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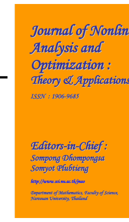

SOME FIXED POINT RESULTS FOR UNIFORMLY QUASI-LIPSCHITZIAN MAPPINGS IN CONVEX METRIC SPACES

ISA YILDIRIM*,[1], SAFEER HUSSAIN KHAN[2] AND MURAT OZDEMIR[3]

[1,3] Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey

[2] Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar

ABSTRACT. In this paper, an iteration process for approximating common fixed points of two uniformly quasi Lipschitzian mappings in convex metric spaces is defined. Without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with asymptotically (quasi-)nonexpansive mappings, some convergence theorems are also proved. The results presented generalize, improve and unify some recent results.

KEYWORDS: Uniformly quasi-Lipschitzian mappings; Common fixed points; Convex metric spaces.
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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. We also denote by $F(T)$ the set of fixed points of T and by $F = F(T) \cap F(S)$ the set of common fixed points of two mappings T and S.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be asymptotically nonexpansive, if there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \le k_n d(x, y), \quad \forall x, y \in X, \ n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then T is said to be asymptotically quasi-nonexpansive, if there exists $k_n \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \le k_n d(x, p), \quad \forall x \in X, \ p \in F(T), \ n \in \mathbb{N}.$$

T is said to be uniformly quasi-Lipschitzian, if there exists a constant $L > 0$ (called Lipschitz constant) such that

$$d(T^n x, p) \le L d(x, p), \quad \forall x \in X, \ p \in F(T), \ n \in \mathbb{N}.$$

* *Corresponding author.*
Email address : isayildirim@atauni.edu.tr (I. Yildirim), safeer@qu.edu.qa (S. H. Khan), mozdemir@atauni.edu.tr (M. Ozdemir).
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Remark 1.1. If $F(T) \neq \emptyset$, it follows from the above definitions that each asymptotically nonexpansive mapping must be an asymptotically quasi-nonexpansive, and each asymptotically quasi-nonexpansive mapping must be a uniformly quasi-Lipschitzian, where $L = \sup_{n\geq 0}\{k_n\} < \infty$. But the converse may not necessarily hold.

The approximation problems concerned with the fixed points of the asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively by many authors in recent years. Takahashi [4] introduced the notion of a convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. A normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space (see [4]). Later on, Tian [5] gave some sufficient and necessary conditions such that Ishikawa iteration process for an asymptotically quasi-nonexpansive mapping converges to a fixed point in a convex metric space. Liu et al. [3] and Wang and Liu [6] gave some sufficient and necessary conditions for Ishikawa iteration process with errors to approximate common fixed points of two uniformly quasi-Lipschitzian mappings in a convex metric space. Also, Chang et al. [1], Khan and Abbas [2], Yildirim and Khan [8] and other authors have studied fixed point theorems in convex metric spaces.

We recall the following which can be found in [5].

Let (X,d) be a metric space.

- A mapping $W : X^3 \times [0,1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition: for any $(x,y,z;a,b,c) \in X^3 \times [0,1]^3$ with $a+b+c=1$, and $u \in X$:
$$d(W(x,y,z;a,b,c),u) \leq ad(x,u) + bd(y,u) + cd(z,u).$$
- If (X,d) is a metric space with a convex structure W, then (X,d) is called a convex metric space.
- Let (X,d) be a convex metric space, a nonempty subset E of X is said to be convex, if $W(x,y,z;a,b,c) \in E$, $\forall (x,y,z) \in E^3$, $(a,b,c) \in [0,1]^3$ with $a+b+c=1$.

Recently, Wang and Liu [6] considered the following iteration process for uniformly quasi-Lipschitzian mappings S and T in convex metric spaces:

$$\begin{aligned} x_{n+1} &= W(x_n, S^n y_n, u_n; a_n, b_n, c_n), \\ y_n &= W\left(x_n, T^n x_n, v_n; a_n', b_n', c_n'\right) \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}, \{c_n'\}$ are six sequences in $[0,1]$ with $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$, $n \in \mathbb{N}$ and $\{u_n\}, \{v_n\}$ are two sequences in X satisfying condition: For any nonnegative integers $n, m, 0 \leq n < m$, if $\delta(A_{nm}) > 0$, then

$$\max_{n\leq i,j\leq m}\{d(x,y) : x \in \{u_i, v_i\}, y \in \{x_j, y_j, Sy_j, Tx_j, u_j, v_j\}\} < \delta(A_{nm}),$$

where $A_{nm} = \{x_i, y_i, Sy_i, Tx_i, u_i, v_i : n \leq i \leq m\}$,

$$\delta(A_{nm}) = \sup_{x,y\in A_{nm}} d(x,y).$$

They also proved convergence of the iteration process (1.1) to a common fixed point of S and T.

Motivated by the above studies, we introduce, in this paper, an iteration process to approximate common fixed points for two uniformly quasi-Lipschitzian mappings as follows:

Let (X,d) be a convex metric space with a convex structure W. Let $S,T : X \to X$ be uniformly quasi-Lipschitzian mappings with respective Lipschitz constants $L_1 > 0$ and $L_2 > 0, \{a_n\}, \{b_n\}, \{c_n\}$ be three sequences in $[0,1]$ with $a_n + b_n + c_n = 1$, $n \in \mathbb{N}$. For any given $x_0 \in X$, define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = W\left(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n\right). \tag{1.2}$$

While acknowledging the process (1.1) due to Wang and Liu, we underscore that our process

- is independent of (1.1) due to Wang and Liu : none reduces to the other.
- is one-step process as compared with the two-step process (1.1) and still able to compute common fixed points.
- being one-step process is simpler than (1.1).

Having introduced this process, we use it to prove some strong convergence results for quasi-Lipschitzian mappings. Moreover, as opposed to Wang and Liu [6], some convergence theorems are proved for asymptotically (quasi-)nonexpansive mappings without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such mappings.

In order to prove our main results, the following lemma will be needed:

Lemma 1.2. [9] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative numbers such that*

$$a_{n+1} \le (1 + b_n)\, a_n, \quad n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

2. MAIN RESULTS

In what follows, we take $L = \max\{L_1, L_2\}$ where $L_1 > 0$ and $L_2 > 0$ are Lipschitz constants of the quasi-Lipschitzian mappings S and T respectively.

Theorem 2.1. *Let (X,d) be a convex metric space, E be a nonempty closed convex subset of X and $S,T : E \to E$ be uniformly quasi-Lipschitzian mappings. Let the sequence $\{x_n\}$ be as in (1.2) with the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ in $[0,1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

If $F \neq \emptyset$, then:

(1) *for all $p \in F$ and for each $n \in \mathbb{N}$,*

$$d\left(x_{n+1}, p\right) \le \left(1 + L\left(1 - a_n\right)\right) d\left(x_n, p\right),$$

(2) *there exists a constant $M > 0$ such that, for all $n, m \in \mathbb{N}$ and for every $p \in F$,*

$$d\left(x_{n+m}, p\right) \le M d\left(x_n, p\right).$$

Proof. (1) For any $p \in F$, from (1.2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W\left(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n\right), p) \\ &\le a_n d(x_n, p) + b_n d(S^n x_n, p) + c_n d(T^n x_n, p) \\ &\le a_n d(x_n, p) + b_n L_1 d(x_n, p) + c_n L_2 d(x_n, p) \\ &\le \left(a_n + b_n L + c_n L\right) d(x_n, p) \end{aligned}$$

$$\leq \quad (1 + L(1 - a_n))\, d(x_n, p). \tag{2.1}$$

This completes the proof of (1).

(2) It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the inequality (2.1), we have

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + L(1 - a_{n+m-1}))\, d(x_{n+m-1}, p) \\ &\leq e^{L(1-a_{n+m-1})} d(x_{n+m-1}, p) \\ &\leq e^{L(1-a_{n+m-1})} \left[(1 + L(1 - a_{n+m-2}))\, d(x_{n+m-2}, p) \right] \\ &\leq e^{L[(1-a_{n+m-1})+(1-a_{n+m-2})]} d(x_{n+m-2}, p) \\ &\vdots \\ &\leq M d(x_n, p), \end{aligned} \tag{2.2}$$

where $M = e^{L \sum_{k=0}^{\infty}(1-a_k)}$. This completes the proof of (2). □

Now we give the main theorems of this paper. Our first theorem deals with uniformly quasi-Lipschitzian mappings.

Theorem 2.2. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be uniformly quasi-Lipschitzian mappings and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty} (1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Proof. The necessity is obvious. Thus, we will only prove the sufficiency. From Theorem 2.1, we have

$$d(x_{n+1}, F) \leq (1 + L(1 - a_n))\, d(x_n, F).$$

As $\sum_{n=0}^{\infty}(1 - a_n) < \infty$, therefore $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 1.2. But by hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, so for each $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, F) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.3}$$

Thus, there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.4}$$

From (2.2) and (2.4), we obtain

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \\ &\leq M d(x_n, p_1) + d(x_n, p_1) \\ &= (M+1)\, d(x_n, p_1) \\ &< (M+1) \left(\frac{\varepsilon}{M+1} \right) \\ &= \varepsilon, \end{aligned}$$

for all $n, m \geq n_1$. Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset E of the complete metric space X, therefore, it must converge to a point of E. Suppose $\lim_{n\to\infty} x_n = p$; we prove that $p \in F$. To this end, we only need to prove that F is closed because

$$d(p, F) = \lim_{n\to\infty} d(x_n, F) = 0. \tag{2.5}$$

Let $p_n \in F$ be a sequence such that $\lim_{n\to\infty} p_n = p^*$. We show that $p^* \in F$. In fact,

$$\begin{aligned} d(Sp^*, p^*) &\leq d(p^*, p_n) + d(Sp^*, p_n) \\ &\leq d(p^*, p_n) + Ld(p^*, p_n) \\ &= (1+L)\, d(p^*, p_n) \end{aligned}$$

yields that $d(Sp^*, p^*) = 0$. Similarly, $d(Tp^*, p^*) = 0$. Thus $p^* \in F$ and so F is closed. Thus by (2.5), $p \in F$. This completes the proof. □

In the following results concerned with asymptotically (quasi-)nonexpansive mappings, we do not need "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such type of mappings.

Theorem 2.3. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. $\{k_n\}, \left\{k_n^{'}\right\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = \lim_{n\to\infty} k_n^{'} = 1$; therefore there exist $L_1 > 0$ and $L_2 > 0$ such that $L_1 = \sup_{n\geq 0}\{k_n\} < \infty$ and $L_2 = \sup_{n\geq 0}\left\{k_n^{'}\right\} < \infty$. In this case, S and T are uniformly quasi-Lipschitzian mappings with $L_1 > 0$ and $L_2 > 0$. Hence, Theorem 2.3 can be proven by Theorem 2.2. □

Theorem 2.4. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Remark 2.1. All the results proved in this paper can also be proved for the iteration process with error terms. In this case our main iteration process (1.2) looks like

$$x_{n+1} = W(x_n, S^n x_n, T^n x_n, u_n; a_n, b_n, c_n, d_n), \tag{2.6}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$, $n \in \mathbb{N}$.

Remark 2.2. (i) From computational point of view, our iteration processes (1.2) and (2.6) are simpler than iteration processes of Chang et al. [1], Liu et al. [3], Wang and Liu [6].

(ii) Our results also generalize results of Yao et al. [7] to two uniformly quasi-Lipschitzian mappings in convex metric spaces.

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