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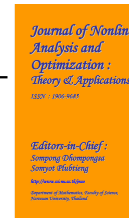


# RECENT FIXED POINT THEOREMS FOR T-CONTRACTIVE MAPPINGS AND T-WEAK (ALMOST) CONTRACTIONS IN METRIC AND CONE METRIC SPACES ARE NOT REAL GENERALIZATIONS

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**ABSTRACT.** The purpose of this research article is to show that recent fixed point theorems obtained in metric and cone metric spaces for T-contractive mappings and TW-contractions are equivalent to previously existing theorems in the literature; hence are redundant. We also show that Proposition 2.5 of [4] is invalid

**KEYWORDS**: Fixed point theorem; $T$-contraction; $T$-contractive mapping; $T$-Kannan mapping; TCaterjea mapping; $T-$Zamfirescu mapping; $T-$weak contraction; Cone metric space.
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## 1. INTRODUCTION

In 2007 Huang and Zhang [10] introduced the notion of cone metric spaces, replacing the set R of real numbers by an ordered real Banach space E as the codomain of a metric and they defined several notions related to sequences in cone metric spaces and proved their properties. They also generalized the famous Banach contraction principle and Kannan's fixed point theorem to such spaces. Subsequently, many authors studied fixed point theory of various kinds of self mappings defined on cone metric spaces.

Many authors [17–24] have noticed that fixed point results in cone metric spaces can be obtained from the existing results in the usual metric space setting. For instance Du [17] obtained the equivalence between three fixed point theorems and their metric space versions. I.D. Arandelovic and D.J keckic [24] also proved that a large number of generalizations of fixed point results to topological vector space valued cone metric spaces (and hence to cone metric spaces) are not real generalizations but a complicated way to formulate a result that is a special case of an old one. But none of these authors proved that all fixed point results that are

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provable in cone metric spaces are reducible to (or obtainable from) corresponding fixed point results in metric spaces.

In [5] Berinde introduced the notion of weak contraction which he renamed later as almost contraction in [6] and proved that such mappings have a fixed point. But the fixed point may not be unique. The class of almost contractions includes Kannan mappings, Chaterjea mappings, Zamfirescu mappings and some quasi-contractions as special cases.

In [4], A. Beiranvand, S. Moradi, M.Omid and H. Pazadeh introduced the notion of T-Banach contraction and extended the Banach contraction principle [3] to such contraction types. In the same paper, they introduced the notion of T-contractive mapping and extended one of Edelstein fixed point theorems ([8], Remark 3.1) to T-contractive maps. In [12], S.Moradi introduced T-Kannan mappings and extended Kannan's fixed point theorem [11] to such maps. All these extensions were done in the setting of metric spaces.

In [13–15], R. Morales and E. Rojas studied T-contractive mappings, T-Kannan contractions, T-Chaterjea contractions, T-Zamfirescu contractions and T-weak (almost) contractions in cone metric spaces.

In [9], Haghi et al showed that some recent fixed point theorems which are supposed to be generalizations of previously existing theorems are not real generalizations. Also, Aydi et al [2] showed that the fixed point theorem for TB-contractions which was obtained by Beiranvand et al [4] is equivalent to the Banach contraction principle.

In this paper we show that the fixed point theorem recently obtained for T-contractive mappings in metric spaces ([4], Theorem 2.9) is equivalent to one of Edelstein's fixed point theorems ([8], Remark 3.1). Secondly, we show that the fixed point theorem for T-weak contractions in cone metric spaces ([15], Theorem 3.4) is equivalent to the cone metric space version of Berinde's fixed point theorem for almost contractions ([1], Theorem 2.1 and [6], Theorem 1). As a corollary of the later, we show that the fixed point theorems obtained in [13–15] for T-Kannan, T-Chaterjea and T-Zamfirescu mappings are not real generalizations. We also provide a counter example to disprove Proposition 2.5 of [4].

## 2. PRELIMINARIES AND NOTATIONS

**Definition 2.1.** ([10]) Let $E$ be a real Banach space and $P \subseteq E$. The subset $P$ of $E$ is called a cone if:

(i) P is closed, nonempty and nontrivial (i.e., $P \neq \{0\}$);
(ii) ax + by $\in$ P for all $x, y \in P$ and nonnegative real numbers a and b;
(iii) P $\bigcap$ (-P) = {0}.

A cone P of a real Banach space E induces a partial ordering on E as follows. Define $x \preceq y$ if and only if $y - x \in P$ for every $x, y \in E$. Then $\preceq$ is a partial ordering on E.

**Notation**: For x, y $\in$ E, we write $x \prec y$ if $x \preceq y$ and $x \neq y$. Likewise, we write $x \ll y$ if $y - x \in intP$, where int P denotes the interior of P.

**Definition 2.2.** ([10]): Let E be a real Banach space and $P \subseteq E$ be a cone. The cone P is called normal if there is a number $K > 1$ such that for all $x, y \in E, 0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying the above inequality is called the normal constant of P. The cone P is called a solid cone if $intP \neq \phi$.

Throughout this paper let E denote a real Banach space and P denote a solid cone of E. Moreover, let $\preceq$ represents the partial ordering on E induced by P.

**Definition 2.3.** ([10]) Let X be a nonempty set. Suppose a mapping $d : X \times X \to E$ satisfies

(d1) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) d(x, y) = d(y, x) and
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric for X and the pair (X, d) is called a cone metric space.

**Definition 2.4.** ([10]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}, n = 1, 2, \cdots$ be a sequence in X. Then we say

(i) $\{x_n\}$ converges to x if for every $c \in intP$ there exists a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.
(ii) $\{x_n\}$ is a Cauchy sequence if for every $c \in intP$ there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence in (X, d) is convergent in (X, d).

**Definition 2.5.** [4, 12–15] Let (X, d) be a cone metric space and T, S be two self maps of X. The mapping S is said to be:

(i) T-Banach contraction (TB- contraction) if there exists $k \in [0, 1)$

$$d(TSx, TSy) \preceq kd(Tx, Ty). \tag{2.1}$$

(ii) T- Contractive mapping if

$$d(TSx, TSy) \prec d(Tx, Ty) \forall x, y \in X with\ x \neq y. \tag{2.2}$$

(iii) T-Kannan contraction (TK- contraction) if there exists $b \in [0, 1/2)$

$$d(TSx, TSy) \preceq b[d(Tx, TSx) + d(Ty, TSy)]. \tag{2.3}$$

(iv) T- Chaterjea contraction (TC -contraction) if there exists $c \in [0, 1/2)$

$$d(TSx, TSy) \preceq c[d(Tx, TSy) + d(Ty, TSx)] \forall x, y \in X. \tag{2.4}$$

(v) T- Zamfirescu contraction (TZ- contraction) if there are real numbers a, b and c with $0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$ such that for all $x, y \in X$ at least one of the following conditions hold:

$$\begin{aligned} (TZ1) &: d(TSx, TSy) \preceq ad(Tx, Ty) \\ (TZ2) &: d(TSx, TSy) \preceq b[d(Tx, TSx) + d(Ty, TSy)] \\ (TZ3) &: d(TSx, TSy) \preceq c[d(Tx, TSy) + d(Ty, TSx)] \end{aligned} \tag{2.5}$$

(vi) T-Weak contraction(TW-contraction) if there exists real numbers $a \in (0, 1)$ and $b \geq 0$ such that

$$d(TSx, TSy) \preceq ad(Tx, Ty) + bd(Ty, TSx) \forall x, y \in X. \tag{2.6}$$

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**Proposition 2.6.** *([15]) Let (X, d) be a cone metric space and T, S be two self maps of X.*

(i) *If S is a TB-contraction, then S is a T-weak contraction.*
(ii) *If S is a TK-contraction, then S is a T-weak contraction.*

(iii) *If S is a TC-contraction, then S is a T-weak contraction.*
(iv) *If S is a TZ-contraction, then S is a T-weak contraction.*

**Definition 2.7.** ([13]) Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T : X \to X$. Then T is said to be

(i) continuous if for every sequence $(x_n)$ in X and x ∈ X, $lim_{n \longrightarrow \infty} x_n = x$ implies that
$$lim_{n \longrightarrow \infty} Tx_n = Tx;$$
(ii) sequentially convergent if the following holds: For every sequence $(y_n)$ in X, if T($y_n$) is convergent, so is $(y_n)$.
(iii) subsequentially convergent if we have, for every sequence $(y_n)$ in X, if $T(y_n)$ is convergent, then $(y_n)$ has a convergent subsequence.

**Theorem 2.8.** *([8]) Let (X, d) be a compact metric space and S be a self mapping of X satisfying the condition $d(Sx, Sy) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

A Cone metric space version of Theorem 2.8 is given in [10]. On the other hand, Beiranvand et al [4] extended Theorem 2.8 to T-contractive mappings as follows.

**Theorem 2.9.** *([4]) Let (X, d) be a compact metric space and S, T be self mappings of X such that T is injective, continuous and S is a T-contractive mapping. Then S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

Morales et al [13] extended Theorem 2.9 to cone metric spaces as follows.

**Theorem 2.10.** *([13])Let (X, d) be a compact cone metric space, P be a normal cone with normal constant K and $T, S : X \to X$ functions such that T is injective, continuous and S is T- contractive mapping. Then,*

(i) *S has a unique fixed point;*
(ii) *For any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point of S.*

Berinde proved the following theorem.

**Theorem 2.11.** *([5]) Let (X, d) be a complete metric space and S be a weak (almost) contraction. Then,*

(i) *S has a fixed point;*
(ii) *For any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to a fixed point of S. Further if, for some $\theta \in (0, 1)$ and $L_1 \in [0, \infty)$, S satisfies $d(TSx, TSy) \preceq \theta d(Tx, Ty) + L_1 d(Tx, TSy)$ for all x, y∈ $X$, then S has a unique fixed point.*

A Cone metric space version of Theorem 2.11 is given in [1]. It is stated as follows.

**Theorem 2.12.** *([1]) Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ a weak contraction (i.e., there exists a constant $a \in (0, 1)$ and some $b \geq 0$ such that $d(Tx, Ty) \preceq ad(x, y) + bd(y, Tx)$ for all $x, y \in X$. Then T has a fixed point in X.*

Morales et al proved the following result, which can be thought as a common generalization of Theorem 2.11 and Theorem 2.12.

**Theorem 2.13.** *([15]) Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K and $T, S : X \to X$ functions such that T is injective, continuous and S is a continuous T-weak contraction. Then,*

(i) *If T is subsequentially convergent, then S has a fixed point;*
(ii) *If T is sequentially convergent, then the sequence of iterates $\{S^n x_0\}$ converges to a fixed point of S for any $x_0 \in X$.*

## 3. MAIN RESULTS

We start with a disproof of proposition 2.5 of [4], which is stated as follows.

**Proposition 3.1.** *([4]) If (X, d) is a compact metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

**Disproof**: If X has at least two elements, then the proposition is invalid; in view of the following simple example. Let (X, d) be a compact metric space and T be a constant map on X (i.e., there exists $z \in X$ such that Tx = z $\forall x \in X$ ). Clearly, T is continuous. Consider the alternate sequence

$$y_n := \begin{cases} x & if\ n = 0, 2, 4, \cdots \\ y & if\ n = 1, 3, 5, \cdots ; \end{cases}$$

where x, y $\in X$ and $x \neq y$. Since $(Ty_n)$ is a constant sequence, so it is convergent. However, $(y_n)$ is not convergent. Thus, Proposition 2.5 of [4] is invalid. □

**Theorem 3.2.** *Theorem 2.8 is equivalent to Theorem 2.9.*

**Proof**:

Part I (Theorem 2.9 ⇒ Theorem 2.8):

If T is the identity mapping on X, then Theorem 2.9 reduces to Theorem 2.8.

Part II (Theorem 2.8 ⇒ Theorem 2.9):

Define $\delta(x, y) := d(Tx, Ty) \forall x, y \in X$. Then $\delta$ is a metric on X. We now show that $(X, \delta)$ is a compact metric space. Let $(x_n)$ be a sequence in X. Since (X, d) is compact, so there exist a subsequence $(x_{n_i})$ of $(x_n)$ and an element y of X such that $x_{n_i}$ converges to y (with respect to d), i.e. $d(x_{n_i}, y) \to 0$ as $i \to \infty$. Since T is continuous (w.r.t d), so $d(Tx_{n_i}, Ty) \to 0$. This implies that $\delta(x_{n_i}, y) \to 0$. Hence $x_{n_i}$ converges to y with respect to $\delta$, too. Therefore $(X, \delta)$ is compact. Furthermore, condition (2.2) reduces to $\delta(Sx, Sy) < \delta(x, y)$ for all $x, y \in X$ with $x \neq y$. By Theorem 2.8, S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point. □

A similar argument may be used to establish Theorem 3.3 below.

**Theorem 3.3.** *Theorem 2.10 is equivalent to the cone metric space version of Theorem 2.8.*

**Theorem 3.4.** *Theorem 2.13 is equivalent to Theorem 2.12.*

**Proof:**

Part I (Theorem 2.13 ⇒ Theorem 2.12):

If T is the identity mapping on X, then Theorem 2.13 reduces to Theorem 2.12.

Part II (Theorem 2.12 ⇒ Theorem 2.13):

Define $\delta(x, y) := d(Tx, Ty)$ for all $x, y \in X$. Then $\delta$ is a cone metric on X. We now show that $(X, \delta)$ is a complete cone metric space. Let $\{x_n\}$ be a Cauchy sequence in $(X, \delta)$. From the definition of $\delta$, this implies that $\{Tx_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $y \in X$ such that $d(Tx_n, y) \to 0$ as n $\to \infty$. But T is sequentially convergent, and then there exists $x \in X$, such that $d(x_n, x) \to 0$ as $n \to \infty$. Since T is continuous, this implies that $d(Tx_n, Tx) \to 0$ as $n \to \infty$, that is, $\delta(x_n, x) \to 0$ as $n \to \infty$. This proves that $(X, \delta)$ is complete. Furthermore, condition (2.6) reduces to $\delta(Sx, Sy) \preceq ad(x, y) + b\delta(x, Sy)$ for all $x, y \in X$. By Theorem 2.12, S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point. □

**Corollary 3.5.**

(i) *Theorem 1 in [10] is equivalent the cone metric space version of Theorem 2.6 in [4].*
(ii) *Theorem 3 in [10] is equivalent to Theorem 3.1 in [14].*
(iii) *Theorem 4 in [10] is equivalent to Theorem 3.5 in [14].*
(iv) *Theorem 3.2 in [15] is equivalent to the cone metric space version of Zamfirescu fixed point theorem.*

**Proof** Since TB-contractions, TK-contractions, TC-contractions and TZ-contractions are TW-contractions, so corollary 3.5 follows easily from Theorem 3.4. □

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