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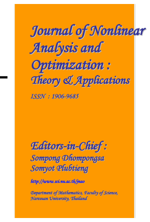


# PPF DEPENDENT FIXED POINT THEOREMS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN BANACH SPACES

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**ABSTRACT.** In this paper, we prove the existence of the PPF dependent fixed point theorems in the Razumikhin class for rational type contraction mappings where the domain and range of the mappings are not the same. We also use this result to prove the PPF dependent coincidence point theorems. Our results extend and generalize some results of Bernfeld *et al.* in [S. R. Bernfeld, V. Lakshmikatham and Y. M. Reddy, Fixed point theorems of operators with PPF dependence in Banach spaces, Applicable Anal. 6 (1977), 271–280.].

**KEYWORDS**: PPF fixed points; Razumikhin classes; Rational type contraction.
**AMS Subject Classification**: 47H09 47H10

## 1. INTRODUCTION

The applications of fixed point theory are very important in diverse disciplines of mathematics since they can be applied for solving various problems, for instance, equilibrium problems, variational problems, and optimization problems. The Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to the importance, generalizations of Banach's contraction mapping principle have been investigated heavily by many mathematicians. One of the most interesting is extension of Banach's contraction mapping principle in case of non-self mappings.

In 1997, Bernfeld *et al.* [1] introduced the concept of PPF dependent fixed point or the fixed point with PPF dependence which is a one type of fixed point for mappings that have different domains and ranges. They also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. The PPF dependent fixed point theorems are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data

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and future consideration. Afterward, a number of papers appeared in which PPF dependent fixed point theorems have been discussed (see [3, 4, 5] and references therein).

On the other hand, Dass and Gupta [2] and Jaggi [6] were first to establish the existence of fixed point theorems using contractive conditions involving rational expressions. To the best of our knowledge, there is no discussion so far concerning the PPF dependent fixed point theorems via rational contractive conditions.

In this paper, we will introduce the rational type contraction non-self mapping and also establish the existence of PPF dependent fixed point theorems for such mapping in Razumikhin class. Furthermore, we apply this result to the existence of PPF dependence coincidence point theorems. Our results extend some result in [1].

## 2. PRELIMINARIES

Throughout this paper, let $E$ denotes a Banach space with the norm $\|\cdot\|_E$, $I$ denotes a closed interval $[a, b]$ in $\mathbb{R}$, and $E_0 = C(I, E)$ denotes the set of all continuous $E$-valued functions on $I$ equips with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E.$$

A point $\phi \in E_0$ is said to be *a PPF dependent fixed point* or *a fixed point with PPF dependence* of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in $E_0$ is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

It is easy to see that if the function $\widetilde{\phi} \in E_0$ is a constant function, then $\widetilde{\phi} \in \mathcal{R}_c$.

The class $\mathcal{R}_c$ is algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on $E_0$ generated by the norm $\|\cdot\|_{E_0}$.

The Razumikhin class play an important role in proving the existence of PPF fixed points with different domain and range of the mappings in abstract spaces.

**Definition 2.1** (Bernfeld *et al.* [1]). The mapping $T : E_0 \to E$ is called Banach type contraction if there exists a real number $\alpha \in [0, 1)$ such that

$$\|T\phi - T\xi\|_E \leq \alpha\|\phi - \xi\|_{E_0} \tag{2.1}$$

for all $\phi, \xi \in E_0$.

The following PPF dependent fixed point theorem is proved in Bernfeld *et al.* [1].

**Theorem 2.2** (Bernfeld *et al.* [1]). *Let $T : E_0 \to E$ be a Banach type contraction. If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

## 3. PPF DEPENDENT FIXED POINT THEOREMS

First of all, we introduce the definition of the rational type contraction mappings.

**Definition 3.1.** The mapping $T : E_0 \to E$ is called rational type contraction if there exist real numbers $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $c \in I$ such that

$$\|T\phi - T\xi\|_E \leq \alpha\|\phi - \xi\|_{E_0} + \frac{\beta\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{3.1}$$

for all $\phi, \xi \in E_0$.

It is easy to see that every Banach type contraction mapping is rational type contraction mapping, but the converse is necessarily not true.

Next, we prove PPF dependent fixed point theorems for rational type contraction mappings.

**Theorem 3.2.** *Let $T : E_0 \to E$ be a rational type contraction mapping. If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{3.2}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof.* Let $\phi_0$ be an arbitrary function in $\mathcal{R}_c \subseteq E_0$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. Since $\mathcal{R}_c$ is algebraically closed with respect to difference, we have

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$.

Next, we will show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$.

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \\
&= \|T\phi_{n-1} - T\phi_n\|_E \\
&\leq \alpha\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \frac{\beta\|\phi_{n-1}(c) - T\phi_{n-1}\|_E \|\phi_n(c) - T\phi_n\|_E}{1 + \|T\phi_{n-1} - T\phi_n\|_E} \\
&= \alpha\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \frac{\beta\|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E} \\
&\leq \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \beta\|\phi_{n-1}(c) - \phi_n(c)\|_E \\
&= \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \beta\|\phi_{n-1} - \phi_n\|_{E_0} \\
&= (\alpha + \beta)\|\phi_{n-1} - \phi_n\|_{E_0}.
\end{aligned}$$

Hence, by repeated application of the above relation yields

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n \|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$, where $k = \alpha + \beta$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} \\
&\quad + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of $E_0$, we get $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$.

Now we prove that $\phi^*$ is a PPF dependent fixed point of $T$. From the assumption of rational type contraction of $T$, we get

$$\|T\phi^* - \phi^*(c)\|_E \leq \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E$$

$$
\begin{aligned}
&= \|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \alpha\|\phi^* - \phi_{n-1}\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|T\phi^* - T\phi_{n-1}\|_E} \\
&\quad + \|\phi_n - \phi^*\|_{E_0} \\
&= \alpha\|\phi^* - \phi_{n-1}\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|T\phi^* - \phi_n(c)\|_E} \\
&\quad + \|\phi_n - \phi^*\|_{E_0}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E = 0$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.

Finally, we prove the uniqueness of PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Let $\phi^*$ and $\xi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$. Therefore,

$$
\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E \\
&= \|T\phi^* - T\xi^*\|_E \\
&\leq \alpha\|\phi^* - \xi^*\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E \|\xi^*(c) - T\xi^*\|_E}{1 + \|T\phi^* - T\xi^*\|_E} \\
&= \alpha\|\phi^* - \xi^*\|_{E_0}.
\end{aligned}
$$

Since $\alpha < 1$, we have $\|\phi^* - \xi^*\|_{E_0} = 0$ and hence $\phi^* = \xi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Remark 3.3.** If the Razumikhin class $\mathcal{R}_c$ is not topologically closed, then the limit of the sequence $\{\phi_n\}$ in Theorem 3.2 may be outside of $\mathcal{R}_c$. Therefore, a PPF dependent fixed point of $T$ may not be unique.

By applying Theorem 3.2, we obtain the following corollaries.

**Corollary 3.4.** *Let $T : E_0 \to E$ and there exists a real number $\alpha \in [0, 1)$ such that*

$$\|T\phi - T\xi\|_E \leq \alpha\|\phi - \xi\|_{E_0} \tag{3.3}$$

*for all $\phi, \xi \in E_0$.*

*If there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{3.4}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

**Corollary 3.5.** *Let $T : E_0 \to E$ and there exists a real number $\beta \in [0, 1)$ and $c \in I$ such that*

$$\|T\phi - T\xi\|_E \leq \frac{\beta\|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{3.5}$$

*for all $\phi, \xi \in E_0$.*

*If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{3.6}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

## 4. PPF DEPENDENT COINCIDENCE POINT THEOREMS

**Definition 4.1.** Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of $T$ and $S$ if $T\phi = (S\phi)(c)$ for some $c \in I$.

Next, we introduce the condition of the rational type contraction for a pair of two mappings.

**Definition 4.2.** Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. The ordered pair $(T, S)$ is said to satisfy the condition of rational type contraction if there exist real numbers $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $c \in I$ such that

$$\begin{aligned} \|T\phi - T\xi\|_E &\leq \alpha\|S\phi - S\xi\|_{E_0} \\ &\quad + \frac{\beta\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \end{aligned} \tag{4.1}$$

for all $\phi, \xi \in E_0$.

It easy to see that if $(T, S)$ satisfie the condition of rational type contraction and $S$ is identity mapping, then $T$ is a rational type contraction mapping.

Now, we apply Theorem 3.2 to the PPF dependent coincidence point theorem.

**Theorem 4.3.** *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

*$(i)$: $(T, S)$ satisfies the condition of rational type contraction;*

*$(ii)$: $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

*If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then $T$ and $S$ have a PPF dependent coincidence point.*

*Proof.* Consider the mapping $S : E_0 \to E_0$. We obtain that there exists $F_0 \subseteq E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since

$$T(F_0) \subseteq T(E_0) \subseteq E,$$

we can define a mapping $\mathcal{A} : S(F_0) \to E$ by

$$\mathcal{A}(S\phi) = T\phi \tag{4.2}$$

for all $\phi \in F_0$. Since $S|_{F_0}$ is one-to-one, then $\mathcal{A}$ is well-defined.

From (4.2) and the condition of rational type contraction of $(T, S)$, we have

$$\begin{aligned} \|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E &\leq \alpha\|S\phi - S\xi\|_{E_0} \\ &\quad + \frac{\beta\|(S\phi)(c) - \mathcal{A}(S\phi)\|_E\|(S\xi)(c) - \mathcal{A}(S\xi)\|_E}{1 + \|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E} \end{aligned}$$

for all $S\phi, S\xi \in S(E_0)$. This shows that $\mathcal{A}$ is a rational type contraction mapping.

Now, we use Theorem 3.2 with a mapping $\mathcal{A}$, then there exists a unique PPF dependent fixed point $\varphi \in S(F_0)$ of $\mathcal{A}$, that is, $\mathcal{A}\varphi = \varphi(c)$. Since $\varphi \in S(F_0)$, we can find $\omega \in F_0$ such that $\varphi = S\omega$. Therefore, we get

$$T\omega = \mathcal{A}(S\omega) = \mathcal{A}\varphi = \varphi(c) = (S\omega)(c).$$

This implies that $\omega$ is a PPF dependent coincidence point of $T$ and $S$. This completes the proof. □

By applying Theorem 4.3, we obtain the following corollaries.

**Corollary 4.4.** *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

*$(i)$: there exists a real number $\alpha \in [0, 1)$ such that*

$$\|T\phi - T\xi\|_E \leq \alpha\|S\phi - S\xi\|_{E_0} \tag{4.3}$$

*for all $\phi, \xi \in E_0$;*

*$(ii)$: there exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

*If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then $T$ and $S$ have a PPF dependent coincidence point in $\mathcal{R}_c$.*

**Corollary 4.5.** *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

$(i)$: *there exists a real number $\beta \in [0, 1)$ and $c \in I$ such that*

$$\|T\phi - T\xi\|_E \leq \frac{\beta\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{4.4}$$

*for all $\phi, \xi \in E_0$;*

$(ii)$: $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

*If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then $T$ and $S$ have a PPF dependent coincidence point in $\mathcal{R}_c$.*

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