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LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED EQUATIONS UNDER GENERAL SOURCE CONDITION

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ABSTRACT. Analogues to the procedure adopted by Scherzer et.al (1993) for choosing the regularization parameter in Tikhonov regularization of nonlinear ill-posed equations of the form $F(x) = y$, Tautenhahn (2002) considered an a posteriori parameter choice strategy for Lavrentiev regularization in the case of monotone F, and derived order optimal error estimates under Hölder type source conditions. In this paper, we derive order optimal error estimates under a general source condition so that the results are applicable for both mildly and exponentially ill-posed problems. Results in this paper generalize results of Tautenhahn (2002) and also extend results of Nair and Tautenhahn (2004) to the nonlinear case.

KEYWORDS: Lavrentiev regularization; Inverse problems; Ill-posed problems; Discrepancy principle; Monotone operator.
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1. INTRODUCTION

In this paper we are interested in finding a stable approximate solution for an ill-posed equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset X \to X$ is a nonlinear operator and X is a Hilbert space. We shall denote the inner product and the corresponding norm X by $\langle ., . \rangle$ and $\|.\|$ respectively.

We assume that (1.1) has a solution, say $x^\dagger$ and for $\delta \geq 0$, $y^\delta \in X$ is an available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$

We also assume that the operator F possesses a Fréchet derivative in a neighbourhood of $x^\dagger$, i.e, there exists $r > 0$ such that the Fréchet derivative $F'(x)$ exists for every $x \in B_r(x^\dagger) := \{u \in X : \|u - x^\dagger\| < r\}$.

Since (1.1) is ill-posed, the solution $x^\dagger$ need not depend continuously on the data. So, in order to obtain stable approximate solutions, it is required to regularize (1.1).

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Tikhonov regularization is one of the widely used regularization methods which has been extensively studied in the literature (cf. [2], [3], [6], [7], [9], [13]). In this method, the regularized solution is obtained by minimizing the Tikhonov functional

$$J_{\alpha,\delta}(x) := \|F(x) - y^\delta\|^2 + \alpha\|x - \bar{x}\|^2, \quad x \in D(F), \tag{1.3}$$

for each $\alpha > 0$, where $\bar{x} \in D(F)$ is a known initial approximation of $x^\dagger$. As the given operator is Fréchet differentiable, a minimum for the functional $J_{\alpha,\delta}$ in (1.3), if exists, is a solution of the associated *Euler -Lagrange equation*

$$F'(x)^*(F(x) - y^\delta) + \alpha(x - \bar{x}) = 0, \tag{1.4}$$

where $F'(x)^*$ is the adjoint of operator $F'(x)$, the Fréchet derivative of F at x.

Now, suppose that the given operator F is monotone, i.e.,

$$\langle F(x_2) - F(x_1), x_2 - x_1\rangle \geq 0 \qquad \forall\, x_1, x_2 \in D(F). \tag{1.5}$$

Then to get a regularized solution for (1.1), one can use an equation simpler than (1.4), namely,

$$F(x) + \alpha(x - \bar{x}) = y^\delta. \tag{1.6}$$

This method, in which the regularized solution is obtained by solving the equation (1.6), is known as Lavrentiev regularization. The existence and uniqueness of the solution of (1.6) can be asserted from the proof of Theorem 11.2 in [1] by making use of the hemicontinuity and the monotonicity of F. Note that the equation (1.6) does not involve Fréchet derivatives of F at any point. However, for deriving the error estimates, we shall make use of an equivalent form of (1.6), namely,

$$x_\alpha^\delta = \bar{x} + (A_{\alpha,\delta} + \alpha I)^{-1}[y^\delta - F(x_\alpha^\delta) + A_\alpha^\delta(x_{\alpha,\delta} - \bar{x})], \tag{1.7}$$

where $A_{\alpha,\delta} := F'(x_\alpha^\delta)$.

After getting a regularized solution by solving (1.6) for each $\alpha > 0$, the next important aspect is to choose the regularization parameter $\alpha := \alpha(\delta)$ such that $x_\alpha^\delta \to x^\dagger$ as $\delta \to 0$. This choice may be a priori or a posteriori. Due to the practical applicability, a posteriori parameter strategy gains importance over a priori one. One such procedure is proposed by Scherzer et.al (cf. [9]) for Tikhonov regularization. For Lavrentiev regularization (1.7), Tautenhahn (cf. [11]) considered an analogous a posteriori strategy in which α is required to satisfy the equation

$$\|R_{\alpha,\delta}[F(x_\alpha^\delta) - y^\delta]\| = c\delta, \tag{1.8}$$

where $R_{\alpha,\delta} = \alpha(F'(x_\alpha^\delta) + \alpha I)^{-1}$ and $c > 0$ is an appropriate constant, and derived an order optimal error estimate under the assumption that the solution satisfies a *Hölder type source condition*. It is to be mentioned that Hölder type source conditions, though considered in the literature are suitable for mildly ill-posed problems, they are not applicable for many of the severely ill-posed cases where a *logarithmic type source condition* is sometimes more suitable (See [4], [5]).

In [8], Lavrentiev regularization for linear ill-posed problem is considered under a general source condition and optimal error estimates are obtained under the discrepancy principle of the form (1.8). Such general source conditions are useful for mildly and severely ill-posed problems, in particular for both Hölder type and logarithmic type source conditions. It is the purpose of this paper to extend the above analysis to the case of nonlinear ill-posed problems so that the result can be applied to a wider class of problems.

We note that for deriving the error estimates, Tautenhahn [11] used the assumption that there exists a constant $k_0 > 0$ such that for every $x \in D(F)$ and $v \in X$, there exists an element $k(x, x^\dagger, v) \in X$ satisfying

$$(F'(x) - F'(x^\dagger))v = F'(x^\dagger)k(x, x^\dagger, v), \qquad \|k(x, x^\dagger, v)\| \leq k_0\|v\|. \tag{1.9}$$

However, for deriving an estimate for the error $\|x_\alpha - x^\dagger\|$, using the notation $M_\alpha := \int_o^1 F'(x^\dagger + t(x_\alpha - x^\dagger))dt$, the following relation has been used:

$$\|(M_\alpha + I)^{-1}(F'(x^\dagger) - M_\alpha)u\| \leq k_0\|(M_\alpha + I)^{-1}M_\alpha\| \, \|u\|.$$

(cf. [11], step following (3.7) in the proof of Theorem 3.3). The above relation does not seem to follow from the above assumption (1.9). What follows from the assumption (1.9) is the relation

$$\|(M_\alpha + I)^{-1}(F'(x^\dagger) - M_\alpha)u\| \leq k_0\|(M_\alpha + I)^{-1}F'(x^\dagger)\| \, \|u\|.$$

It is also the purpose of this paper to fill the above apparent gap in the analysis in [11] by using the following alternate assumption on the nonlinearity of F, which has been used in [6], [7], [13] for Tikhonov regularization, so as to suit for analysis under a general source condition as well.

Assumption 1.1. *There exists a constant $k_0 > 0$ such that for every $x, u \in B_r(x^\dagger)$ and $v \in X$, there exists an element $g(x, u, v) \in X$ satisfying*

$$(F'(x) - F'(u))v = F'(u)g(x, u, v), \qquad \|g(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

It is shown in [9] that some parameter identification problems and nonlinear Hammerstein operator equation does satisfy Assumption 1.1.

2. ERROR ESTIMATE FOR LAVRENTIEV REGULARIZATION

Recall that, in Lavrentiev regularization, regularized solution x_α^δ is obtained by solving the nonlinear equation (1.6). As we have already spelt out in the last section, we shall assume that F is Fréchet differentiable in a neighbourhood $B_r(x^\dagger)$, where $x^\dagger$ is a solution of (1.1), and F is also a monotone operator so that for every $x \in B_r(x^\dagger)$, $F'(x)$ is a positive self adjoint operator and $F'(x) + \alpha I$ has continuous inverse for every $\alpha > 0$.

We derive bounds for the term $\|x_\alpha^\delta - x^\dagger\|$ under the following source condition:

Assumption 2.1. *There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \longrightarrow (0, \infty)$ with $a \geq \|F'(x^\dagger)\|$ satisfying*

(i) $\lim\limits_{\lambda \longrightarrow 0} \varphi(\lambda) = 0$,

(ii) *there exists $c_\varphi > 0$ such that*

$$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{(\lambda + \alpha)} \leq c_\varphi \varphi(\alpha) \quad \forall \, \alpha \in \, (0, a],$$

(iii) *there exist $\rho > 0$ and $v \in X$ with $\|v\| \leq \rho$ such that*

$$\bar{x} - x^\dagger = \varphi(F'(x^\dagger)\,)v.$$

Assumption 2.1, known as a general source condition, is similar to the one considered in [8] for linear case. It can be seen easily that it includes both the well known source conditions namely, the Hölder type source condition, that is, with $\varphi(\lambda) = \lambda^\nu$, $0 < \lambda \leq 1$, and the logarithmic source condition, that is, with $\varphi(\lambda) = [\log(1/\lambda)]^{-\nu}$, $\nu > 0$.

2.1. **General Error estimate.** We find out error bound for $\|x_\alpha^\delta - x_\alpha\|$ and $\|x_\alpha - x^\dagger\|$ so that a bound for $\|x_\alpha^\delta - x^\dagger\|$ is obtained by triangle inequality. First we quote a result from [11] for the error bound for $\|x_\alpha^\delta - x_\alpha\|$.

Theorem 2.1. *Let (1.5) hold and x_α be the solution of the (1.6) with y in place of y^δ. Then*

(i) $\|x_\alpha^\delta - x_\alpha\| \le \delta/\alpha$,
(ii) $\|x_\alpha - x^\dagger\| \le \|\bar{x} - x^\dagger\|$,
(iii) $\|F(x_\alpha^\delta) - F(x_\alpha)\| \le \delta$.

Remark 2.2. From the relation $\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x^\dagger\|$ and Theorem 2.1 we obtain that

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \|\bar{x} - x^\dagger\|.$$

Therefore, we see that equation (1.7) is meaningful if r in $B_r(x^\dagger)$ satisfies the relation

$$r > \frac{\delta}{\alpha} + \|\bar{x} - x^\dagger\|.$$

Our next result deals with error bound for $\|x_\alpha - x^\dagger\|$. We shall denote

$$A := F'(x^\dagger), \qquad A_\alpha := F'(x_\alpha).$$

Theorem 2.2. *Let the Assumption 1.1, 2.1 and (1.5) hold, and let $k_0\|\bar{x} - x^\dagger\| < 2$. Then*

$$\|x_\alpha - x^\dagger\| \le \tilde{c}_\varphi \varphi(\alpha)\rho, \tag{2.1}$$

where $\tilde{c}_\varphi = c_\varphi(1 + k_0\|\bar{x} - x^\dagger\|)/(1 - k_0\|\bar{x} - x^\dagger\|/2)$.

Proof. Denote $A_\alpha := F'(x_\alpha)$ and $A := F'(x^\dagger)$. From (1.6), we have

$$x_\alpha = \bar{x} + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - \bar{x})].$$

We observe that

$$\begin{aligned}
x_\alpha - x^\dagger &= \bar{x} - x^\dagger + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - \bar{x})] \\
&= \bar{x} - x^\dagger + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger + x^\dagger - \bar{x})] \\
&= \alpha(A_\alpha + \alpha I)^{-1}(\bar{x} - x^\dagger) + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= \alpha(A_\alpha + \alpha I)^{-1}(\bar{x} - x^\dagger) + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= \alpha(A + \alpha I)^{-1}(\bar{x} - x^\dagger) + \alpha((A_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1})(\bar{x} - x^\dagger) \\
&\quad + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= v_\alpha + (A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha \\
&\quad + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)]
\end{aligned}$$

where $v_\alpha := \alpha(A + \alpha I)^{-1}(\bar{x} - x^\dagger)$. From Assumption 2.1, we have

$$\|v_\alpha\| = \|\alpha(A + \alpha I)^{-1}[\varphi(A)]v\| \le c_\varphi \rho\varphi(\alpha). \tag{2.2}$$

Thus,

$$\|x_\alpha - x^\dagger\| \le c_\varphi \rho\varphi(\alpha) + a_\alpha + b_\alpha, \tag{2.3}$$

$$\begin{aligned}
a_\alpha &:= \|(A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha\| \\
b_\alpha &:= \|(A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)]\|.
\end{aligned}$$

Now, let us find estimates for the quantities a_α and b_α. By Assumption 1.1, we have $(A - A_\alpha)v_\alpha = A_\alpha g(x^\dagger, x_\alpha, v_\alpha)$, with

$$\|g(x^\dagger, x_\alpha, v_\alpha)\| \le k_0 \|x_\alpha - x^\dagger\| c_\varphi \rho \varphi(\alpha).$$

Thus,

$$\begin{aligned} a_\alpha := \|(A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha\| &\le \|(A_\alpha + \alpha I)^{-1} A_\alpha g(x^\dagger, x_\alpha, v_\alpha)\| \\ &\le k_0 \|x^\dagger - x_\alpha\| c_\varphi \rho \varphi(\alpha). \end{aligned}$$

For obtaining a bound for b_α, we first observe from fundamental theorem of calculus and the Assumption 1.1 that

$$\begin{aligned} F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger) &= \int_0^1 [F'(x_\alpha + t(x^\dagger - x_\alpha)) - A_\alpha](x^\dagger - x_\alpha)dt \\ &= A_\alpha \int_0^1 g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)dt, \end{aligned} \tag{2.4}$$

where

$$\|g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)\| \le k_0 \|x_\alpha - x^\dagger\|^2 t.$$

Using (2.4), we get

$$b_\alpha := \|(A_\alpha + \alpha I)^{-1} A_\alpha \int_0^1 g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)dt\| \le \frac{k_0}{2}\|x_\alpha - x^\dagger\|^2.$$

Hence, we get

$$\|x_\alpha - x^\dagger\| \le c_\varphi \varphi(\alpha)\rho + c_\varphi k_0 \|x_\alpha - x^\dagger\| \varphi(\alpha)\rho + \frac{k_0 \|x_\alpha - x^\dagger\|^2}{2}.$$

Using $\|x_\alpha - x^\dagger\| \le \|\bar{x} - x^\dagger\|$, we get

$$\|x_\alpha - x^\dagger\| \le c_\varphi \varphi(\alpha)\rho + c_\varphi k_0 \|\bar{x} - x^\dagger\| \varphi(\alpha)\rho + \frac{k_0 \|\bar{x} - x^\dagger\| \|x_\alpha - x^\dagger\|}{2}.$$

Hence,

$$\|x_\alpha - x^\dagger\| \le c_\varphi \left(\frac{1 + k_0 \|\bar{x} - x^\dagger\|}{1 - k_0 \|\bar{x} - x^\dagger\|/2} \right) \varphi(\alpha)\rho.$$

□

Combining Theorem 2.1 and Theorem 2.2 we obtain a bound for $\|x_\alpha^\delta - x^\dagger\|$ as in the following theorem.

Theorem 2.3. *Under the assumptions of Theorem 2.2,*

$$\|x_\alpha^\delta - x^\dagger\| \le \hat{c}_\varphi \left(\frac{\delta}{\alpha} + \rho\varphi(\alpha) \right),$$

where $\hat{c}_\varphi := max\{\tilde{c}_\varphi, 1\}$.

2.2. **A priori parameter choice.** We note that

$$\frac{\delta}{\alpha} = \rho\varphi(\alpha) \iff \frac{\delta}{\rho} = \psi(\varphi(\alpha)),$$

where $\psi : (0, \varphi(a)] \longrightarrow (0, a\varphi(a)]$ is defined as

$$\psi(\lambda) := \lambda \varphi^{-1}(\lambda),$$

for $\lambda \in (0, \varphi(a)]$. Our next theorem gives error bound for $\|x_\alpha^\delta - x^\dagger\|$ under an a-priori parameter choice.

Theorem 2.4. *Let the assumptions of Theorem 2.2 be satisfied. If the regularization parameter is chosen as* $\alpha = \varphi^{-1}\psi^{-1}(\delta/\rho)$ *with* ψ *defined by* $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ *for* $\lambda \in (0, \varphi(a)]$, *then*

$$\|x_\alpha^\delta - x^\dagger\| \leq \hat{c}_\varphi \rho\, \psi^{-1}(\delta/\rho), \tag{2.5}$$

where $\hat{c}_\varphi := max\{\tilde{c}_\varphi, 1\}$.

3. ERROR ESTIMATE UNDER AN A POSTERIORI CHOICE OF PARAMETER

Throughout this section we assume that the regularization parameter is chosen according to the discrepancy principle (1.8). The following lemma, proved in [11] ensures the existence of the regularization parameter α for which (1.8) holds.

Lemma 3.1 ([11], Proposition 4.1)**.** *Let the monotonicity property (1.5) be satisfied and* $\|F(\bar{x}) - y^\delta\| \geq c\delta$ *with* $c > 1$. *Then there exists an* $\alpha \geq \beta_0 := (c-1)\delta/\|\bar{x} - x^\dagger\|$ *satisfying (1.8).*

As in the last section, we use the notations $A := F'(x^\dagger)$ and $A_\alpha := F'(x_\alpha)$. We shall also use the notations

$$R_\alpha := \alpha(A_\alpha + \alpha I)^{-1}, \qquad R_\alpha^\delta := \alpha(A_\alpha^\delta + \alpha I)^{-1}.$$

For obtaining the main result of this section, we first prove some lemmas.

Lemma 3.2. *Let the Assumption 1.1 and assumptions in Lemma 3.1 hold and* $\alpha := \alpha(\delta)$ *is chosen according to (1.8). Then*

$$\|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\| \geq \frac{(c-2)\delta}{1 + k_1}, \tag{3.1}$$

where $k_1 = k_0 c\|\bar{x} - x^\dagger\|/(c-1)$.

Proof. By (1.8), we have

$$\|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta)\| = c\delta.$$

Now consider

$$\begin{aligned}
&|c\delta - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\
&\quad = \quad |\|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta)\| - \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\
&\quad \leq \quad \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta - (F(x_\alpha) - y))\| \\
&\quad \leq \quad \|F(x_\alpha^\delta) - F(x_\alpha)\| + \|y^\delta - y\|.
\end{aligned}$$

Using (1.2) and Theorem 2.1 we get

$$|c\delta - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \leq 2\delta$$

which gives

$$(c-2)\delta \leq \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \leq (c+2)\delta. \tag{3.2}$$

Now let

$$\begin{aligned}
a &= \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \\
b &= \|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\|.
\end{aligned}$$

Then

$$\begin{aligned}
a \quad &\leq \quad b + \|\alpha((A_\alpha^\delta + \alpha I)^{-1} - (A + \alpha I)^{-1})(F(x_\alpha) - y)\| \\
&\leq \quad b + \|(A_\alpha^\delta + \alpha I)^{-1}(A - A_\alpha^\delta)\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y))\| \\
&\leq \quad b + \|(A_\alpha^\delta + \alpha I)^{-1}A_\alpha^\delta g(x^\dagger, x_\alpha^\delta, \alpha(A + \alpha I)^{-1}(F(x_\alpha) - y))\| \\
&\leq \quad b + k_0\|x_\alpha^\delta - x^\dagger\|\|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\| \\
&\leq \quad (1 + k_1)b
\end{aligned}$$

where $k_1 = k_0 c\|\bar{x} - x^\dagger\|/(c-1)$. Now (3.2) gives

$$(c-2)\delta \le (1+k_1)b$$

which in turn implies

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - y)\| \ge \frac{(c-2)\delta}{1+k_1}. \tag{3.3}$$

□

Lemma 3.3. *Let assumptions of Theorem 2.2 and Lemma 3.1 hold. Then*

(i) $\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - y)\| \le \mu\alpha\varphi(\alpha)\rho$,
(ii) $\alpha \ge \varphi^{-1}\psi^{-1}(\xi\delta/\rho)$,

where $\mu = \tilde{c}_\varphi\left(1 + k_0\|\bar{x} - x^\dagger\|/2\right)$ *with* $\tilde{c}_\varphi$ *as in Theorem 2.2 and* $\xi = (c-2)/(1+k_1)\mu$.

Proof. By Assumption 1.1, we know that for every $x, z \in B_r(x^\dagger)$ and $u \in X$,

$$F'(x)u = F'(z)[u + F'(z)g(x,z,u)], \qquad \|g(x,z,u)\| \le k_0\|u\|\|x-z\|.$$

Hence,

$$\begin{aligned} F(x_\alpha) - F(x^\dagger) &= \int_0^1 F'(x^\dagger + t(x_\alpha - x^\dagger))(x_\alpha - x^\dagger)dt \\ &= A(x_\alpha - x^\dagger) + \int_0^1 Ag(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)dt \\ &= A\left((x_\alpha - x^\dagger) + \int_0^1 g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)\right)dt. \end{aligned}$$

Hence,

$$\begin{aligned} & \|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \\ = \; & \|\alpha(A+\alpha I)^{-1}A\left((x_\alpha - x^\dagger) + \int_0^1 g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)\right)dt\| \\ \le \; & \alpha\|(A+\alpha I)^{-1}A\|\left(\|x_\alpha - x^\dagger\| + \int_0^1 \|g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)dt\|\right) \\ \le \; & \alpha\left(\|x_\alpha - x^\dagger\| + \frac{k_0\|x_\alpha - x^\dagger\|^2}{2}\right) \\ \le \; & \alpha\|x_\alpha - x^\dagger\|\left(1 + \frac{k_0\|x_\alpha - x^\dagger\|}{2}\right). \end{aligned}$$

Using Theorems 2.1 and 2.2, we have

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \le \tilde{c}_\varphi\left(1 + \frac{k_0\|\bar{x} - x^\dagger\|}{2}\right)\alpha\varphi(\alpha)\rho,$$

where $\tilde{c}_\varphi = c_\varphi(1 + k_0\|\bar{x} - x^\dagger\|)/(1 - k_0\|\bar{x} - x^\dagger\|/2)$. Thus,

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \le \mu\alpha\varphi(\alpha)\rho, \tag{3.4}$$

where $\mu = \tilde{c}_\varphi\left(1 + k_0\|\bar{x} - x^\dagger\|/2\right)$. In view of the relation (3.1), we get

$$\frac{(c-2)\delta}{1+k_1} \le \mu\alpha\varphi(\alpha)\rho$$

which implies, using the definition of ψ,

$$\psi(\varphi(\alpha)) = \alpha\varphi(\alpha) \ge \frac{(c-2)\delta}{(1+k_1)\mu\rho} = \frac{\xi\delta}{\rho}.$$

Thus,
$$\alpha \geq \varphi^{-1}\psi^{-1}(\xi\delta/\rho).$$
This completes the proof. □

Lemma 3.4. *Let Assumption 1.1 be satisfied and* $k_0\|\bar{x} - x^\dagger\| < 1$. *Then for all* $0 < \alpha_0 \leq \alpha$
$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{\|R_\alpha(F(x_\alpha) - y)\|}{(1 - k_0\|\bar{x} - x^\dagger\|)\alpha_0}. \tag{3.5}$$

Proof. From (1.6) we know that
$$(F(x_\alpha) - y) + \alpha(x_\alpha - \bar{x}) = 0 \tag{3.6}$$
$$(F(x_{\alpha_0}) - y) + \alpha_0(x_{\alpha_0} - \bar{x}) = 0. \tag{3.7}$$
Hence,
$$\alpha_0(x_\alpha - x_{\alpha_0}) = (\alpha - \alpha_0)(\bar{x} - x_\alpha) + \alpha_0(\bar{x} - x_{\alpha_0}) - \alpha(\bar{x} - x_\alpha).$$
Using (3.6) and (3.7), we get
$$\alpha_0(x_\alpha - x_{\alpha_0}) = \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + F(x_{\alpha_0}) - F(x_\alpha).$$
Adding $A_\alpha(x_\alpha - x_{\alpha_0})$ on both sides of the above equation, we get
$$(A_\alpha + \alpha_0 I)(x_\alpha - x_{\alpha_0}) = \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + (F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0}))$$
which implies
$$x_\alpha - x_{\alpha_0} = \frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y) + (A_\alpha + \alpha_0 I)^{-1}(F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})).$$
We first observe from fundamental theorem of calculus and the Assumption 1.1 that
$$\begin{aligned} F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0}) &= \int_0^1 [F'(x_\alpha + t(x_{\alpha_0} - x_\alpha)) - A_\alpha](x_{\alpha_0} - x_\alpha)dt \\ &= A_\alpha \int_0^1 g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha) \\ & \quad dt. \end{aligned} \tag{3.8}$$

Thus,
$$\|x_\alpha - x_{\alpha_0}\| \leq \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| + c_\alpha,$$
where
$$c_\alpha = \|(A_\alpha + \alpha_0 I)^{-1} A_\alpha \int_0^1 g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha)dt\|.$$
Using the estimate $\|(A_\alpha + \alpha_0 I)^{-1}A_\alpha\| \leq 1$ and again Assumption 1.1, we have
$$\begin{aligned} c_\alpha &\leq \int_0^1 \|g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha)\|dt \\ &\leq \int_0^1 k_0\|x_{\alpha_0} - x_\alpha\|^2 t\, dt \\ &\leq \frac{k_0\|x_{\alpha_0} - x_\alpha\|^2}{2} \end{aligned}$$
Since $\|x_\alpha - x^\dagger\| \leq \|\bar{x} - x^\dagger\|$ and
$$\|x_{\alpha_0} - x_\alpha\| \leq \|x_{\alpha_0} - x^\dagger\| + \|x^\dagger - x_\alpha\| \leq 2\|\bar{x} - x^\dagger\|,$$

we have

$$c_\alpha \le k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|.$$

Thus,

$$\|x_\alpha - x_{\alpha_0}\| \le \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| + k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|.$$

We observe that

$$\begin{aligned}\|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| &\le \|(A_\alpha + \alpha_0)^{-1}(F(x_\alpha) - y)\| \\ &= \frac{1}{\alpha_0}\|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}R_\alpha(F(x_\alpha) - y)\| \\ &\le \|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}\|\frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0}.\end{aligned}$$

Note that

$$\|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}\| \le \sup_{\lambda \ge 0}\frac{\alpha_0(\lambda + \alpha)}{\alpha(\lambda + \alpha_0)}.$$

But

$$\frac{\alpha_0(\lambda + \alpha)}{\alpha(\lambda + \alpha_0)} \le \frac{\alpha\lambda + \alpha_0\alpha}{\alpha(\lambda + \alpha_0 I)} = 1.$$

Thus, we have

$$\|x_\alpha - x_{\alpha_0}\| \le \frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0} + k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|$$

so that using the assumption $k_0\|\bar{x} - x^\dagger\| < 1$, we obtain

$$\|x_\alpha - x_{\alpha_0}\| \le \frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0(1 - k_0\|\bar{x} - x^\dagger\|)}.$$

□

Next lemma gives a bound for $\|R_\alpha(F(x_\alpha) - y)\|$.

Lemma 3.5. *Let assumptions of Lemma 3.3 hold. Then*

$$\|R_\alpha(F(x_\alpha) - y)\| \le \beta\delta,$$

where $\beta = (c + 2)[1 + k_0\|\bar{x} - x^\dagger\|/(c - 1)]$.

Proof. Let $a = \|R_\alpha(F(x_\alpha) - y)\|$ and $b = \|R_\alpha^\delta(F(x_\alpha) - y)\|$. We note that

$$\begin{aligned} a &\le \|R_\alpha^\delta(F(x_\alpha) - y)\| + \|(R_\alpha - R_\alpha^\delta)[F(x_\alpha) - y]\| \\ &= b + \|(A_\alpha + \alpha I)^{-1}(A_\alpha^\delta - A_\alpha)\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \\ &= b + k_0\|x_\alpha^\delta - x_\alpha\|\|R_\alpha^\delta(F(x_\alpha) - y)\|.\end{aligned}$$

Using $\|x_\alpha^\delta - x_\alpha\| \le \delta/\alpha$, we get

$$a \le (1 + k_0\delta/\alpha)b$$

and using Lemma 3.1, we have

$$a \le \left(1 + \frac{k_0\|\bar{x} - x^\dagger\|}{c - 1}\right)b. \tag{3.9}$$

We observe that

$$\begin{aligned} b &= \|R_\alpha^\delta(F(x_\alpha) - y)\| \\ &= \|R_a^\delta(F(x_\alpha) - y^\delta + y^\delta - y)\| \\ &\le \|R_\alpha^\delta(F(x_\alpha) - y^\delta)\| + \|y^\delta - y\|.\end{aligned}$$

From (1.2), (1.8), we get

$$\|R_\alpha^\delta(F(x_\alpha)-y)\| \le (c+2)\delta. \tag{3.10}$$

Using (3.9) and (3.10), we obtain

$$a = \|R_\alpha(F(x_\alpha)-y)\| \le (c+2)\left(1+\frac{k_0\|\bar{x}-x^\dagger\|}{c-1}\right)\delta.$$

□

We now give our main result.

Theorem 3.1. *Let assumptions of Lemma 3.3 hold. If, in addition, $k_0\|\bar{x}-x^\dagger\| \le 1$, then*

$$\|x_\alpha^\delta - x^\dagger\| \le \kappa_\varphi \psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho, \tag{3.11}$$

where

$$\kappa_\varphi = ((1-k_0\|\bar{x}-x^\dagger\|)^{-1} + \tilde{c}_\varphi + 1/\xi)$$

and $\eta = max\{\beta, \xi\}$ with $\tilde{c}_\varphi, \xi, \beta$, as in Theorem 2.2, Lemma 3.3 and 3.5 respectively.

Proof. Let $\Phi(\lambda) := \varphi^{-1}\psi^{-1}(\lambda)$ and $\alpha_0 := \Phi(\beta\delta/\rho)$ with $\beta = (c+2)[1+k_0\|\bar{x}-x^\dagger\|/(c-1)]$.

First consider the case when $\alpha(\delta) \le \alpha_0$. We have

$$\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x^\dagger\|.$$

Using Theorem 2.1 and 2.2, we have

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \tilde{c}_\varphi\varphi(\alpha)\rho,$$

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \tilde{c}_\varphi\varphi(\alpha_0)\rho. \tag{3.12}$$

Next assume that $\alpha(\delta) \ge \alpha_0$. In this case, using Lemma 3.4, Theorem 2.1 and 2.2 in

$$\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x_{\alpha_0}\| + \|x_{\alpha_0} - x^\dagger\|,$$

we get

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \frac{\|R_\alpha(F(x_\alpha)-y)\|}{(1-k_0\|\bar{x}-x^\dagger\|)\alpha_0} + \tilde{c}_\varphi\varphi(\alpha_0)\rho. \tag{3.13}$$

Since the error bound in (3.12) is smaller than the error bound in (3.13), the error bound for the latter case will be the error bound for the $\|x_\alpha^\delta - x^\dagger\|$, for any $\alpha \in (0,a]$.

Using Lemma 3.5, we get

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \frac{\beta\delta}{(1-k_0\|\bar{x}-x^\dagger\|)\alpha_0} + \tilde{c}_\varphi\varphi(\alpha_0)\rho.$$

Using Lemma 3.3 in the first term of right hand side and using the value of α_0 in the second and last term, we obtain

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\Phi(\frac{\xi\delta}{\rho})} + \frac{\beta\delta}{(1-k_0\|\bar{x}-x^\dagger\|)\Phi\left(\frac{\beta\delta}{\rho}\right)} + \tilde{c}_\varphi\psi^{-1}\left(\frac{\beta\delta}{\rho}\right)\rho. \tag{3.14}$$

But, since $\varphi^{-1}(\lambda) = \frac{1}{\lambda}\psi(\lambda)$, we have $\Phi(\lambda) = \varphi^{-1}\psi^{-1}(\lambda) = \lambda/\psi^{-1}(\lambda)$ so that

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\psi^{-1}(\xi\delta/\rho)\rho}{\xi} + \frac{\psi^{-1}(\beta\delta/\rho)\rho}{(1-k_0\|\bar{x}-x^\dagger\|)} + \tilde{c}_\varphi\psi^{-1}\left(\frac{\beta\delta}{\rho}\right)\rho.$$

Hence,

$$\|x_\alpha^\delta - x^\dagger\| \le \left(\frac{1}{\xi} + \frac{1}{(1-k_0\|\bar{x}-x^\dagger\|)} + \tilde{c}_\varphi\right)\psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho.$$

where $\eta = max\{\beta, \xi\}$. Thus,

$$\|x_\alpha^\delta - x^\dagger\| \leq \kappa_\varphi \psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho$$

with $\kappa_\varphi := (1 - k_0\|\bar{x} - x^\dagger\|)^{-1} + \tilde{c}_\varphi + 1/\xi$. □

4. EXAMPLE

Here we give an example, taken from [10], of a nonlinear ill-posed operator equation for the purpose of illustration of the Assumption 1.1 which is a modified form of a condition considered by Tautenhan [11].

For $x \in L^2(0,1)$, let

$$[F(x)](t) := \int_0^1 k(s,t)x^3(s)ds, \quad t \in (0,1),$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases}$$

Then for all $x, y \in L^2(0,1)$ with $x(t) > y(t)$ for $t \in (0,1)$, we have

$$\langle F(x) - F(y),\, x - y\rangle = \int_0^1 \Big(\int_0^1 k(t,s)(x^3 - y^3)(s)ds\Big)(x-y)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$[F'(x)v](t) = 3\int_0^1 k(t,s)x^2(s)v(s)ds, \quad x, v \in L^2(0,1), \quad t \in (0,1).$$

Now, let us restrict the domain of F to

$$D(F) := \{x \in L^2(0,1) : x > c \text{ a.e. }\}$$

for some constant $c > 0$. Then for $u \in D(F)$ and $v, w \in L^2(0,1)$, we have

$$[(F'(x) - F'(u))v](t) = 3\int_0^1 k(t,s)u^2(s)\frac{[x^2(s) - u^2(s)]}{u^2(s)}v(s)ds, \quad t \in (0,1).$$

Thus, for $x, u \in D(F)$ and $v \in L^2(0,1)$,

$$[F'(x) - F'(u)]v = F'(u)g(x,u,v),$$

where

$$g(x,u,v)(s) = \frac{[x^2(s) - u^2(s)]}{u^2(s)}v(s) = \frac{[x(s)+u(s)]\,[x(s)-u(s)]}{u^2(s)}v(s).$$

Observe that

$$\|g(v,u,w)\|_2 \leq \frac{1}{c^2}\|x+u\|_2\|x-u\|_2\|v\|_2.$$

So Assumption 1.1 is satisfied if $k_0 > 0$ is taken such that

$$\frac{\|x+u\|_2}{c^2} \leq k_0 \quad \text{for all} \quad x, y \in B_r(x^\dagger).$$

If we take

$$y(t) = \frac{6\sin(\pi t) + \sin^3(\pi t)}{9\pi^2}, \quad t \in (0,1),$$

then the exact solution is $x^\dagger(t) = \sin(\pi t)$, $t \in (0,1)$. If we use

$$x_0(t) = \sin(\pi t) + \frac{3(t\pi^2 - t^2\pi^2 + \sin^2(\pi t))}{4\pi^2}$$

as the initial guess, then $x_0 - \hat{x} = \varphi(F'(x^\dagger))\frac{1}{4}$ with $\varphi(\lambda) = \lambda$.

5. CONCLUDING REMARKS

Laverntiev regularization of nonlinear ill-posed operator equation $F(x) = y$ is considered when F is monotone and Fréchet differentiable in the neighbourhood of a solution $x^\dagger$. Order optimal error estimates are derived under a general source condition by choosing the regularization parameter a priori and a posteriori manners. The results of this paper generalize the results in [11], [12], fills an apparent gap in the analysis in [11] by using an alternate assumption on the nonlinearity of F, and extend the results in [8] to nonlinear case.

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