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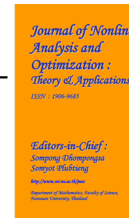

GENERALIZED MIXED GENERAL VECTOR VARIATIONAL-LIKE INEQUALITIES IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this work, we consider and study new kinds of generalized mixed general vector variational-like inequalities in real topological vector spaces. We use the Ferro minimax theorem to discuss the existence of weak and strong solutions for the generalized mixed general vector variational-like inequality problems.

KEYWORDS: Generalized mixed general vector variational-like inequality; Weak solution; Strong solution; Ferro minimax theorem; Topological vector space.
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1. INTRODUCTION

Variational inequalities were introduced and considered by Stampacchia [1] in the early sixties. It has been shown that a wide class of linear and nonlinear problem arising in various branches of mathematical and engineering sciences can be studied within the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions by using novel techniques. In 1980, Giannessi [2] initiated the vector variational inequality in finite dimensional Euclidean space. Since then, Chen *et al.* [3], Lee *et al.* [4, 5], Khan and Salahuddin [6], Yang [7], Ding [8], Ding and Tarafdar [9, 10], Peng [11], Usman and Khan [12], and Irfan and Ahmad [13] have investigated vector variational inequalities in abstract spaces.

The variational-like inequality also known as the pre-variational inequalities is one of the generalized form of variational inequalities. The variational-like inequalities and generalized variational-like inequalities are powerful tools for studying nonconvex optimization problems and nonconvex and nondifferentiable optimization problems respectively, see [2, 14].

In 1988, Noor [15] introduced and studied general variational inequality in Hilbert spaces which can be used to study both odd-order and even-order free and moving boundary value problems. Since then, many authors have further studied various generalizations of general variational inequalities in Hilbert spaces and Banach spaces respectively. For example, see [16–22].

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It is well known that the variational inequality theory and equilibrium problems have wide applications in finance, economics, transportation, optimization and operation research, and the solution sets for variational inequalities are of considerable interest [23–26].

Let X, Y be arbitrary real Hausdorff topological vector spaces. Let 2^Y denotes the family of all nonempty subsets of Y and $L(X,Y)$ the space of all continuous linear mappings from X to Y. Let K be a nonempty set of $X, C : K \to 2^Y$ a set valued mapping such that for each $x \in K, C(x)$ is proper closed convex pointed cone with apex at the origin and $int\ C(x) \neq \emptyset$. The mappings $g : K \to K, A : K \times L(X,Y) \to L(X,Y), T : K \to 2^{L(X,Y)}, h : K \times K \longrightarrow Y$ and $\eta : K \times K \longrightarrow K$ are given. For each $x \in K$, we define the relations $\leq_{C(x)}$ and $\nleq_{C(x)}$ as follows:

(i) $z \leq_{C(x)} y \Leftrightarrow y - z \in C(x)$,
(ii) $z \nleq_{C(x)} y \Leftrightarrow y - z \notin C(x)$.

Similarly we can define the relations $\leq_{intC(x)}$ and $\nleq_{int\ C(x)}$ if we replace the set $C(x)$ by $intC(x)$. If the mapping $C(x)$ is constant, then we write $C(x)$ as C.

Inspired and motivated by recent works of authors see [27–30, 32–34], in this paper, we consider the following generalized mixed general vector variational-like inequality problem (GMGVVLIP): find $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \nleq_{int\ C(\bar{x})} 0. \tag{1.1}$$

Such solution $\bar{x}$ is called a weak solution of the GMGVVLIP (1.1). If $\bar{s}$ does not depend on y, then GMGVVLIP (1.1) reduces to the following problem: find $\bar{x} \in K$ and $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \nleq_{int\ C(\bar{x})} 0,\ \forall\ y \in K. \tag{1.2}$$

Such solution $\bar{x}$ is called a strong solution of the GMGVVLIP (1.2).

If $Y = \mathcal{R}$ and $C(x) = (-\infty, 0]$, then $X^* = L(X, \mathcal{R})$ and $T : K \longrightarrow 2^{X^*}$ is a mapping from K into 2^{X^*} and the GMGVVLIP (1.1) reduces to the following generalized mixed general variational-like inequality problem: find $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \geq 0. \tag{1.3}$$

We remark that if $\bar{s}$ does not depend on y, then the solution of problem (1.3) is called strong solution.

Definition 1.1 Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \to 2^\Sigma$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. For any $x \in K, \psi : K \to \Sigma$ is said to be

(i) $C(x)$-convex iff
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} t\psi(x_1) + (1-t)\psi(x_2)$ for every $x_1, x_2 \in K$ and $t \in [0, 1]$,

(ii) properly quasi $C(x)$-convex iff we have either
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} \psi(x_1)$ or
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} \psi(x_2)$ for every $x_1, x_2 \in K$ and $t \in [0, 1]$.

Definition 1.2 Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \to 2^\Sigma$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. Further, let A be a nonempty subset of Σ, then for any fixed $x \in K$,

(i) a point $z \in A$ is called a minimal point A with respect to the cone $C(x)$ iff $A \cap (z - C(x)) = \{z\}; Min^{C(x)}A$ is the set of all minimal points of A with respect to the cone $C(x)$;

(ii) a point $z \in A$ is called a maximal point A with respect to the cone $C(x)$ iff $A \cap (z + C(x)) = \{z\}; Max^{C(x)}A$ is the set of all maximal points of A with respect to the cone $C(x)$;

(iii) a point $z \in A$ is called a weakly minimal point of A with respect to the cone $C(x)$ iff $A \cap (z - intC(x)) = \emptyset; Min_w^{C(x)}A$ is the set of all weakly minimal points of A with respect to the cone $C(x)$;

(iv) a point $z \in A$ is called a weakly maximal point of A with respect to the cone $C(x)$ iff $A \cap (z + intC(x)) = \emptyset$; $Max_w^{C(x)} A$ is the set of all weakly maximal points of A with respect to the cone $C(x)$.

Definition 1.3 Let X, Y be real topological vector spaces. The set valued mapping $T : X \to 2^Y$ is a closed mapping iff the following holds:
the net $(x_\alpha) \to x_0, (y_\alpha) \to y_0,\ y_\alpha \in T(x_\alpha) \Rightarrow y_0 \in T(x_0)$.

Lemma 1.1[35] *Let K be a nonempty subset of a Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM mapping such that for any $y \in K, G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$.*

Lemma 1.2[9] *Let X and Y be Hausdorff topological vector spaces and $L(X, Y)$ be the topological vector space under the σ-topology. Then, the bilinear mapping*

$$\langle \cdot, \cdot \rangle : L(X, Y) \times X \longrightarrow Y$$

is continuous on $L(X, Y) \times X$, where $\langle l, x\rangle$ denotes the evaluation the linear operator $l \in L(X, Y)$ at $x \in X$.

Lemma 1.3[33] *Let X, Y, Z be the real topological vector spaces, let K and C be two nonempty subsets of X and Y respectively. Let $F : K \times C \to 2^Z,\ T : K \to 2^Y$ be the set valued mappings. If both F and T are upper semicontinuous with nonempty compact values, then the multivalued mapping $G : K \to 2^Z$ defined by*

$$G(x) = \bigcup_{y \in T(x)} F(x, y) = F(x, T(x))$$

is upper semicontinuous with nonempty compact values.

2. EXISTENCE OF WEAK SOLUTIONS FOR THE GMGVVLIP (1.1)

Theorem 2.1 *Let X, Y be the real Hausdorff topological vector spaces, K a nonempty closed convex subset of $X, C : K \to 2^Y$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and int $C(x) \neq \emptyset$. Given the mappings $A : K \times L(X, Y) \to L(X, Y), h : K \times K \to Y, \eta : K \times K \longrightarrow K, g : K \to K, T : K \to 2^{L(X,Y)}$ and $\upsilon : K \times K \to Y$, suppose that*

(i) *$0 \leq_{C(x)} \upsilon(x, x)$ for all $x \in K$;*

(ii) *for each $x \in K$, there is an $s \in T(x)$ such that for all $y \in K$*

$$\upsilon(x, y) - \langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{C(x)} 0;$$

(iii) *for each $x \in K$, the set $\{y \in K : 0 \not\leq_{C(x)} \upsilon(x, y)\}$ is convex;*

(iv) *there is a nonempty compact convex subset D of K such that for every $x \in K \backslash D$, there is a $y \in D$ such that for all $s \in T(x)$*

$$\langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0;$$

(v) *for each $y \in K$, the set*

$$\{x \in K : \langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x)\}$$

is open in K.

Then there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ is a solution of the problem (1.1).

Proof Define a set-valued mapping $\Omega : K \to 2^D$ by

$$\Omega(y) = \{x \in D : \exists\ s \in T(x) \text{ such that } \langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \not\leq_{intC(x)} 0\},$$

for all $y \in K$. From condition (v), we know that for each $y \in K$, the set $\Omega(y)$ is closed in K and hence it is compact in D because of the compactness of D. Next we claim that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property, then whole intersection $\bigcap_{y\in K} \Omega(y)$ is nonempty and any element in the intersection $\bigcap_{y\in K} \Omega(y)$ is a solution of (1.1). For any given nonempty finite subset N of K, let $D_N = conv\{D \cup N\}$, the convex hull of $D \cup N$. Then D_N is a compact convex subset of K. Define the set-valued mappings $S, R : D_N \to 2^{D_N}$ respectively by

$$S(y) = \{x \in D_N : \exists\, s \in T(x) \text{ such that } \langle A(x,s), \eta(y, g(x))\rangle + h(g(x), y) \not\leq_{intC(x)} 0\},$$
$$R(y) = \{x \in D_N : 0 \leq_{C(x)} \upsilon(x,y)\}, \text{ for each } y \in D_N.$$

From the conditions (i) and (ii), we have

$$0 \leq_{C(y)} \upsilon(y,y) \text{ for all } y \in D_N, \tag{2.1}$$

and for each $y \in K$ there is an $s \in T(y)$ such that

$$\upsilon(y,y) - \langle A(y,s), \eta(y, g(y))\rangle + h(g(y), y) \leq_{C(y)} 0.$$

Hence

$$0 \leq_{C(y)} \langle A(y,s), \eta(y, g(y))\rangle + h(g(y), y)$$

and then $y \in S(y)$ for all $y \in D_N$. We can easily see that S has closed value in D_N. Since for each $y \in D_N, \Omega(y) = S(y) \cap D$. If we prove that whole intersection of the family $\{S(y) : y \in D_N\}$ is nonempty, we can deduce that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property because $N \subset D_N$ and due to the condition (iv). In order to deduce the conclusion of our theorem we can apply Lemma 1.1 if we claim that S is a KKM-mapping. Indeed if S is not a KKM- mapping neither is R since $R(y) \subset S(y)$ for each $y \in D_N$, then there is a nonempty finite subset M of D_N such that

$$conv\ M \not\subset \bigcup_{u\in M} R(u).$$

Thus there is an element $\bar{u} \in conv\ M \subset D_N$ such that $\bar{u} \notin R(u)$ for all $u \in M$, that is

$$0 \not\leq_{C(\bar{u})} \upsilon(\bar{u}, u), \text{ for all } u \in M.$$

By (iii) we have

$$\bar{u} \in convM \subset \{y \in K : 0 \not\leq_{C(\bar{u})} \upsilon(\bar{u}, y)\}$$

and hence

$$0 \not\leq_{C(\bar{u})} \upsilon(\bar{u}, \bar{u}),$$

which contradicts (2.1). Hence R is a KKM-mapping and so is S. Therefore there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

. That is, $\bar{x} \in K$ is a solutions of the problem (1.1). This completes the proof.

If we further assume that $h : K \times K \longrightarrow Y$ is continuous on $K, \eta : K \times K \longrightarrow K$ is also continuous, let the mappings $A : K \times L(X,Y) \to L(X,Y), g : K \to K$ be continuous and $T : K \to 2^{L(X,Y)}$ be upper semicontinuous with nonempty compact values. Then, by using Lemma 1.2 and Lemma 1.3, we can prove that the condition (v) of Theorem 2.1 is satisfied. Hence we have the following result.

Theorem 2.2 *Let X, Y be real Hausdorff topological vector spaces, K a nonempty closed convex subset of $X, C : K \to Y$ a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. Let the mappings $A : K \times L(X,Y) \to L(X,Y), h : K \times K \to Y, \eta : K \times K \to K$ and $g : K \to K$ be continuous. Let $T : K \to 2^{L(X,Y)}$ be the upper semicontinuous with nonempty compact values and $\upsilon : K \times K \to Y$. Suppose that*

(i) *the conditions (i)-(iv) of Theorem 2.1 hold;*

(*ii*) *the mapping* $W : K \longrightarrow 2^Y$ *defined by* $W(x) = Y \setminus (-intC(x)),\ \forall\, x \in K$ *is upper semicontinuous.*

Then there exists $\bar{x} \in K$ *such that for each* $y \in K$, *there exists* $\bar{s} \in T(\bar{x})$ *satisfying*

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ *is a solution of the problem (1.1).*

Proof For each fixed $y \in K$, define the mappings $F : L(X,Y) \times K \longrightarrow Y$ and $G : K \longrightarrow 2^Y$ by

$$F(s,x) = \langle s, \eta(y, g(x))\rangle + h(g(x), y) \text{ and}$$
$$G(x) = \bigcup_{s \in T(x)} F(s,x).$$

It follows from the continuity of the mapping A, h, η, g and Lemma 1.2 that the mapping F is continuous. Since T is upper semicontinuous with nonempty compact values, it follows from Lemma 1.3 that G is also upper semicontinuous on K with nonempty compact values. We claim that for each $y \in Y$, the set $M = \{x \in K : G(x) \nsubseteq (-intC(x))\}$ is closed in K. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in M and $x_\lambda \longrightarrow x_0$. Then we have $x_\lambda \in K$, $x_0 \in K$ and $G(x_\lambda) \nsubseteq (-intC(x))$ for each $\lambda \in \Lambda$. Hence, for each $\lambda \in \Lambda$, there exists $u_\lambda \in G(x_\lambda)$ such that $u_\lambda \notin (-intC(x))$ and hence $u_\lambda \in K \setminus (-intC(x))$. Noting that the set $L = \{x_\lambda\}_{\lambda\in\Lambda} \bigcup \{x_0\}$ is compact and G upper semicontinuous with compact values we have $G(L)$ is compact. Since $\{u_\lambda\}_{\lambda\in\Lambda} \subseteq G(L)$, without any loss of generality, we can assume $u_\lambda \longrightarrow u_o$. By the upper semicontinuous of the mappings G and W, we have $u_o \in G(x_o)$ and so $u_o \notin (-intC(x_o))$. Hence $G(x_o) \nsubseteq (-intC(x_o))$, $x_0 \in M$ and M is a closed set. It follows that the set

$$\{x \in K : \langle A(x,s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall s \in T(x)\}$$
$$= \{x \in K : G(x) \subseteq -intC(x)\} = K \setminus M$$

is open in K. Then all the conditions of Theorem 2.1 hold. By Theorem 2.1, there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ is a solutions of the problem (1.1).

Theorem 2.3 *Let* $X, Y, K, C, A, h, \eta, g, T$ *be the same as in Theorem 2.1. Assume that for each* $x \in K, h$ *is* $C(x)$*-convex in* K *such that*

(*i*) *for each* $x \in K$, h *is* $C(x)$*-convex in the second argument;*

(*ii*) η *is affine at first argument;*

(*iii*) *for each* $x \in K$ *there is an* $s \in T(x)$ *such that*

$$\langle A(x,s), \eta(x, g(x))\rangle + h(g(x), x) \not\leq_{intC(x)} 0;$$

(*iv*) *there is a nonempty compact convex subset* D *of* K *such that for every* $x \in K\setminus D$ *there is* $y \in D$ *such that*

$$\langle A(x,s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x);$$

(*v*) *for each* $y \in K$, *the set*

$$\{x \in K : \langle A(x,s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x)\}$$

is open in K.

Then, there exists $\bar{x} \in K$ *such that for each* $y \in K$, *there exists* $\bar{s} \in T(\bar{x})$ *satisfying*

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ *is a solutions of the problem (1.1).*

Proof For any given nonempty finite subset N of K let $D_N = conv(D \cup N)$. Then D_N is a nonempty compact convex subset of K. Define $\Omega : K \longrightarrow 2^D$ and $S : D_N \to 2^{D_N}$ as in the proof of Theorem 2.1. We note that for each $x \in D_N, S(x)$ is nonempty and closed since

$x \in S(x)$ by conditions (iii) and (v). For each $y \in K, \Omega(y)$ is nonempty and compact in D. Next we claim that S is a KKM-mapping. Indeed if not there is a nonempty finite subset M of D_N such that

$$conv\ M \not\subset \bigcup_{x \in M} S(x).$$

Then there is an $x^* \in convM \subset D_N$ such that

$$\langle A(x^*, s), \eta(x, g(x^*))\rangle + h(g(x^*), x) \leq_{intC(x^*)} 0,\ \forall\ x \in M,\ s \in T(x^*).$$

Since η is affine in the first argument and h is $C(x^*)$-convex in the second variable, the mapping

$$x \to \langle A(x^*, s), \eta(x, g(x^*))\rangle + h(g(x^*), x), \forall s \in T(x^*)$$

is also $C(x^*)$-convex on D_N. Hence we can deduce that

$$\langle A(x^*, s), \eta(x^*, g(x^*))\rangle + h(g(x^*), x^*) \leq_{intC(x^*)} 0, \text{ for all } s \in T(x^*).$$

This contradict the condition (iii). Therefore S is a KKM-mapping by Lemma 1.4, we have

$$\bigcap_{x \in D_N} S(x) \neq \emptyset.$$

Note that for any $u \in \bigcap_{x \in D_N} S(x)$, we have $u \in D$ by condition (iv). Hence, we have

$$\bigcap_{y \in N} \Omega(y) = \bigcap_{y \in N} (S(y) \cap D) \neq \emptyset$$

for each nonempty finite subset N of K. Therefore the whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty. Let $\bar{x} \in \bigcap_{y \in K} \Omega(y)$. Then $(\bar{x}, \bar{s})$ is a solution of problem (1.1).

3. EXISTENCE OF STRONG SOLUTIONS FOR THE GMGVVLIP (1.2)

Theorem 3.1 *Let X be a real Banach space, Y, K, C, η, A, h, g and υ be the same as in Theorem 2.2. Under the assumptions of Theorem 2.2, we have a weak solution $\bar{x}$ of the GMGVVLIP (1.1) with $\bar{s} \in T(\bar{x})$. In addition, if K is compact, $x \to Y \backslash \{-intC(x)\}$ a closed mapping on $K, T(\bar{x})$ is convex, h is $C(\bar{x})$-convex in the second argument and continuous on K, the mappings $A : K \times L(X, Y) \to L(X, Y), g : K \to K$ are continuous, $\eta : K \times K \longrightarrow K$ is continuous and affine in the first argument, $T : K \to 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values and the mapping $s \to -\langle A(x, s), \eta(x, g(\bar{x}))\rangle$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume that*

$$(L^*)\ Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset$$

$$Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} + C(\bar{x}), \forall s \in T(\bar{x}).$$

Assume also that

(i) *for any fixed $x \in K$, if*

$$\delta \in Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}$$

and δ can not be compared with

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)$$

which does not equal to δ, then

$$\delta \not\leq_{intC(\bar{x})} 0,$$

(ii) if

$$Max^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x\in K} \{\langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y\backslash(-intC(\bar{x})),$$

there exists an $s \in T(\bar{x})$ such that

$$Min_w^{C(\bar{x})} \bigcup_{x\in K} \{\langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y\backslash(-intC(\bar{x})).$$

Then $\bar{x}$ is a strong solution of the GMGVVLIP (1.2), that is there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \ \forall\, x \in K.$$

Furthermore, the set of all strong solutions of problem (1.2) is compact.

Proof Since η is affine in the first argument and h is $C(\bar{x})$-convex in the second argument on K, the mapping

$$x \to \langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)$$

is also $C(\bar{x})$-convex on K. Since the mapping

$$s \to -\langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle$$

is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $\bar{x} \in K$, it follows that the mapping

$$s \to -\langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x), \text{ for each } x \in K$$

is also properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $\bar{x} \in K$. From Theorem 2.1, we know that $\bar{x} \in K$ such that (1.1) holds for all $x \in K$ and for some $\bar{s} \in T(\bar{x})$. Then

$$\forall \gamma \in Min^{C(\bar{x})} \bigcup_{x\in K} Max^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} \{\langle A(\bar{x},s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\},$$

by (i), we have

$$\gamma \nleq_{intC(\bar{x})} 0.$$

From condition (L^*), the convexity of $T(\bar{x})$, and the Ferro Minimax Theorem [27] we have, for every

$$\alpha \in \text{Max}^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} \text{Min}_w^{C(\bar{x})} \bigcup_{x\in K} \{\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}, \ \alpha \nleq_{intC(\bar{x})} 0.$$

This implies that

$$\text{Max}^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} \text{Min}_w^{C(\bar{x})} \bigcup_{x\in K} \{\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y\backslash(-intC(\bar{x})).$$

From (ii) there is an $\bar{s} \in T(\bar{x})$ such that

$$\text{Min}_w^{C(\bar{x})} \bigcup_{x\in K} \{\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y\backslash(-intC(\bar{x})).$$

Hence we know that

$$\forall \rho \in \bigcup_{x\in K} \{\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\},$$

therefore

$$\rho \nleq_{int\ C(\bar{x})} 0.$$

Hence there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \text{ for all } x \in K,$$

such that $\bar{x}$ is a strong solution of the GMGVVLIP (1.2).
Finally to see that the solution set of the GMGVVLIP (1.2) is compact, it is sufficient to show that the solution set is closed due to the coercivity condition (iv) of Theorem 2.1. To this end

let Γ denote the solution set of the GMGVVLIP (1.2). Suppose that a net $\{x_\lambda\} \subset \Gamma$ which converges to p. Fix any $y \in K$, for each λ, there is $s_\lambda \in T(x_\lambda)$ such that

$$\langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle + h(g(x_\lambda), y) \nleq_{intC(x_\lambda)} 0. \tag{3.1}$$

Since T is upper semicontinuous with nonempty compact values and the set $\{x_\lambda\} \cup \{p\}$ is compact, it follows that $T(\{x_n\} \cup \{p\})$ is compact. Therefore, without loss of generality, we may assume that the sequence $\{s_\lambda\}$ converges to some s. Then $s \in T(p)$ and

$$h(g(x_\lambda), y) - \langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle \notin intC(x_\lambda).$$

This implies that

$$h(g(x_\lambda), y) - \langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle \in Y \backslash (-intC(x_\lambda)).$$

By the continuity of A, η, g and h and Lemma 1.2, we have

$$h(g(p), y) - \langle A(x, s), \eta(y, g(p))\rangle$$
$$= \lim_{x \to \infty} h(g(x_n), y) - \langle A(x_n, s_n), \eta(y, g(x_n))\rangle \in Y \backslash (-intC(p)).$$

Then we obtain

$$\langle A(x, s), \eta(y, g(p))\rangle + h(g(p), y) \nleq_{intC(p)} 0.$$

Hence $p \in \Gamma$ and Γ is closed.

Theorem 3.2 *Let X be a real Banach space, let Y, K, C, A, h, g, η and T be as in Theorem 2.2. Under the assumption of Theorem 2.2, we have a weak solution $\bar{x}$ of the problem (1.1) with $\bar{s} \in T(\bar{x})$. In addition, if $T(\bar{x})$ is convex, h is $C(\bar{x})$-convex with respect to the first variable, $x \to Y \backslash (-intC(x))$ a closed mapping on K and the mappings $A : K \times L(X, Y) \to L(X, Y), g : K \to K, h : K \times K \to Y$ are continuous. Suppose that $T : K \to 2^{L(X,Y)}$ is upper semi continuous with nonempty compact values and the mapping $s \to -\langle A(x, s), \eta(x, g(\bar{x}))\rangle$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume for any nonempty compact subset M of K :*

$$(L^*)\ Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset$$
$$Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} + C(\bar{x}), \forall s \in T(\bar{x}).$$

Assume also that

(i) *for any fixed $x \in M$, if*

$$\delta \in Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}$$

and δ can not be compared with

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x),$$

which is not equal to δ, then

$$\delta \nleq_{int\ C(\bar{x})} 0,$$

(ii) *if*

$$Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{\bar{x} \in M} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y \backslash (-intC(\bar{x})).$$

Then $(\bar{x}, \bar{s})$ is a strong solution of the problem (1.2), that is there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \text{ for all } x \in K.$$

Furthermore, the set of all strong solutions of the problem (1.2) is compact.

proof Let $\bar{B}(0, r) = \{x \in X : \|x\| \leq r\}$ for each $r > 0$, then the set $K_r = \bar{B}(0, r) \cap K$ is compact in X. If $K_r \neq \emptyset$ and we replace K by K_r, in Theorem 3.1, all the conditions of Theorem 3.1 hold. Hence by Theorem 3.1 there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(z, g(\bar{x}))\rangle + h(g(\bar{x}), z) \nleq_{int\ C(\bar{x})} 0, \text{ for all } z \in K_r. \tag{3.2}$$

Let us choose $r > \|g(\bar{x})\|$. Since g is continuous and convex for any $x \in K$, choose $t \in (0, 1)$ small enough such that $(1-t)\bar{x} + tx \in K_r$. Putting $z = (1-t)\bar{x} + tx$ in (3.3), we have

$$\langle A(\bar{x}, \bar{s}), \eta((1-t)\bar{x} + tx, g(\bar{x}))\rangle + h(g(\bar{x}), (1-t)\bar{x} + tx) \nleq_{intC(\bar{x})} 0.$$

We note that

$$\begin{aligned} & t\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), (1-t)\bar{x} + tx) \\ \leq_{C(\bar{x})} & \ t\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + (1-t)h(g(\bar{x}), \bar{x}) + th(g(\bar{x}), x) \\ = & \ t\{\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}, \end{aligned}$$

which implies that

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \text{ for all } x \in K.$$

This completes the proof.

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References

1. Stampacchia, G.: Forms bi linearies coercives sur les ensembles convexes. C. R. Acad. Sci. Paris, **258**, 4413-4416 (1964)
2. Giannessi, F.: Theorem of alternative, quadratic program and complementarity problems: In R.W. Cottle, F. Giannessi, J.L. Lions (eds) Variational Inequalities and Complementarity Problems, pp. 151-186, Wiley, New York, (1980)
3. Chen, G.Y., Cheng, G.M.: Vector variational inequalities and vector optimization. In Lecture Notes in Economics and Mathematical Systems, 285: 408–456 (1987)
4. Lee, G.M., Lee, B.S., Chang, S.S.: On vector quasivariational inequalities. J. Math. Anal. Appl. **203**, 626-639 (1996)
5. Lee, G.M., Kim, D.S., Cho, S,J.: Generalized vector variational inequalities and fuzzy extensions. Appl. Math. Lett. **6**(6), 47–51 (1993)
6. Khan, M.F., Salahuddin.: On generalized vector variational like inequalities. Nonlinear Anal. **59**, 879-889 (2004)
7. Yang, X.Q.: Generalized convex functions and vector variational inequality. J. Optim. Theory Appl. **79**, 563-580 (1993)
8. Ding, X.P.: The generalized vector quasi-variational-like inequalities. Computers and Mathematics with Applications **37**, 57-67 (1999)
9. Ding X.P., Tarafdar, E.: Generalized vector variational-like inequalities without monotonicity. In: Vector Variational Inequalities and Vector Equilibrium, Mathematical Theories. (Eds. by Giannessi F.), Kluwer Academic Publishers, 113-123, (2000)
10. Ding X.P., Tarafdar, E.: Generalized vector variational-like inequalities with C_x-η-pseudomonotone mappings. In: Vector Variational Inequalities and Vector Equilibrium, Mathematical Theories. (Eds. by Giannessi F.), Kluwer Academic Publishers, 125-140, (2000)
11. Peng, J.W.: Equilibrium problems on W space. Mathematica Applicata **12**(3), 81–87 (1999)
12. Usman, F., Khan, S.A.: A generalized mixed vector variational-like inequality problem. Nonlinear Anal. **71**(11), 5354-5362 (2009)
13. Irfan, S.S., Ahmad R.: Generalized multivalued vector variational-like inequalities. J. Glob. Optim. **46**(1), 25-30 (2010)

14. Daniele, P., Giannessi F., Maugeri, A.: Equilibrium problems and variational models. Nonconvex Optimization and its Applications, Vol. 68, Kluwer Academic Massachusetts, (2003)
15. Noor, M.A.: Multivalued general equilibrium problems. J. Math. Anal. Appl. **283**, 140-149 (2003)
16. Moudafi. A., Théra, M.: Proximal and dynamical approaches to equilibrium problems, in: Lecture Notes in Economics and Mathematical Systems, Berlin, Springer-Verlag, **477** 187-201 (1999)
17. Kazmi, K. R., Khan, F. A.: Existence and iterative approximation of solutions of generalized mixed equilibrium problems. Comput. Math. Appl. **56** 1314-1321 (2008)
18. Huang, N. J., Lan, H. Y., Cho, Y. J.: Sensitivity analysis for nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings. J. Comput. Appl. Math. **196** 608-618 (2006)
19. Ding, X. P.: Iterative algorithm of solutions for generalized mixed implicit equilibrium-like problems. Appl. Math. Comput. **162**(2) 799–809 (2005)
20. Ding, X. P., Lin, Y. C., Yao, J. C.: Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems. Appl. Math. Mech. **27**(9) 1157-1164 (2006)
21. Ding, X. P.: Approximation solvability of system of generalized mixed implicity equilibrium problems in Banach spaces. J. Sichuan Normal Univ. (N.S.) **34**(1) 1-9 (2011)
22. László, S.: Some existence results of solutions for general variational inequalities. J. Optim. Theory Appl. **150** 425-443 (2011)
23. Blum, E., Ottli, W.: From Optimization and Variational inequalities to Equilibrium problems. Math. Stud. **63**, 123-145 (1994)
24. Fang, Y.P., Huang, N.J.: Strong vector variational inequalities in Banach spaces. Appl. Math. Lett. **19**, 362-368 (2006)
25. Mishra, S.K., Noor, M.A.: On vector variational like inequality problems. J. Math. Anal. Appl. **311**, 69-75 (2005)
26. Ding, X.P., Salahuddin : Generalized vector mixed general quasi-variational-like inequalities in Hausdorff topological vector spaces. Optim. Lett. DOI 10.1007/s11590-012-0464-x (2012)
27. Ferro, F.: A minimax theorem for vector valued functions. J. Optim. Theory Appl. **60**, 19-31 (1989)
28. Huang, N.J., Fang, Y.P.: On vector variational inequalities in reflexive Banach spaces. J. Global Optim. **32**, 495-505 (2005)
29. Khaliq, A., Rashid, M.: On generalized vector quasi variational like inequality problems. Fixed Point Theo. Appl. **3**, 243-255 (2005)
30. Lee, B.S., Khan, M.F., Salahuddin : Generalized vector variational type inequalities. Comp. Math. Appl. **55**(6), 1164-1169 (2008)
31. Lin, L.J., Yu, Z.T.: On some equilibrium problems for multimapping. J. Comput. Appl. Math. **129**, 171-183 (2001)
32. Lin, Y.C.: On F-implicit generalized vector variational inequalities. J. Optim. Theory Appl. **142**, 557-568 (2009)
33. Zeng, L.C., Lin, Y.C., Yao, J.C.: On weak and strong solutions of F-implicit generalized variational inequalities with applications. Appl. Math. Lett., **19**, 684-689 (2004)
34. Zeng, L.C., Yao, J.C.: Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces. J. Global Optim. **36**, 483-497 (2006)
35. Fan, K.: A generalization of Tychonoff fixed point theorems. Math. Ann. **142**, 305-310 (1961)