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MINIMAX PROGRAMMING WITH (G, α)-INVEXTY

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ABSTRACT. In this paper, we deal with the minimax programming (P) under the differentiable (G, α)-invexity which was proposed in [J. Nonlinear Anal. Optim. 2(2): 305-315]. With the help of auxiliary programming problem (G-P), some new Kuhn-Tucker necessary conditions, namely for G-Kuhn-Tucker necessary conditions, is presented for the minimax programming (P). Also G-Karush-Kuhn-Tucker sufficient conditions under (G, α)-invexity assumptions are obtained for the minimax programming (P). Making use of these optimality conditions, we construct a dual problem (DI) for (P) and establish weak, strong and strict converse duality theorems between problems (P) and (DI).

Keywords: (G, α)-invexity; minimax programming, optimal solution, G-Kuhn-Tucker necessary optimality conditions
AMS Subject Classification: 90C29, 90C46.

1. INTRODUCTION

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 10]. One of these concepts, invexity, was introduced by Hanson in [7]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [6] introduced the concept of pre-invex functions which is a special case of invexity.

Recently, Antczak extended further invexity to G-invexity [3] for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In [4], Antczak defined

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vector G-invex (G-incave) functions with respect to η, and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification [4]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems [5]. A number of new vector duality problems such as G-Mond–Weir, G-Wolfe and G-mixed dual vector problems to the primal one were also defined in [5].

Motivated by [4, 5, 9], we [12] presented the vector (G, α)-invexity concept. In this sequel, we deal with nonlinear minimax programming problems with the vector (G, α)-invexity, and the nonlinear minimax programming problem is presented as follows.

$$\text{(P)} \qquad \min \sup_{y \in Y} \phi(x, y)$$
$$\text{subject to} \quad g_j(x) \leq 0, \ j \in M = \{1, \cdots, m\}$$

where Y is a compact subset of $\mathbb{R}^p$, $\phi(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ $(j \in M)$. Let E be the set of feasible solutions of problem (P); in other words, $E = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in M\}$. For convenience, let us define the following sets for every $x \in E$.

$$J(x) = \{j \in M \mid \ g_j(x) = 0\}, \ Y(x) = \left\{ y \in Y | \varphi(x, y) = \sup_{z \in Y} \varphi(x, z) \right\}.$$

The rest of the paper is organized as follows. In Section 2, we present concepts regarding to vector (G, α)-invexity. In Section 3, we present G-Karush-Kuhn-Tucker sufficient and necessary optimality conditions for the minimax fractional mathematical programming problems. When the sufficient conditions are utilized, dual problem is formulated and duality results are presented in Section 4.

2. Vector (G, α)-invex functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convenience for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$$x > y \text{ if and only if } x_i > y_i, \ \text{for } i = 1, 2, \cdots, n;$$
$$x \geqq y \text{ if and only if } x_i \geq y_i, \ \text{for } i = 1, 2, \cdots, n;$$
$$x \geqslant y \text{ if and only if } x_i \geq y_i, \ \text{for } i = 1, 2, \cdots, n, \ \text{but } x \neq y;$$
$$x \not> y \text{ is the negation of } x > y.$$

We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Let $g = (g_1, \cdots, g_m) : X \to \mathbb{R}^m$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$; let $I_{g_i}(x)$ be the range of g_i, that is, the image of X under g_i for each $i \in M$. Further, suppose that $G_g = (G_{g_1}, \cdots, G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ be a vector-valued function such that $G_{g_i} : I_{g_i}(X) \to \mathbb{R}$ is strictly increasing on $I_{g_i}(X)$ for each $i \in M$. The following Definition 2.1 is taken from [12]

Definition 2.1. Let $g = (g_1, \cdots, g_m) : X \to \mathbb{R}^m$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$; let $I_{g_i}(x)$ be the range of g_i for each $i \in M$. If there exist a differentiable vector-valued function $G_g = (G_{g_1}, \cdots, G_{g_m}) : R \to \mathbb{R}^m$ such that any its component $G_{g_i} : I_{g_i}(X) \to \mathbb{R}$ is strictly increasing on $I_{g_i}(X)$, a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ and real functions $\alpha_i : X \times X \to \mathbb{R}_+ (i \in M)$ such that, for all $x \in X$ $(x \neq u)$,

$$G_{g_i}(g_i(x)) - G_{g_i}(g_i(u)) \geq \alpha_i(x, u) G'_{g_i}(g_i(u)) \nabla g_i(u) \eta(x, u), i \in M, \tag{2.1}$$

then g is said to be a (strictly) vector (G_g, α)-invex function at u on X (with respect to η) (or shortly, (G_g, α)-invex function at u on X), where $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_m)^T$. If (2.1) is satisfied for each $u \in X$, then g is vector (G_g, α)-invex on X with respect to η.

Remark 2.2. In order to define an analogous class of (strictly) vector (G_g,α)-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

We note that the (G_g,α)-invex function is a generalization of α-invex and G_g-invex function.

For convenience, we need the following nonlinear fractional programming problem (G-P).

$$(G\text{-P})\quad \min \sup_{y\in Y} G_\phi(\phi(x,y))$$
$$s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)), \cdots, G_{g_m}(g_m(x))) \leqq G_g(0),$$

where $G_g(0) := (G_{g_1}(0), G_{g_2}(0), \cdots, G_{g_m}(0))$. We denote by $X_{G\text{-}P} = \{x \in \mathbb{R}^n \mid G_g g(x) \leq G_g(0)\}$, $J'(\bar{x}) := \{j \in M : G_{g_j} g_j(\bar{x}) = G_{g_j}(0)\}$. If function G_{g_j} is strictly increasing on $I_{g_j}(X)$ for each $j \in M$, then $X_P = X_{G\text{-}P}$ and $J(\bar{x}) = J'(\bar{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (CVP) or (G-CVP) by the notations E and $J(\bar{x})$, respectively.

Theorem 2.3. *Let G_ϕ be a strictly increasing function defined on $I_\phi(X,Y)$, G_{g_j} be a strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$, and $0 \in I_{g_j}(X)$, $j \in M$. Then $\bar{x}$ is an optimal solution for (P) if and only if $\bar{x}$ is also an optimal solution for (G-P).*

Proof “if” part, we prove that if $\bar{x}$ is an optimal solution for (G-P), then $\bar{x}$ is an optimal solution for (P). On the contrary, let $\bar{x}$ be an optimal solution for (G-P) but not an optimal solution for (P). Define

$$f(x) := \sup_{y\in Y} \phi(x,y),$$

Then there exists $x_0 \in E$ such that

$$f(x_0) < f(\bar{x}).$$

This means that

$$\phi(x_0,y) < \phi(\bar{x},z), \forall\, y \in Y(x_0), \forall\, z \in Y(\bar{x}).$$

Note that the strictly monotonicity of G_ϕ, we have

$$G_\phi(\phi(x_0,y)) < G_\phi(\phi(\bar{x},z)), \forall\, y \in Y(x_0), \forall\, z \in Y(\bar{x}).$$

This contradicts to the assumption that $\bar{x}$ is an optimal solution for (G-P).

The proof of “only if” part is similar to “if ” part, we omitted it.

3. Optimality conditions in minimax programming

In [4], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problem. Under G-invexity assumptions, he considered also G-Karush-Kuhn-Tucker sufficient optimality conditions for this kind of multiobjective programming problem. Here, we firstly present some G-Kuhn-Tucker necessary optimality conditions for differentiable minimax programming problem through an auxiliary programming problem. After that, we give some sufficient optimality conditions under (G,α)-invexity. We shall use the following Theorem 3.1 proved by Schmitendrof in [13].

Theorem 3.1. *Let x^* be an optimal solution to the minimax problem (P). Moreover, the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* \geqslant 0$ $(i = 1, \ldots, q^*)$ and $\mu_j^* \geqslant 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* \nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0,$$
$$\mu_j^* g_j(x^*) = 0, j \in M,$$
$$\sum_{i=1}^{q^*} \lambda_i^* = 1.$$

Furthermore, if α is the number of nonzero λ_i^*, and β is the number of nonzero μ_j^*, then

$$1 \leq \alpha + \beta \leq n + 1.$$

Theorem 3.2 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let x^* be an optimal solution to the minimax problem (P). Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Moreover, the vectors $G'_{g_j}(g_j(x^*))\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* $(1 \leq q^* \leq n+1)$ and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* > 0$ $(i = 1, \ldots, q^*)$ and $\mu_j^* \geq 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*))\nabla g_j(x^*) = 0, \tag{3.1}$$

$$\mu_j^*(G_{g_j}(g_j(x^*)) - G_{g_j}(0)) = 0, j \in M, \tag{3.2}$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1. \tag{3.3}$$

Proof Since x^* is an optimal solution to the minimax problem (P), we can choose $y_i \in Y(x^*), i = 1, \cdots, q^*$ such that they satisfy Theorem 3.1. For each y_i, we consider the programming problem (P$_{y_i}$) as follows.

$$(\mathrm{P}_{y_i}) \qquad \min \quad \phi(x, y_i)$$
$$\text{subject to} \quad g_j(x) \leq 0,\ j \in M = \{1, \cdots, m\}.$$

It is evident that x^* is an optimal solution to (P$_{y_i}$). Using similar arguments as in the proof of Theorem 2.3, we can prove that x^* is an optimal solution to $(G\text{-P}_{y_i})$

$$(G\text{-P}_{y_i}) \quad \min \quad G_\phi(\phi(x, y_i))$$
$$s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)), \cdots, G_{g_m}(g_m(x))) \leqq G_g(0).$$

Thus, there exist $\lambda_i > 0$, $\mu_{ji} \geq 0$ $(j \in M)$ such that

$$\lambda_i \nabla_x \left(G_\phi(\phi(x^*, y_i))\right) + \sum_{j=1}^{m} \mu_{ji} \nabla \left(G_{g_j}(g_j(x^*))\right) = 0, \tag{3.4}$$

$$\mu_{ji}\left(G_{g_j}(g_j(x^*)) - G_{g_j}(0)\right) = 0, j \in M.$$

Note that

$$\nabla_x \left(G_\phi(\phi(x^*, y_i))\right) = G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i),$$
$$\nabla \left(G_{g_j}(g_j(x^*))\right) = G'_{g_j}(g_j(x^*))\nabla g_j(x^*), j \in M,$$

One obtains from (3.4) that

$$\lambda_i G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_{ji} G'_{g_j}(g_j(x^*))\nabla g_j(x^*) = 0, \tag{3.5}$$

Therefore, one obtains from (3.5) that

$$\sum_{i=1}^{q^*} \lambda_i G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \left(\sum_{i=1}^{q^*} \mu_{ji}\right) \nabla \left(G_{g_j}(g_j(x^*))\right) = 0,$$

or

$$\sum_{i=1}^{q^*} \frac{\lambda_i}{\sum_{j=1}^{q^*} \lambda_j} G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \left(\frac{\sum_{i=1}^{q^*} \mu_{ji}}{\sum_{i=1}^{q^*} \lambda_i}\right) \nabla \left(G_{g_j}(g_j(x^*))\right) = 0.$$

Let $\lambda_i^* = \frac{\lambda_i}{\sum_{j=1}^{q^*} \lambda_j}$ and $\mu_j^* = \frac{\sum_{i=1}^{q^*} \mu_{ji}}{\sum_{i=1}^{q^*} \lambda_i}$ in the above equation. Then we can deduce the required results.

Theorem 3.3 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let x^* be an optimal solution to the minimax problem (P). Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X,Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ such that $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$. Moreover, the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* $(1 \leq q^* \leq n+1)$ and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* > 0$ $(i = 1, \ldots, q^*)$ and $\mu_j^* \geq 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* G'_\phi(\phi(x^*, y_i)) \nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*)) \nabla g_j(x^*) = 0,$$

$$\mu_j^* G_{g_j}(g_j(x^*)) = G_{g_j}(0), j \in M,$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1.$$

Proof Since $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$, and the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then we can deduce that the vectors $G'_{g_j}(g_j(x^*))\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Now, from Theorem 3.2, we obtain the required results.

Next, we establish the sufficient optimality conditions for the minimax programming problems (P). In the following theorem, we assume that functions constituting the considered nonlinear optimization problem (P) are (G, α)-invex, and we prove that a feasible point $\bar{x}$, at which the G-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled, is an optimal solution.

Theorem 3.4. *Let x^* be a feasible point for (P), G_ϕ be a differentiable and strictly increasing function defined on $I_\phi(X,Y)$, G_{g_j} be a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at x^*. Further, assume that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex with respect to η at x^* on X for $i = 1, \cdots, q^*$, g is vector (G_g, β)-invex with respect to the same function η at x^* on X. Then x^* is an optimal solution to (P).*

Proof Suppose, contrary to the result, that x^* is not an optimal solution for (P). Hence, there exists $x_0 \in X$ such that

$$\sup_{y \in Y} \phi(x_0, y) < \phi(x^*, y_i), i = 1, \cdots, q^*.$$

Thus,

$$\phi(x_0, y_i) < \phi(x^*, y_i), i = 1, \cdots, q^*.$$

Since G_ϕ is strictly increasing on $I_\phi(X,Y)$, then

$$G_\phi(\phi(x_0, y_i)) < G_\phi(\phi(x^*, y_i)), i = 1, \cdots, q^*. \tag{3.6}$$

By the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j, we have

$$G_\phi(\phi(x_0, y_i)) - G_\phi(\phi(x^*, y_i)) \geq \alpha_i(x_0, x^*) G'_\phi(\phi(x^*, y_i)) \nabla_x \phi(x^*, y_i) \eta(x_0, x^*), i = 1, \cdots, q^*, \tag{3.7}$$

$$G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(x^*)) \geq \beta_j(x_0, x^*) G'_{g_j}(g_j(x^*)) \triangledown g_j(x^*) \eta(x_0, x^*), j \in M \tag{3.8}$$

Multiplying (3.7) and (3.8) by λ_i^* and μ_j^* for $i = 1, \cdots, q^*$ and $j \in M$, respectively, we get

$$\left(\sum_{i=1}^{q^*} \bar{\lambda}_i G'_\phi(\phi(x^*, y_i)) \triangledown \phi(x^*, y_i) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(x^*)) \triangledown g_j(x^*) \right) \eta(x_0, x^*) < 0$$

which contradicts the G-Karush-Kuhn-Tucker necessary optimality condition (3.1). Hence, x^* is an optimal solution for (P), and the proof is complete.

4. Duality theorems

Making use of the optimality conditions of the preceding section, we present dual problem (DI) to the minimax problem (P), and establish weak, strong and strict converse duality theorems. For convenience, we use the following notation.

$$K(x) \quad = \{(q,\, \lambda,\, \bar{y}) \in \mathrm{N} \times \mathbb{R}^q_+ \times \mathbb{R}^{mq} | 1 \leqslant q \leqslant n+1, \lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q_+ \text{ with } \textstyle\sum_{i=1}^q \lambda_i = 1,\ \bar{y} = (y_1, \ldots, y_q) \text{ with } y_i \in Y(x), i = 1, \ldots, q\}.$$

$H_1(q,\, \lambda,\, \bar{y})$ denotes the set of all triplets $(z, \mu, \nu) \in R^n \times R^n_+ \times R_+$ satisfying

$$\sum_{i=1}^q \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^m \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) = 0, \tag{4.1}$$

$$\phi(z, y_i) \geq \nu, i = 1,\, 2,\, \cdots,\, q, \tag{4.2}$$

$$\mu_j g_j(z) \geq 0, j \in M, \tag{4.3}$$

$$y_i \in Y(z),\ (q,\, \lambda,\, \bar{y}) \in K(z).$$

Our dual problem (DI) can be stated as follows.

$$\text{(DI)} \max_{(q,\, \lambda,\, \bar{y}) \in K(z)} \sup_{(z, \mu, \nu) \in H_1(q,\, \lambda,\, \bar{y})} \nu$$

Note that if $H_1(q,\, \lambda,\, \bar{y})$ is empty for some triplet $(q,\, \lambda,\, \bar{y}) \in K(z)$, then $\sup_{(z,\mu,\nu) \in H_1(q,\, \lambda,\, \bar{y})} \nu = -\infty$.

Theorem 4.1 (Weak duality). *Let x and $(z, \mu, \nu, q, \lambda,\, \bar{y}\,)$ be (P)-feasible and (DI)-feasible, respectively; let G_ϕ be a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, and G_{g_j} be a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex at z for $i = 1, \ldots, q$, g_j is (G_{g_j}, β_j)-invex at z for $j \in M$. Then*

$$\sup_{y \in Y} \phi(x,\, y) \geqslant \nu.$$

Proof Suppose to the contrary that $\sup_{y \in Y} \phi(x,\, y) < \nu$. Therefore, we obtain

$$\phi(x, y) < \nu \leq \phi(z, y_i),\ \forall\, y \in Y.$$

Thus

$$\phi(x, y_i) < \phi(z, y_i),\ i = 1, \ldots, q.$$

Note that

$$g_j(x) \leq 0,\ \mu_j g_j(z) \geq 0,\ \mu_j \geq 0,\ j \in M.$$

By the increase of G_ϕ and G_{g_j}, we obtain

$$\sum_{i=1}^q \lambda_i \frac{G_\phi(\phi(x, y_i)) - G_\phi(\phi(z, y_i))}{\alpha(x, z)} + \sum_{j=1}^m \mu_j \frac{G_{g_j}(g_j(x)) - G_{g_j}(g_j(z))}{\beta_j(x, z)} < 0. \tag{4.4}$$

Similar to the proof of Theorem 3.4, by (4.4) and the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j for $i = 1, \cdots, q$ and $j \in M$, we have

$$\left(\sum_{i=1}^q \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^m \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) \right) \eta(x, z) < 0.$$

Thus, we have a contradiction to (4.1). So $\sup_{y \in Y} \phi(x,\, y) \geqslant \nu$.

Theorem 4.2 (Strong duality). *Let x^* be an optimal solution of (P). Suppose that G_ϕ is a strictly increasing differentiable function defined on $I_\phi(X, Y)$, G_{g_j} is a strictly increasing differentiable function defined on $I_{g_j}(X)$ for each $j \in M$. Moreover, the vectors $G'_{g_j}(g_j(x^*)) \nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. If the hypothesis of Theorem 4.1 holds for all (DI)-feasible points $(z,\, \mu,\, \nu,\, q,\, \lambda,\, \bar{y}\,)$, then there exists $(q^*,\, \lambda^*,\, \bar{y}^*) \in K$, $(x^*,\, \mu^*,\, \nu^*) \in H_1(q^*,\, \lambda^*,\, \bar{y}^*)$ such that $(q^*,\, \lambda^*,\, \bar{y}^*,\, x^*,\, \mu^*,\, \nu^*)$ is a (DI) optimal solution, and the two problems (P) and (DI) have the same optimal values.*

Proof By Theorem 3.2, there exists $\nu^* = \phi(x^*, y_i^*)$ $(i = 1, \cdots, q^*)$, satisfying the requirements specified in the theorem, such that $(q^*, \lambda^*, \bar{y}^*, x^*, \mu^*, \nu^*)$ is a (DI) feasible solution and $\nu^* = \phi(x^*, y_i^*)$, then the optimality of this feasible solution for (DI) follows from Theorem 4.1.

Theorem 4.3 (Strict converse duality). *Let $\bar{x}$ and $(z, \mu, \nu, q, \lambda, \bar{y})$ be optimal solutions of (P) and (DI), respectively. Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex at z for $i = 1, \ldots, q$, g_j is (G_{g_j}, β_j)-invex at z for $j \in M$. Then $\bar{x} = z$; that is, z is a (P)-optimal solution and $\sup_{y \in Y} \phi(\bar{x}, y) = \nu$.*

Proof Suppose to the contrary that $\bar{x} \neq z$. Using similar arguments as in the proof of Theorem 3.4, we have

$$0 = \left(\sum_{i=1}^{q} \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) \right) \eta(\bar{x}, z)$$
$$< \sum_{i=1}^{q} \lambda_i \frac{G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i))}{\alpha_i(\bar{x}, z)} + \sum_{j=1}^{m} \mu_j \frac{G_{g_j}(g_j(\bar{x})) - G_{g_j}(g_j(z))}{\beta_j(\bar{x}, z)}$$

and

$$\sum_{j=1}^{m} \mu_j \frac{G_{g_j}(g_j(\bar{x})) - G_{g_j}(g_j(z))}{\beta_j(\bar{x}, z)} \leq 0.$$

Therefore,

$$\sum_{i=1}^{q} \lambda_i \frac{G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i))}{\alpha_i(\bar{x}, z)} > 0.$$

From the above inequality, we can conclude that there exists a certain $i_0 \in \{1, \ldots, q\}$, such that

$$G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i)) > 0.$$

It follows that

$$\sup_{y \in Y} \phi(\bar{x}, y) \geq \phi(\bar{x}, y_{i_0}) > \phi(z, y_{i_0}) > \nu. \tag{4.5}$$

On the other hand, we know from Theorem 4.1 that

$$\sup_{y \in Y} \phi(\bar{x}, y) = \nu.$$

This contradicts to (4.5).

5. Conclusion

This paper deals with the minimax programming under (G_f, α)-invexity assumptions which was introduced in [12]. Note that this invexity unifies the G-invexity and α-invexity presented in [4] and [9], respectively. By constructing auxiliary mathematical programmings (G-P), we have discussed the relations between programming problems (G-P) and (P). We have proved G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable minimax programming problem (P). We pointed out that our statement of the so-called G-Kuhn-Tucker necessary optimality conditions established in this paper is more general than the classical Kuhn-Tucker necessary optimality conditions found in the literature. Also, we have proved the sufficiency of the introduced G-Karush-Kuhn-Tucker (G-Kuhn-Tucker) necessary optimality conditions for minimax programming problem (P) involving (G, α)-invexity. Making use of the optimality conditions presented in Section 3, we present dual problem (DI) to the minimax problem (P), and establish weak, strong and strict converse duality theorems.

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