y_l^i) + \Phi_i(z) - e_i h_i(z, y_l^i) + e_i \Psi_i(z) \right\}.$$

That is,

$$H(x) - H(z) < 0.$$

Since $b_0(x,z) > 0$ and $V < 0 \Rightarrow \phi_0(V) < 0$, we get

$$b_0(x,z)\phi_0(H(x) - H(z)) < 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ at z, we obtain

$$0 > b_0(x,z)\phi_0(H(x) - H(z))$$

$$\geq F\left(x, z; \alpha_1(x,z) \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i) \right) + \rho_1 d^2(\theta(x,z)),$$

$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial \Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial \Psi_i(z),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p.$$

Since $\alpha_1(x, z) > 0$, by the sublinearity of F, we obtain

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i)\right) + \frac{\rho_1 d^2(\theta(x, z))}{\alpha_1(x, z)} < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial \Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial \Psi_i(z),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p. \tag{4.5}$$

Utilizing the feasibility of x and (4.3), we have

$$\sum_{j=1}^{r} u_j (g_j(x) - g_j(z)) \leq 0.$$

Since $b_1(x, z) \geq 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, we get

$$b_1(x, z) \phi_1 \left(\sum_{j=1}^{r} u_j (g_j(x) - g_j(z))\right) \leq 0.$$

From $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at z, we obtain

$$0 \geq b_1(x, z) \phi_1 \left(\sum_{j=1}^{r} u_j (g_j(x) - g_j(z))\right) \geq F\left(x, z; \alpha_2(x, z) \sum_{j=1}^{r} u_j \gamma_j\right)$$
$$+ \rho_2 d^2(\theta(x, z)), \quad \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r.$$

Since $\alpha_2(x, z) > 0$, by the sublinearity of F, we obtain

$$F\left(x, z; \sum_{j=1}^{r} u_j \gamma_j\right) + \frac{\rho_2 d^2(\theta(x, z))}{\alpha_2(x, z)} \leq 0, \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r. \tag{4.6}$$

On adding (4.5), (4.6) and with the sublinear functional F satisfying condition D, we get

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j \gamma_j + K\eta\right)$$
$$+ \left(\frac{\rho_1}{\alpha_1(x, z)} + \frac{\rho_2}{\alpha_2(x, z)}\right) d^2(\theta(x, z)) < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial \Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial \Psi_i(z),$$
$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r.$$

By the assumption $\dfrac{\rho_1}{\alpha_1(x, z)} + \dfrac{\rho_2}{\alpha_2(x, z)} \geq 0$, we have

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j \gamma_j + K\eta\right) < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial \Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial \Psi_i(z),$$
$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r,$$

which contradicts the relation (4.1). Therefore the proof is completed. □

Corollary 4.2. *Let x and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(., y_l^i) + \Phi_i(.) \right. -e_i(h_i(., y_l^i) \left. -\Psi_i(.)) \right\}$ is strongly $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is strongly $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$. Then the following can not hold:*

$$E_i(x) \leq e_i, \text{ for } i = 1, 2, ..., p,$$

and

$$E_k(x) < e_k, \text{ for at least one } k \in \{1, 2, ..., p\}.$$

Proof. Under the assumptions of this corollary, we know that inequality (4.4) holds. So, we get the corollary from Theorem 4.1. □

Theorem 4.3 (Strong duality)**.** *Assume that $\bar{x}$ is efficient solution to (GFPP) and let (GFPP) satisfies Calmness Constraints Qualification [6] at $\bar{x}$. Then, there exist $\bar{\lambda} \in R^p, \bar{u} \in R^r, \bar{e} \in R^p_+, (\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$, and $\bar{K} > 0$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is feasible solution to (GFMD). Further, if the hypothesis of weak duality theorem 4.1 holds for all feasible $(z, \lambda, u, e, K, s, t, y)$ to (GFMD), then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is an efficient solution to (GFMD) and the two objectives have the same optimal values.*

Proof. By Theorem 2.5, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}) \in H_1(\bar{s}, \bar{t}, \bar{y})$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is feasible for (GFMD). Since (GFPP) and (GFMD) have the same objective values, the optimality of this feasible solution follows from weak duality Theorem 4.1. □

Theorem 4.4 (Strict converse duality)**.** *Let $\bar{x}$ and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(., y_l^i) \right. +\Phi_i(.) -e_i(h_i(., y_l^i) - \Psi_i(.)) \left. \right\}$ is $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, and let the inequalities*

(a) $\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x})) \right\} < 0,$

(b) $\dfrac{\rho_1}{\alpha_1(\bar{x}, z)} + \dfrac{\rho_2}{\alpha_2(\bar{x}, z)} \geq 0,$

hold. Then, $\bar{x} = z$; that is, z is optimal to (GFPP).

Proof. Suppose to the contrary that $\bar{x} \neq z$. By the feasibility of $\bar{x}$ and $(z, \lambda, u, e, K, s, t, y)$ to (GFPP) and (GFMD), respectively and the hypothesis (a), we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x})) \right\} < 0$$
$$\leq \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(z, y_l^i) + \Phi_i(z) - e_i \left(h_i(z, y_l^i) - \Psi_i(z) \right) \right\},$$

and

$$\sum_{j=1}^{r} u_j (g_j(\bar{x}) - g_j(z)) \leq 0.$$

That is,

$$H(\bar{x}) - H(z) < 0,$$

and

$$\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z)) \leq 0.$$

Since $b_0(\bar{x}, z) > 0$, $b_1(\bar{x}, z) \geq 0$, $V < 0 \Rightarrow \phi_0(V) < 0$, and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, we get

$$b_0(\bar{x}, z)\phi_0(H(\bar{x}) - H(z)) < 0,$$

and

$$b_1(\bar{x}, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z))\right) \leq 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ and $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at z, we have

$$\begin{aligned} 0 &> b_0(\bar{x}, z)\phi_0(H(\bar{x}) - H(z)) \\ &\geq F\left(\bar{x}, z; \alpha_1(\bar{x}, z)\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s} t_l^i(\xi_l^i + e_i\eta_l^i)\right) + \rho_1 d^2(\theta(\bar{x}, z)), \\ &\qquad \forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z), \\ &\qquad l = 1, 2, ..., s, i = 1, 2, ..., p, \end{aligned}$$

and

$$\begin{aligned} 0 \geq b_1(\bar{x}, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z))\right) &\geq F\left(\bar{x}, z; \alpha_2(\bar{x}, z)\sum_{j=1}^{r} u_j\gamma_j\right) \\ &+ \rho_2 d^2(\theta(\bar{x}, z)), \quad \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r. \end{aligned}$$

Since $\alpha_1(\bar{x}, z) > 0$, $\alpha_2(\bar{x}, z) > 0$, by the sublinearity of F, above inequalities imply

$$\begin{aligned} F\left(\bar{x}, z; \sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s} t_l^i(\xi_l^i + e_i\eta_l^i)\right) + \frac{\rho_1 d^2(\theta(\bar{x}, z))}{\alpha_1(\bar{x}, z)} < 0, \\ \forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z), \\ l = 1, 2, ..., s, i = 1, 2, ..., p, \end{aligned} \tag{4.7}$$

and

$$F\left(\bar{x}, z; \sum_{j=1}^{r} u_j\gamma_j\right) + \frac{\rho_2 d^2(\theta(\bar{x}, z))}{\alpha_2(\bar{x}, z)} \leq 0, \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r. \tag{4.8}$$

On adding (4.7), (4.8) and with the sublinear functional F satisfying condition D, we get

$$\begin{aligned} F\left(\bar{x}, z; \sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s} t_l^i(\xi_l^i + e_i\eta_l^i) + \sum_{j=1}^{r} u_j\gamma_j + K\eta\right) \\ + \left(\frac{\rho_1}{\alpha_1(\bar{x}, z)} + \frac{\rho_2}{\alpha_2(\bar{x}, z)}\right) d^2(\theta(\bar{x}, z)) < 0, \\ \forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z), \\ \forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r. \end{aligned}$$

By the assumption $\frac{\rho_1}{\alpha_1(\bar{x},z)} + \frac{\rho_2}{\alpha_2(\bar{x},z)} \geq 0$, we have

$$F\left(\bar{x}, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j \gamma_j + K\eta\right) < 0,$$

$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$

$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r,$$

which contradicts the relation (4.1). Therefore the proof is completed. □

5. CONCLUDING REMARK

This paper addressed the sufficient optimality conditions for generalized minimax fractional programming problems involving generalized $(F, \alpha, \rho, d, \theta)$-univex function. For the class of problems, we formulated a dual model and proved weak, strong and strict converse duality theorems. The question arises whether the second and higher order dual and duality theorems for the considered problems hold. This would be the task of our forthcoming works.

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