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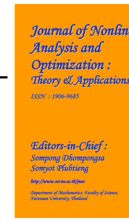

WEAK CONVERGENCE OF FIXED POINT ITERATIONS IN METRIC SPACES

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ABSTRACT. The concept of convergence in normed spaces is extended to metric spaces; and weak convergence of fixed point iterations of contractions on metric spaces is obtained in this article.

KEYWORDS: Directed set; Fixed point iteration; Semi metric.
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1. INTRODUCTION

It is true in a non zero normed space $(X, \|\cdot\|)$ that $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\} = \sup\{\sup\{|f(x)| : f \in F\} : F$ is a finite nonempty subset of the set $\{g \in X^* : \|g\| = 1\}\}$. Here the collection of all finite nonempty subsets of $\{g \in X^* : \|g\| = 1\}$ is a directed set under the inclusion relation. This article is to consider metrics of the type $d(x,y) = \sup\{d_i(x,y) : i \in I\}$ on a nonempty set X, when each d_i is a semi metric (i.e., $d_i(x,y) = 0$ need not imply $x = y$; following the book [1], p.100) on X, for every i in a directed set $(I, \leq)$; and when $d_i \leq d_j$ whenever $i \leq j$. Convergence of a fixed point iteration through each d_i is considered as weak convergence. For some results in connection with weak convergence for fixed point results in nonlinear functional analysis see [2, 3, 5, 6].

The following two results (see [4]) are fundamental results which are applied to obtain extensions for weak convergence. Many other generalized results can also be applied to obtain results on weak convergence. If d_i is a semi metric on a nonempty set X, then X is said to be d_i-complete or (X, d_i) is said to be complete, if for a sequence $(x_n)_{n=1}^{\infty}$ in X such that $d_i(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, there is a point x in X such that $d_i(x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$.

Theorem 1.1. *Suppose d_i is a semi metric on a nonempty set X such that (X, d_i) is complete. Let $T : X \longrightarrow X$ be a given function such that $d_i(T^2(x), T(x)) \leq kd_i(T(x), x), \forall x \in X$, for some $k \in (0,1)$. Fix $x_0 \in X$ and define $x_1, x_2, \ldots,$ by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots.$ Then there is a point x^* in X such that*

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$d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$ and $d_i(T(x^*), x^*) = 0$. Moreover, if d_i is a metric, then the fixed point of T is unique.

Theorem 1.2. *Suppose (X, d) is a nonempty compact metric space. Let $T : X \longrightarrow X$ be a function such that $d(T(x), T(y)) < d(x, y)$ whenever $d(x, y) \neq 0$. Then T has a unique fixed point x^*. Moreover, if $x_0 \in X$ is fixed and $x_1, x_2, \ldots$ are defined by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$, then $d(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$.*

2. MAIN RESULTS

Let X be a nonempty metric space with a metric d. Suppose $(d_i)_{i\in I}$ is a family of semi metrics on X such that $d(x, y) = \sup_{i\in I} d_i(x, y), \forall x, y \in X$. Suppose further that $(I, \leq)$ is a directed set such that $d_i(x, y) \leq d_j(x, y), \forall x, y \in X$, whenever $i \leq j$ in I. These things are assumed in the following two results. The next theorem 2.1 assumes that one more condition is satisfied.

Consider a nonempty set of the form $A_i = \{y \in X : d_i(x_i, y) = 0\}$, for some $x_i \in X$. If a set of this form A_i is called an i-zero set, and if there is a collection $(A_i)_{i\in I}$ of i-zero sets such that $A_i \supseteq A_j$ whenever $i \leq j$ in I, then it is assumed in the next theorem 2.1 that $\bigcap_{i\in I} A_i \neq \emptyset$.

Theorem 2.1. *Let $(k_i)_{i\in I}$ be a given family of numbers in the open interval $(0, 1)$. Let $T : X \longrightarrow X$ be a mapping such that $d_i(T^2(x), T(x)) \leq k_i d_i(T(x), x)$, $\forall x \in X, \forall i \in I$. Suppose further that each (X, d_i) is complete, for every $i \in I$. Then there is a unique fixed point x^* of T in X. Moreover, if $x_0 \in X$ is fixed and $x_1, x_2, \ldots$ are defined by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$, then $d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \in I$.*

Proof. Fix $x_0 \in X$, and define $x_1, x_2, \ldots$ in X by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$. Then, by theorem 1.1, for each $i \in I$, there is a point x_i^* in X such that $d_i(x_n, x_i^*) \longrightarrow 0$ as $n \longrightarrow \infty$ and $d_i(T(x_i^*), x_i^*) = 0$.

Write $A_i = \{x \in X : d_i(x, x_i^*) = 0\}$, an i-zero set, for every $i \in I$. For $i \leq j$ in I, if $x \in A_j$, then

$$\begin{aligned} 0 &\leq d_i(x, x_i^*) \\ &\leq d_i(x, x_j^*) + d_i(x_j^*, x_n) + d_i(x_n, x_i^*) \\ &\leq d_j(x, x_j^*) + d_j(x_j^*, x_n) + d_i(x_n, x_i^*) \\ &= d_j(x_j^*, x_n) + d_i(x_n, x_i^*); \end{aligned}$$

and the right hand side tends to zero as n tends to infinity. Thus $A_j \subseteq A_i$, whenever $i \leq j$ in I. So, by assumption, $\bigcap_{i\in I} A_i \neq \emptyset$. Suppose $x^* \in \bigcap_{i\in I} A_i$. Since $0 \leq d_i(x^*, x_n) \leq d_i(x^*, x_i^*) + d_i(x_i^*, x_n) = d_i(x_i^*, x_n)$, then $d_i(x^*, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$. Also, $0 \leq d_i(T(x^*), x^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) + d_i(x_i^*, x^*) \leq k_i d_i(x^*, x_i^*) + 0 + 0 = 0, \forall i \in I$, imply that $T(x^*) = x^*$. Moreover, if $y^* = T(y^*)$ for some $y^* \in X$, then $0 \leq d_i(x^*, y^*) = d_i(T(x^*), T(y^*)) \leq k_i d_i(x^*, y^*), \forall i \in I$, imply that $x^* = y^*$. This proves the theorem. □

Note that the assumption made before the statement of the theorem 2.1 is not necessary in the previous theorem, if X is a compact metric space.

Lemma 2.2. *Suppose (X, d) is compact. Let $T : X \longrightarrow X$ be a mapping such that $d_i(T(x), T(y)) < d_i(x, y)$ whenever $d_i(x, y) \neq 0$, with $x, y \in X$ and $i \in I$. Then T has a unique fixed point x^* in X. Moreover, if $x_0 \in X$, and if $x_{n+1} = T(x_n)$, for $n = 0, 1, 2, \ldots$, then $d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \in I$.*

Proof. Fix $x_0 \in X$ and define $x_1, x_2, \dots$ in X by $x_{n+1} = T(x_n)$, for $n = 0, 1, 2, \dots$. To each $i \in I$ and to each $x \in X$, let $[x]_i = \{y \in X : d_i(x, y) = 0\}$. Then for given $x, y \in X$, either $[x]_i = [y]_i$ or $[x]_i \cap [y]_i = \emptyset$, for any $i \in I$. Define $\tilde{d}_i([x]_i, [y]_i) = d_i(x, y), \forall x, y \in X$ and define $\tilde{X}_i = \{[x]_i : x \in X\}$, for any $i \in I$. Then $(\tilde{X}_i, \tilde{d}_i)$ is a compact metric space, for any $i \in I$. Define $T_i : (\tilde{X}_i, \tilde{d}_i) \longrightarrow (\tilde{X}_i, \tilde{d}_i)$ by $T_i([x]_i) = T(x), \forall x \in X$, for any $i \in I$. Then $\tilde{d}_i(T_i([x]_i), T_i([y]_i)) < \tilde{d}_i([x]_i, [y]_i)$ whenever $\tilde{d}_i([x]_i, [y]_i) \neq 0$. Then, by theorem 1.2, for each $i \in I$, there is a point x_i^* in X such that $d_i(T(x_i^*), x_i^*) = 0$, $[x_i^*]_i$ is the unique fixed point of T_i, and $d_i(x_n, x_i^*) \longrightarrow 0$ as $n \longrightarrow \infty$. Consider a subnet of $(x_i^*)_{i \in I}$ that converges to some x^* in (X, d). Then $d_i(T(x^*), x^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) + d_i(x_i^*, x^*) \leq 2d_i(x^*, x_i^*) \leq 2d(x^*, x_i^*), \forall i \in I$, imply that $T(x^*) = x^*$. If $y^* = T(y^*)$ for some $y^* \in X$, then $d_i(x^*, y^*) = d_i(T(x^*), T(y^*)) < d_i(x^*, y^*)$, whenever $d_i(x^*, y^*) \neq 0$, for any $i \in I$. This proves the uniqueness of the fixed point of T. Moreover, $d_i(x^*, x_i^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) < d_i(x^*, x_i^*)$ whenever $d_i(x^*, x_i^*) \neq 0$. This proves that $d_i(x^*, x_i^*) = 0, \forall i \in I$. So, for every $i \in I, d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof. □

Example 2.3. Let X be the collection of all bounded continuous real valued functions defined on the real line R. This is a complete metric space under the metric d defined by $d(f, g) = \sup_{x \in R} |f(x) - g(x)|, \forall f, g \in X$. To each $i = 1, 2, \dots$, define $B_i = (-\infty, -1 - \frac{1}{4^i}] \cup [-1 + \frac{1}{4^i}, 1 - \frac{1}{4^i}] \cup [1 + \frac{1}{4^i}, \infty)$, and define $d_i(f, g) = \sup\{|f(x) - g(x)| : x \in B_i\}, \forall f, g \in X$. Then define $T : X \longrightarrow X$ by

$$(T(f))(x) = \begin{cases} \frac{f(x)}{x} & \text{for } |x| \geq 1 \\ xf(x) & \text{for } |x| \leq 1. \end{cases}$$

Note that $d(f, g) = \sup_{i \in I} d_i(f, g), \forall f, g \in X$, with $I = \{1, 2, \dots\}$, which is a directed set under the usual ordering relation. It can be verified that X, d_i, d, and I satisfy the conditions of the theorem 2.1 with $k_n = \max \left\{ \frac{1}{1 + \frac{1}{4^n}}, 1 - \frac{1}{4^n} \right\}$. Here the zero function is the unique fixed point.

This example 2.3 also reveals that the fixed point iteration may not converge strongly with respect to d. But this is not the case when (X, d) is compact. Now the proof of the theorem 2.2 is to be analyzed. The uniqueness part of the proof implies that the net $(x_i^*)_{i \in I}$ converges to x^*. If a subsequence $(z_m)_{m=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ converges to some z^* in (X, d), then $d(z_m, z^*) \longrightarrow 0$ as $m \longrightarrow \infty$, and hence $d_i(z_m, z^*) \longrightarrow 0$ as $m \longrightarrow \infty, \forall i \in I$; whereas $d_i(z_m, x^*) \longrightarrow 0$ as $m \longrightarrow \infty, \forall i \in I$. Thus $d_i(x^*, z^*) = 0, \forall i \in I$ and hence $d(x^*, z^*) = 0$. Thus $x^* = z^*$. So, every subsequence of $(x_n)_{n=1}^{\infty}$ should converge to x^* in the compact metric space (X, d).

Theorem 2.4. *Under the assumptions of lemma 2.2, and for the sequence $(x_n)_{n=1}^{\infty}$ given in lemma 2.2, the following strong conclusion holds:*

$$d(x_n, x^*) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

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