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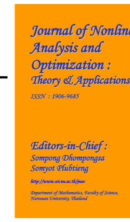

A NOTE ON ULAM-HYERS STABILITY OF A FIXED POINT EQUATION VIA GENERALIZED PICARD OPERATORS

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ABSTRACT. In this note, we introduce new classes operators, which is a generalization of Picard operators, and obtain some Ulam-Hyers stability results for the operators which extend results in [5]. As application, an existence and uniqueness result for an integral equation is given.

KEYWORDS: Ulam-Hyers stability; Generalized Ulam-Hyers stability; Fixed point; Weakly Picard operator.
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1. INTRODUCTION

Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be an operator. The set of fixed points of f will be denoted by $Fix(f) := \{x \in X | x = f(x)\}$. We will denote by $\tilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$, i.e., $\tilde{B}(x_0, r) = \{x \in X | d(x_0, x) \leq r\}$. Following [3] we present the basic notions of weakly Picard operators.
$I(f) := \{Z \subset Y | f(Z) \subset Z, Z \neq \emptyset\}$ - the set of all invariant subsets of f;
$(MI)_f := \bigcup_{Z \in I(f)} Z$ - the maximal invariant subset of f;
$(AB)_f(x^*) := \{x \in Y | f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \xrightarrow{d} x^* \in Fix(f)\}$- the attraction basin of $x^* \in Fix(f)$ with respect to f;
$(AB)_f := \bigcup_{x^* \in Fix(f)} (AB)_f(x^*)$- the attraction basin of f.

Definition 1.1. ([2]) An operator $f : Y \longrightarrow X$ is nonself weakly Picard operator if $Fix(f) \neq \emptyset$ and $(MI)_f = (AB)_f$. If $Fix(f) = \{x^*\}$, then a nonself weakly Picard operator is said to be nonself Picard operator.

Definition 1.2. ([2]) For each nonself weakly Picard operator $f : Y \longrightarrow X$ we define the operator $f^\infty : (AB)_f \longrightarrow Fix(f) \subset (AB)_f$, by $f^\infty(x) = \lim_{n \longrightarrow \infty} f^n(x)$.

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Definition 1.3. ([2]) Let $f : Y \longrightarrow X$ be a nonself weakly Picard operator and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function which is continuous at 0 and $\psi(0) = 0$. The operator f is nonself ψ-weakly Picard operator if

$$d(x, f^\infty(x)) \le \psi(d(x, f(x))), \text{ for all } x \in (MI)_f.$$

In the case that $\psi(t) := ct$ (for some $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is c-weakly Picard operator.

For some examples of nonself weakly Picard operators and ψ-weakly Picard operators, see [2].

If $f : Y \longrightarrow X$ is an operator, let us consider the fixed point equation

$$x = f(x), \ x \in Y \tag{1.1}$$

and the inequation

$$d(y, f(y)) \le \varepsilon. \tag{1.2}$$

Definition 1.4. ([5]) The equation (1.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in (AB)_f$ of (1.2) there exists a solution x^* of the fixed point equation (1.1) such that

$$d(y^*, x^*) \le \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, the equation (1.1) is said to be Ulam-Hyers stable.

In 2009, Rus [5] proved the following result:

Theorem 1.5. *Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be a ψ-weakly Picard operator. Then, the fixed point equation (1.1) is generalized Ulam-Hyers stable. In particular, if f is c-weakly Picard operator, then the equation (1.1) is Ulam-Hyers stable.*

This paper is organized as follows: In Section 2, we extend Theorem 1.5 to wider classes of operators. Examples of such opertors are given. Then, in Section 3, an application to an integral equation is also given.

2. MAIN RESULTS

Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be an operator. For a sequence $S = \{s_n\}$ of selfmaps on X, we define the following notions:
$C(S)_f(x^*) = \{x \in X | s_n(x)$ is defined for all $n \in \mathbb{N}$ and $s_n(x) \xrightarrow{d} x^* \in Fix(f)\}$-the convergence set of S at x^*;
$C(S)_f = \bigcup_{x^* \in Fix(f)} C(S)_f(x^*)$-the convergence set of S.
We will denote the composition $f_n \circ f_{n-1} \circ ... \circ f_j$ simply by $\prod_{i=j}^n f_i = f_n \circ f_{n-1} \circ ... \circ f_j$. In particular, $\prod_{i=1}^n f$ is simply the n-th iterate f^n of f. We now introduce new classes of operators.

Definition 2.1. Let S be a sequence of selfmaps on X. An operator $f : Y \longrightarrow X$ is nonself weakly convergence operator with respect to S (nonself WCO wrpt S) if $Fix(f) \ne \emptyset$ and $(MI)_f = C(S)_f$. If $Fix(f) = \{x^*\}$, then a nonself WCO wrpt S is said to be nonself convergence operator with respect to S (nonself CO wrpt S).

It is obvious that if f is a Picard operator, then it is a CO wrpt $S = \{f^n\}$. The converse is not true, as the following example shows:

Example 2.2. Put $X = [\frac{1}{2}, 2]$ and define a mapping $f : X \longrightarrow X$ by $f(x) = \frac{1}{x}$ for $x \in X$. Then f is a CO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a Picard operator.

Proof. It is easy to see that $Fix(f) = \{1\}$. Let $S = \{((1-\lambda)I + \lambda fI)^n\}$, where I denotes the identity map with $\lambda \in (0,1)$. By Example 4.3 in [1], we get that $(MI)_f = C(S)_f = X$. Therefore, f is a CO wrpt $S = \{f^n\}$. We know that $(MI)_f = X \neq \{1\} = (AB)_f$, so f is not a Picard operator. □

Similarity, if f is a weakly Picard operator, then it is a WCO wrpt $S = \{f^n\}$. The converse is not true.

Example 2.3. Let $X = [0,1]$ and $f : X \longrightarrow X$ be given by $f(x) = x$, for all $x \in (0,1)$ and $f(0) = 1$ and $f(1) = 0$. Then f is a WCO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a weakly Picard operator.

Proof. Let $S = \{((1-\lambda)I + \lambda fI)^n\}$, with $\lambda \in (0,1)$. It is easy to see that $Fix(f) = (0,1)$ and $(MI)_f = C(S)_f = X$. Hence, f is a WCO wrpt S. Since $\{f^n(0)\}, \{f^n(1)\}$ do not converge and $(MI)_f = X \neq (0,1) = (AB)_f$, f is not a weakly Picard operator. □

Definition 2.4. Let $S = \{s_n\}$ be a sequence of selfmaps on X and $f : Y \longrightarrow X$ be a nonself WCO wrpt S. We define the operator $r : C(S)_f \longrightarrow Fix(f) \subset C(S)_f$, by $r(x) = \lim_{n \longrightarrow \infty} s_n(x) \in Fix(f)$.

Definition 2.5. Let $S = \{s_n\}$ be a sequence of selfmaps on X, $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ an increasing function which is continuous at 0 and $\psi(0) = 0$. An operator $f : Y \longrightarrow X$ is said to be a nonself ψ-weakly convergence operator with respect to S (nonself ψ-WCO wrpt S) if it is a nonself WCO wrpt S and

$$d(x, r(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in (MI)_f.$$

In the case that $\psi(t) := ct$ (for some $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is a nonself c-WCO wrpt S.

For sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, if $S = \{\prod_{i=1}^{n} g_i\}$ is a sequence such that

$$g_i = (1-\alpha_i)I + \alpha_i f[(1-\beta_i)I + \beta f]$$

for each $i \in \mathbb{N}$, a nonself ψ-WCO wrpt S is called nonself ψ-weakly Ishikawa type operator associated to sequences $\{\alpha_n\}$ and $\{\beta_n\}$. When $\{\beta_n\} = \{0\}$, a nonself ψ-WCO wrpt S is called nonself ψ-weakly Mann type operator associated to sequences $\{\alpha_n\}$. A nonself ψ-weakly Ishikawa type operator associated to constant sequence is called nonself ψ-weakly Krasnoselskij type operator.

It is easy to see that if $f : Y \longrightarrow X$ is a ψ-weakly Picard operator, then it is a ψ-WCO wrpt $S = \{f^n\}$. The following example shows that the converse is not true.

Example 2.6. For X and f as in Example 2.2, we obtain f is a ψ-WCO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a ψ-weakly Picard operator.

Proof. From Example 2.2, f is CO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$. Consider

$$d(x, r(x)) = |x - 1| \leq |x - \frac{1}{x}| = d(x, f(x)) \leq \psi(d(x, f(x))),$$

where $\psi(t) = at + 1$, $a \geq 1$. Since f is not a Picard operator, it is not a ψ-weakly Picard operator. □

Definition 2.7. If $f : Y \longrightarrow X$ is an operator and $S = \{s_n\}$ be a sequence of selfmaps on X. The equation (1.1) is called generalized Ulam-Hyers stable with respect to S if there exists $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in C(S)_f$ of (1.2) there exists a solution x^* of the fixed point equation (1.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, the equation (1.1) is said to be Ulam-Hyers stable with respect to S.

An Ulam-Hyers stability result is the following:

Theorem 2.8. *Let (X, d) be a metric space, Y be a nonempty subset of X, $f : Y \longrightarrow X$ be a ψ-WCO wrpt S and $S = \{s_n\}$ be a sequence of selfmaps on X. Then, the equation (1.1) is generalized Ulam-Hyers stable with respect to S. In particular, if f is c-WCO wrpt S, then the equation (1.1) is Ulam-Hyers stable with respect to S.*

Proof. Let $\varepsilon > 0$ and $y^* \in C(S)_f$ such that $d(y^*, f(y^*)) \leq \varepsilon$. Since f is ψ-WCO wrpt S, we get

$$d(x, r(x)) \leq \psi(d(x, f(x))),\ x \in (MI)_f.$$

From $(MI)_f = C(S)_f$, we take $x^* := r(y^*)$. Thus, $d(y^*, x^*) \leq \psi(\varepsilon)$. □

The proof presented here based on a standard proof in [5](see [3]). However, we obtain a result for larger classes of operators and the following results are immediate:

Corollary 2.9. *([3]) Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \tilde{B}(x_0, r) \longrightarrow X$ be an α-contraction, such that $d(x_0, f(x_0)) \leq (1 - \alpha)r$. Then, the fixed point equation (1.1) is Ulam-Hyers stable with respect to $S = \{f^n\}$.*

Corollary 2.10. *([3]) Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \tilde{B}(x_0, r) \longrightarrow X$ be an φ-contraction, such that $d(x_0, f(x_0)) \leq r - \varphi(r)$. Suppose also that the function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto. Then, the fixed point equation (1.1) is generalized Ulam-Hyers stable with respect to $S = \{f^n\}$.*

We will present some consequences of Theorem 2.8. We need first some definitions and theorems.

Definition 2.11. ([1],[4]) A mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a comparison function if it is increasing and $\varphi^k(t) \longrightarrow 0$ as $k \longrightarrow +\infty$.
As a consequence, we also have $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.

Definition 2.12. ([1]) Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. An operator $f : H \longrightarrow H$ is said to be

(i) generalized pseudo-contraction if there exists a constant $M > 0$ such that

$$\langle f(x) - f(y), x - y \rangle \leq M \cdot \|x - y\|^2,\ x, y \in H;$$

(ii) Lipschitzian if there exists $L > 0$ such that

$$\|f(x) - f(y)\| \leq L \cdot \|x - y\|,\ x, y \in H.$$

Theorem 2.13. ([1]) *Let K be a nonempty closed convex subset of a real Hilbert space and $f : K \longrightarrow K$ a generalized pseudocontractive and Lipschitzian operator with the corresponding constants M and L fulfilling the conditions*

$$0 < M < 1 \text{ and } M \leq L.$$

Then

(i) *f has an unique fixed point p;*
(ii) *for each x_0 in K, the Kranoselskij iteration $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda f(x_n),\ n = 0, 1, 2, ...$$

converges to p, for all $\lambda \in (0, 1)$ satisfying

$$0 < \lambda < 2(1 - M)/(1 - 2M + L^2);$$

(iii) *Both a priori*

$$\|x_n - p\| \leq \frac{\theta^n}{1-\theta} \cdot \|x_1 - x_0\|,\ n = 1, 2, ...$$

and a posteriori

$$\|x_n - p\| \leq \frac{\theta}{1-\theta} \cdot \|x_n - x_{n-1}\|,\ n = 1, 2, ...$$

estimates hold, with

$$\theta = ((1 - \lambda)^2 + 2\lambda(1 - \lambda)M + \lambda^2 L^2)^{1/2}.$$

Using the previous Theorem, we can prove the following.

Theorem 2.14. *Let K be a nonempty closed convex subset of a real Hilbert space and $f : K \longrightarrow K$ a generalized pseudocontractive and Lipschitzian operator with the corresponding constants M and L fulfilling the conditions*

$$0 < M < 1 \text{ and } M \leq L.$$

Then, the fixed point equation (1.1) is Ulam-Hyers stable with respect to $S = \{((1 - \lambda)I + \lambda f I)^n\}$ where $\lambda \in (0, 1)$ satisfying $0 < \lambda < 2(1 - M)/(1 - 2M + L^2)$.

Proof. Let $S = \{g^n\}$ such that

$$g = (1 - \lambda)I + \lambda f I$$

where $\lambda \in (0, 1)$ satisfying $0 < \lambda < 2(1 - M)/(1 - 2M + L^2)$. By Theorem 2.13, $Fix(f) = \{p\}$, $(MI)_f = C(S)_f = K$ and for each $x \in K$,

$$\|x - p\| \leq \frac{\lambda}{1-\theta} \cdot \|x - f(x)\|,$$

where $\theta = ((1 - \lambda)^2 + 2\lambda(1 - \lambda)M + \lambda^2 L^2)^{1/2}$. Then f is a c-weakly Krasnoselskij type operator with $c := \dfrac{\lambda}{1-\theta} > 0$. Hence, by Theorem 2.8, the fixed point equation (1.1) is Ulam-Hyers stable with respect to S. □

3. Application

Consider the integral equation

$$x(t) = \int_a^b K(t, s, x(s))ds + g(t), \ t \in [a, b]. \tag{3.1}$$

Theorem 3.1. *Assume*

(i) $K : [a, b] \times [a, b] \times \mathbb{R}^n$ *and* $g : [a, b] \longrightarrow \mathbb{R}^n$ *are continuous;*
(ii) K *is Lipschitzian with respect to the third variable, i.e., there exists* $L > 0$ *such that*

$$|K(t, s, u) - K(t, s, v)| \leq L|u - v|, \text{ for each } t, s \in [a, b], u, v \in \mathbb{R}^n;$$

(iii) $\int_a^b K(t, s, u) - K(t, s, v)ds \leq R(u - v)$, *for each* $t \in [a, b], u, v \in \mathbb{R}^n$ *where* $0 < R < 1$ *and* $R \leq L(b - a)$.

Then the following conclusions hold;

(a) *the integral equation (3.1) has a unique solution* x^* *in* $L_2([a, b], \mathbb{R}^n)$,
(b) *there exists a sequence* S *of selfmaps on* X *such that the integral equation (3.1) is Ulam-Hyers stable with respect to* S.

Proof. Let $X := L_2([a, b], \mathbb{R}^n)$ with norm $\|x\| := (\int_a^b x^2(t)dt)^{1/2}$ and inner product $\langle x, y\rangle = \int_a^b x(t)y(t)dt$ for $x, y \in X$. Define $T : X \longrightarrow X$ by

$$Tx(t) := \int_a^b K(t, s, x(s))ds + g(t), \ t \in [a, b].$$

For $x, y \in X$

$$|Tx(t) - Ty(t)| \leq \int_a^b |K(t, s, x(s)) - K(t, s, y(s))|ds \leq L\int_a^b |x(s) - y(s)|ds.$$

Thus

$$|Tx(t) - Ty(t)|^2 \leq L^2(\int_a^b |x(s) - y(s)|ds)^2 \leq L^2 \cdot \int_a^b |x(s) - y(s)|^2 ds \cdot \int_a^b 1ds = L^2(b - a)\|x - y\|^2.$$

We have

$$\begin{aligned} \|Tx(t) - Ty(t)\|^2 &= \int_a^b |Tx(t) - Ty(t)|^2 dt \leq \int_a^b L^2(b - a)\|x - y\|^2 dt \\ &= L^2(b - a)^2\|x - y\|^2. \end{aligned}$$

Therefore T is Lipschitzian operator, i.e.,

$$\|Tx - Ty\| \leq L(b - a)\|x - y\|.$$

Consider

$$\begin{aligned} \langle Tx(t) - Ty(t), x(t) - y(t)\rangle &= \langle \int_a^b K(t, s, x(s)) - K(t, s, y(s))ds, x(t) - y(t)\rangle \\ &= \int_a^b \int_a^b K(t, s, x(s)) - K(t, s, y(s))ds \cdot (x(t) - y(t))dt \\ &\leq R\int_a^b (x(t) - y(t))^2 dt = R\|x - y\|^2. \end{aligned}$$

Hence we obtain T is a generalized pseudocontractive and Lipschitzian operator. The conclusion follows from Theorem 2.14. □

Remark 3.2. Note that the operator f in Example 2.2 is a generalized pseudocontractive and Lipschitzian operator with the corresponding constants $M > 0$ and $L = 4$ but it fails to be Picard operator. This means that the operator T in Theroem 3.1 does not satisfy condition in Theorem 1.5.

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