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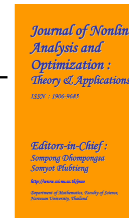

MINIMUM-NORM FIXED POINT OF A FINITE FAMILY OF $\lambda-$STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. Let K be a nonempty closed and convex subset of a real Hilbert space H and for each $1 \leq i \leq N$, let $T_i : K \to K$ be λ_i-strictly pseudocontractive mapping. Then for $\beta \in (0, 2\lambda]$, where $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$, and each $t \in (0, 1)$, it is proved that, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big]$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_N T_N$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$, which converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Moreover, we provide an explicit iteration process which converges strongly to a common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Corresponding results, for a common minimum-norm solution of a finite family of $\alpha-$inverse strongly monotone mappings are also discussed. Our theorems improve several results in this direction.

KEYWORDS: Minimum-norm fixed point; nonexpansive mappings; $\lambda-$strict pseudocontractive mappings; monotone mappings.
AMS Subject Classification: 47H06 47H09 47H10 47J05 47J05.

1. INTRODUCTION

Let K be a nonempty subset of a real Hilbert space H and T be a self-mapping of K. The mapping T is called *Lipschitzian* if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive* and if $L \in [0, 1)$, T is called *contraction*. A mapping T with domain $D(T)$ and range $R(T)$ in H is called $\lambda-$*strictly pseudocontractive* in the terminology of Browder and Petryshyn [2] if for all $x, y \in D(T)$ there exists $\lambda > 0$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - \lambda||x - y - (Tx - Ty)||^2. \quad (1.1)$$

It is obvious that $\lambda-$strictly pseudocontractive mapping is Lipschitzian with $L = \frac{1+\lambda}{\lambda}$. Without loss of generality we may assume $\lambda \in (0, 1)$. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq \lambda||(I - T)x - (I - T)y||^2. \quad (1.2)$$

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Moreover, one can show that (1.1) (and hence (1.2)) is equivalent to the inequality

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, k = (1 - 2\lambda) < 1.$$

It is easy to see that a class of nonexpansive mappings which includes a class of contraction mappings is contained in a class of $\lambda-$ strictly pseudocontractive mappings. However, the converse may not be true (see, [1, 8] for details).

Interest in $\lambda-$strictly pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear $\alpha-$*inverse strongly monotone* mappings where a mapping A with domain $D(A)$ and range in H is called $\alpha-$*inverse strongly monotone* if there exists $\alpha \in (0, 1)$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2, \text{ for every } x, y \in D(A).$$

Observe that A is $\alpha-$inverse strictly monotone if and only if $(I - A)$ is $\lambda-$strictly pseudocontractive, where $\lambda = \alpha$. It is known [9] that $\lambda-$strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. Therefore, it is interesting to develop the algorithms for λ-strictly pseudo-contractive mappings. Consequently, considerable research efforts, especially within the past 20 years or so, have been devoted to iterative methods for approximating fixed points of T when T is nonexpansive or $\lambda-$strictly pseudocontractive (see for example [6, 7, 8, 12, 14, 15, 16, 17] and the references contained therein).

Recently, we notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. In an abstract way, we may formulate such problems as follows:

$$\text{find } x^* \in K \text{ such that } ||x^*|| := \min_{x\in K} ||x||, \tag{1.3}$$

that is, x^* is a metric projection of the origin onto K, $P_K 0$.

A typical example is the split feasibility problem which was introduced by Censor and Elfving [4]. The problem is formulated as finding:

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.4}$$

where C and Q are nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. A split feasibility problem in finite dimensional Hilbert spaces was used for modeling inverse problems which arise in medical image constructions [3] and intensity-modulated radiation therapy [4].

Set

$$\min_{x\in C} \varphi(x) := \min_{x\in C} \frac{1}{2}||Ax - P_Q Ax||^2. \tag{1.5}$$

It is clear that $\bar{x}$ is a solution to the split feasibility problem (1.4) if and only if $\bar{x}$ solves the minimization problem (1.5) with the minimum equal to 0. Now, assume that (1.4) is consistent (i.e., (1.4) has a solution) and let Γ denote the (closed convex) solution set of (1.4) (or equivalently, solution of (1.5)). Then, in this case, Γ has a unique element $\bar{x}$ if and only if it is a solution of the following equation:

$$\bar{x} \in C, \text{ such that } \bigtriangledown \varphi(\bar{x}) = A^*(I - P_Q)A\bar{x} = 0, \tag{1.6}$$

where A^* is the adjoint of A. Let $Tx := (I - \gamma A^*(I - P_Q)A)x$, for any $\gamma > 0$. Then problem (1.6) is equivalent to the fixed point problem equation

$$\bar{x} = T\bar{x} = (I - \gamma A^*(I - P_Q)A)\bar{x}. \tag{1.7}$$

Therefore, finding the solution of the split feasibility problem (1.4) is equivalent to finding the minimization problem (1.5) with the minimum equal to zero if and only it is the minimum-norm of fixed point of the mapping $x \mapsto Tx = (I - \gamma A^*(I - P_Q)A)x$.

Thus, we study the general case of finding the minimum-norm fixed point of $\lambda-$strict pseudocontractive mapping $T : K \longrightarrow K$; that is, we find $x^* \in K$ which satisfies

$$x^* \in F(T) \text{ such that } ||x^*|| = \min\{||x|| : x \in F(T)\}. \tag{1.8}$$

In connection with the iterative approximation of the minimum-norm fixed point of nonexpansive mapping T, Yang *et.al* [12] introduced an implicit scheme given by

$$y_t = \beta T y_t + (1 - \beta)P_K[(1 - t)y_t], t \in (0, 1).$$

They proved that, under appropriate conditions on t and β, the path $\{y_t\}$ converges strongly to the minimum-norm fixed point of T, in real Hilbert spaces. Furthermore, they showed that an explicit scheme given by

$$x_{n+1} = \beta T x_n + (1 - \beta)P_K[(1 - \alpha_n)x_n], n \geq 1,$$

under appropriate conditions on $\{\alpha_n\}$ and β, converges strongly to the minimum-norm fixed point of T.

More recently, Yao and Xu [13] have also introduced and proved that the implicit scheme given by

$$y_t = P_K[(1 - t)T y_t], t \in (0, 1),$$

under appropriate conditions on t, converge strongly to the minimum-norm fixed point of nonexpansive self-mapping T. In addition, they showed that an explicit scheme given by

$$x_{n+1} = P_K[(1 - t_n)T x_n], n \geq 1,$$

under appropriate conditions on $\{t_n\}$, converges strongly to the minimum-norm fixed point of T.

A natural question arises whether we can extend the results of Yang et.al [12] *and Yao and Xu* [13] *to a class of mappings more general than nonexpansive mappings or not?*

Let K be a closed convex subset of a real Hilbert space H and let $T_i : K \to K, i = 1, 2, ..., N$ be λ_i-strictly pseudocontractive mapping.

It is our purpose in this paper to prove that for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = P_K\big[(1 - t)(\beta T y_t + (1 - \beta)y_t)\big]$, which converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Moreover, we provide an explicit iteration process which converges strongly to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Finally, we also give a numerical example which support our results. Our theorems improve several results in this direction.

2. PRELIMINARIES AND NOTATIONS

In what follows we shall make use of the following lammas.

Lemma 2.1. [10] *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \leq 0, \forall z \in K.$$

Lemma 2.2. [15] *Let K be a closed and convex subset of a real Hilbert space H. Let $T_i : K \longrightarrow K, i = 1, 2, ..., N,$ be λ_i-strictly pseudocontractive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_N T_N$, where $\theta_1 + \theta_2 + ... + \theta_N = 1$. Then T is λ-strictly pseudocontractive with $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$ and $F(T) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.3. [11] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n \longrightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.4. [18] *Let H be a real Hilbert space, K be a closed convex subset of H and $T : K \longrightarrow K$ be a $\lambda-$strictly pseudo-contractive mapping, then*
(i) $F(T)$ is closed convex subset of K;
(ii) $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x$ and $Tx_n - x_n \longrightarrow 0$, as $n \longrightarrow \infty$, then $x = T(x)$.

Lemma 2.5. [19] *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \longrightarrow K$ be $\lambda-$strictly pseudocontractive mapping and $T_\beta x := \beta Tx + (1 - \beta)x$. Then for $\beta \in (0, 2\lambda]$, and $x, y \in K$ we have that*

$$||T_\beta x - T_\beta y|| \leq ||x - y||.$$

Lemma 2.6. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle.$$

3. MAIN RESULTS

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be λ_i- strictly pseudocontractive mapping with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Then, for $\beta \in (0, 2\lambda]$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = P_K\big[(1 - t)(\beta T y_t + (1 - \beta) y_t)\big], \tag{3.1}$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$.

Proof. For $\beta \in (0, 2\lambda]$ and each $t \in (0, 1)$, let $T_t(y) := P_K\big[(1 - t)(\beta Ty + (1 - \beta)y)\big]$. Then using nonexpansiveness of P_K, Lemma 2.2 and Lemma 2.5 we have for $x, y \in K$ that

$$||T_t x - T_t y||^2 \;=\; ||P_K\big[(1 - t)(\beta Tx + (1 - \beta)x)\big] - P_K\big[(1 - t)(\beta Ty + (1 - \beta)y)\big]||^2$$

$$\begin{aligned} &\leq (1-t)||\beta(Tx-Ty)+(1-\beta)(x-y)||^2 \\ &\leq (1-t)||x-y||^2, \end{aligned} \tag{3.2}$$

and hence

$$||T_t x - T_t y|| \leq \sqrt{(1-t)}||x-y||.$$

Thus, we get that T_t is a contraction mapping on K and hence T_t has a unique fixed point, y_t, in K. This means that the equation

$$y_t := P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big]. \tag{3.3}$$

has a unique solution for each $t \in (0,1)$. Furthermore, since $F(T) \neq \emptyset$ and $\beta \in (0, 2\lambda]$, for $y^* \in F(T)$, we have from (3.1), convexity of $||.||^2$ and Lemma 2.5 that

$$\begin{aligned} ||y_t - y^*||^2 &= ||P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big] - y^*||^2 \\ &\leq ||(1-t)[\beta(Ty_t - y^*) + (1-\beta)(y_t - y^*)] - ty^*||^2 \\ &\leq ||(1-t)||\beta(Ty_t - y^*) + (1-\beta)(y_t - y^*)||^2 + t||y^*||^2 \\ &\leq (1-t)||y_t - y^*||^2 + t||y^*||^2 \end{aligned}$$

which implies that

$$||y_t - y^*|| \leq ||y^*||.$$

Therefore, $\{y_t\}$ and hence $\{Ty_t\}$ are bounded.

Furthermore, from (3.3) and the fact that P_K is nonexpansive we get that

$$\begin{aligned} ||y_t - Ty_t|| &= ||P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big] - P_K T y_t|| \\ &\leq ||(1-t)(1-\beta)(y_t - Ty_t) - tTy_t|| \\ &\leq (1-t)(1-\beta)||Ty_t - y_t|| + t||Ty_t||, \end{aligned}$$

which implies that

$$||y_t - Ty_t|| \leq \frac{t}{1-(1-\beta)(1-t)}||Ty_t|| \longrightarrow 0, \text{ as } t \longrightarrow 0^+. \tag{3.4}$$

Now, let $z_t := (1-t)(\beta T y_t + (1-\beta)y_t)$. Then from (3.4) we get that

$$||z_t - y_t|| \leq (1-t)\beta||Ty_t - y_t|| + t||y_t|| \longrightarrow 0, \text{ as } t \longrightarrow 0^+. \tag{3.5}$$

In addition, since $\{z_t\}$ is bounded there exists a sequence $\{t_n\} \subset (0,1)$ such that $z_{t_n} \rightharpoonup z$. Thus, from (3.5) we get that

$$y_{t_n} \rightharpoonup z, \text{ as } n \longrightarrow \infty, \tag{3.6}$$

and hence from (3.4) and Lemma 2.4 we have that $z \in F(T)$. Furthermore, from (3.3), nonexpansiveness of P_K and Lemma 2.5 we get that

$$\begin{aligned} ||y_t - y^*||^2 &\leq ||z_t - y^*||^2 \\ &= \langle (1-t)(\beta T y_t + (1-\beta)y_t) - y^*, z_t - y^* \rangle \\ &= (1-t)\langle \beta(Ty_t - Ty^*) + (1-\beta)(y_t - y^*), z_t - y^* \rangle \\ &\quad + t\langle -y^*, z_t - y^* \rangle \\ &\leq (1-t)||y_t - y^*||.||z_t - y^*|| + t\langle -y^*, z_t - y^* \rangle. \\ &\leq (1-t)||z_t - y^*||^2 + t\langle -y^*, z_t - y^* \rangle. \end{aligned}$$

This implies that

$$||z_t - y^*||^2 \leq \langle y^*, y^* - z_t \rangle. \tag{3.7}$$

Since $z \in F(T)$, substituting z for y^* and t_n for t in (3.7) we get that

$$z_{t_n} \longrightarrow z. \tag{3.8}$$

Thus, from (3.7) and (3.8) we have that

$$||z - y^*||^2 \quad \leq \quad \langle y^*, y^* - z\rangle, \text{ as } n \longrightarrow \infty \ ,$$

which is equivalent to the inequality

$$\langle z, y^* - z\rangle \geq 0 \text{ and hence } z = P_F 0.$$

If there is another subsequence $\{z_{t_m}\}$ of $\{z_t\}$ such that $z_{t_m} \rightharpoonup z'$, similar argument gives that $z' = P_F 0$, which implies, by uniqueness of $P_F 0$, that $z' = z$. Therefore, the net $z_t \longrightarrow z = P_F 0$ and hence from (3.5) the net $y_t \longrightarrow z = P_F 0$, which is the minimum-norm fixed point of T. Therefore, from Lemma 2.2, $\{y_t\}$ converges strongly, as $t \longrightarrow 0$, to the common minimum-norm fixed point of $T_i, i = 1, 2, ..., N$. The proof is complete. □

If we assume $T_i's, i = 1, ..., N$ to be nonexpansive mappings we get the following corollary.

Corollary 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be a finite family of nonexpansive mapping with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Then, for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big],$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$.

If in Theorem 3.1, we consider $\{\beta_n\} \subset (a, 2\lambda]$, for some $a > 0$, and $\{t_n\} \subset (0, 1)$ such that $t_n \longrightarrow 0$, and $y_n := y_{t_n}$ the method of proof of Theorem 3.1 provides the following corollary.

Corollary 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be λ_i-strictly pseudocontractive mapping with $\cap_{i=1}^N F(T) \neq \emptyset$. Then, for $\beta_n \subset (a, 2\lambda]$, for some $a > 0$, and $\{t_n\} \subset (0, 1)$, there exists a sequence $\{y_{t_n}\} \subset K$ satisfying the following condition:*

$$y_{t_n} = P_K\big[(1-t_n)(\beta_n T y_{t_n} + (1-\beta_n)y_{t_n})\big], \tag{3.9}$$

and the net $\{y_{t_n}\}$ converges strongly, as $t_n \longrightarrow 0$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$.

We now state and prove a convergence theorem for the common minimum-norm zero of finite family of $\alpha-$inverse strongly monotone mappings.

Theorem 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $A_i : K \longrightarrow H$ be α_i- inverse strongly monotone mapping satisfying $(I - A)(K) \subseteq K$ and $\cap_{i=1}^N N(A_i) \neq \emptyset$. Then, for $\beta \in (0, 2\alpha]$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset H$ satisfying the following condition:*

$$y_t = P_K\big[(1-t)(\beta(I - A)y_t + (1-\beta)y_t)\big], \tag{3.10}$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm zero of $\{A_i : i = 1, 2, ..., N\}$, where $A := \theta_1 A_1 + \theta_2 A_2 + ... + \theta_n A_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\alpha := \min\{\alpha_i : i = 1, 2, ..., N\}$.

Proof. For $x \in K$, let $T_i(x) = (I - A_i)x$, for $i = 1, 2, ..., N$. Then, we get that each T_i is α_i- strictly pseudocontractive self- mapping with $\cap_{i=1}^N N(A_i) = \cap_{i=1} NF(T_i)$. Now, replacing A_i with $(I - T_i)$ we get that scheme (3.10) reduces to (3.1) and hence the conclusion follows from Theorem 3.1. □

Now, we prove strong convergence of an explicit scheme for a common minimum-norm fixed point of λ-strictly pseudocontractive mappings.

Theorem 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \le i \le N$, let $T_i : K \to K$ be λ_i-strictly pseudocontractive mapping with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big], n \ge 1, \tag{3.11}$$

where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0, 1)$, $\beta_n \in [a, 2\lambda]$, for some $a > 0$ and $\lambda := \min\{\lambda_i, i = 1, 2, ..., N\}$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$.

Proof. We note that by Lemma 2.2 we have that T is λ-strictly pseudocontractive and $F(T) = \cap_{n=1}^N F(T_i)$. Let $x^* \in F(T)$. Then from (3.11), using nonexpansiveness of P_K, convexity of $||.||^2$, $\beta_n \in [a, 2\lambda]$ and Lemma 2.5 we get that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big] - P_K x^*||^2 \\
&\le ||(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n) - x^*||^2 \\
&= ||(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n - x^*) - \alpha_n x^*||^2 \\
&\le (1-\alpha_n)||(1-\beta_n)(x_n - x^*) + \beta_n(T x_n - T x^*)||^2 + \alpha_n||x^*||^2 \\
&\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2.
\end{aligned}$$

Thus, by induction we obtain that

$$||x_{n+1} - x^*||^2 \le \max\{||x_0 - x^*||^2, ||x^*||^2\}.$$

Consequently, $\{x_n\}$ and hence $\{T x_n\}$ are bounded. Furthermore, from (3.11) we have that

$$\begin{aligned}
||x_{n+2} - x_{n+1}|| &= ||(1-\alpha_{n+1})(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\
&\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\
&= ||(1-\alpha_{n+1})(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\
&\quad -(1-\alpha_n)(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\
&\quad +(1-\alpha_n)(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\
&\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\
&\le |\alpha_{n+1} - \alpha_n|||\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}|| \\
&\quad +(1-\alpha_n)||\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1} \\
&\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\
&\le M|\alpha_{n+1} - \alpha_n| + (1-\alpha_n)||T_{\beta_{n+1}} x_{n+1} - T_{\beta_n} x_n||,
\end{aligned} \tag{3.12}$$

for some $M > 0$, where $T_{\beta_n} x_n := \beta_n T x_n + (1-\beta_n)x_n$. Furthermore, we have that

$$\begin{aligned}
||T_{\beta_{n+1}} x_{n+1} - T_{\beta_n} x_n|| &\le ||T_{\beta_{n+1}} x_{n+1} - T_{\beta_{n+1}} x_n|| + ||T_{\beta_{n+1}} x_n - T_{\beta_n} x_n|| \\
&\le ||x_{n+1} - x_n|| + ||\beta_{n+1} T x_n + (1-\beta_{n+1})x_n \\
&\quad -(\beta_n T x_n + (1-\beta_n)x_n)||
\end{aligned}$$

$$\begin{aligned} &\leq \quad ||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||x_n-Tx_n|| \\ &\leq \quad ||x_{n+1}-x_n||+M'|\beta_{n+1}-\beta_n|, \end{aligned} \tag{3.13}$$

for some $M'>0$. Then putting (3.13) into (3.12) we obtain that

$$||x_{n+2}-x_{n+1}|| \quad \leq \quad (1-\alpha_n)||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|M''+|\alpha_{n+1}-\alpha_n|M'',$$

for some $M''>0$, and hence using the assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$ and following the method in [6] we get that

$$\lim_{n\longrightarrow\infty}||x_{n+1}-x_n||=0. \tag{3.14}$$

Let $y_n=\beta_nTx_n+(1-\beta_n)x_n$. Then we observe that $\{y_n\}$ is bounded and from (3.11) and the fact that $\alpha_n\longrightarrow 0$, as $n\longrightarrow 0$, we obtain that

$$\begin{aligned} ||x_{n+1}-y_n|| &= ||P_K\big[(1-\alpha_n)(\beta_nTx_n+(1-\beta_n)x_n)\big]-P_Ky_n|| \\ &\leq ||(1-\alpha_n)(\beta_nTx_n+(1-\beta_n)x_n)-y_n|| \\ &\leq \alpha_n||y_n||\longrightarrow 0, \text{ as } n\longrightarrow\infty. \end{aligned} \tag{3.15}$$

Thus, from (3.14) and (3.15) we get that

$$||y_n-x_n||\longrightarrow 0, \text{ as } n\longrightarrow\infty. \tag{3.16}$$

Consequently,

$$\lim_{n\longrightarrow\infty}||x_n-Tx_n||=\lim_{n\longrightarrow\infty}\frac{||x_n-y_n||}{\beta_n}=0. \tag{3.17}$$

Now, let $z_n=(1-\alpha_n)(\beta_nTx_n+(1-\beta_n)x_n)$ and $\tilde{x}:=P_F0$. Then we have that $\{z_n\}$ is bounded. Furthermore, (3.17) and the assumption that $\alpha_n\longrightarrow 0$, as $n\longrightarrow\infty$, give that

$$z_n-x_n=(1-\alpha_n)[\beta_n(Tx_n-x_n)-\alpha_nx_n]\longrightarrow 0, \text{ as } n\longrightarrow\infty. \tag{3.18}$$

Let $\{z_{n_k}\}$ be a subsequence of $\{z_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle\tilde{x},z_n-\tilde{x}\rangle=\lim_{k\longrightarrow\infty}\langle\tilde{x},z_{n_k}-\tilde{x}\rangle,$$

and $z_{n_k}\rightharpoonup z$. Then, from (3.18) we have that $x_{n_k}\rightharpoonup z$. Thus, from (3.17) and Lemma 2.4 we get that $z\in F(T)$. Therefore, by Lemma 2.1 we obtain that

$$\limsup_{n\longrightarrow\infty}\langle\tilde{x},z_n-\tilde{x}\rangle=\lim_{k\longrightarrow\infty}\langle\tilde{x},z_{n_k}-\tilde{x}\rangle=\langle\tilde{x},z-\tilde{x}\rangle\geq 0. \tag{3.19}$$

Now, we show that $x_{n+1}\longrightarrow\tilde{x}$, as $n\longrightarrow\infty$. But from (3.11), Lemma 2.5 and Lemma 2.6 we have that

$$\begin{aligned} ||x_{n+1}-\tilde{x}||^2 &= ||P_K\big[(1-\alpha_n)((1-\beta_n)x_n+\beta_nTx_n)\big]-P_K\tilde{x}||^2 \\ &\leq ||(1-\alpha_n)((1-\beta_n)x_n+\beta_nTx_n)-\tilde{x}||^2 \\ &= ||(1-\alpha_n)((1-\beta_n)(x_n-\tilde{x})+\beta_n(Tx_n-T\tilde{x}))-\alpha_n\tilde{x}||^2 \\ &\leq (1-\alpha_n)||((1-\beta_n)(x_n-\tilde{x})+\beta_n(Tx_n-T\tilde{x})||^2+2\alpha_n\langle-\tilde{x},z_n-\tilde{x}\rangle \\ &\leq (1-\alpha_n)||x_n-\tilde{x}||^2+2\alpha_n\langle-\tilde{x},z_n-\tilde{x}\rangle. \end{aligned} \tag{3.20}$$

Therefore, this with (3.19) and Lemma 2.3 give that $x_n\longrightarrow\tilde{x}$, as $n\longrightarrow\infty$, which is the minimum-norm fixed point of T and hence, by Lemma 2.2, the common minimum-norm fixed point of $\{T_i : i=1,2,...,N\}$. □

If we take $T_i's, i=1,...,N$ to be nonexpansive mappings we get the following corollary.

Corollary 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \to K$ be a finite family of nonexpansive mapping with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big], n \geq 1,$$

where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [a,1)$, for some $a > 0$ and satisfying $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm fixed point of $\{T_i, i = 1, 2, ..., N\}$.

The following corollary is implied from Theorem 3.3.

Theorem 3.4. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $A_i : K \to H$ be α_i-inverse strongly monotone mapping satisfying $(I - A)(K) \subseteq K$ and $\cap_{i=1}^{N} N(A_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in H$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n (I-A)x_n)\big], n \geq 1, \tag{3.21}$$

where $A := \theta_1 A_1 + \theta_2 A_2 + ... + \theta_n A_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [a, 2\alpha]$, for some $a > 0$ and $\alpha := \min\{\alpha_i, i = 1, 2, ..., N\}$ satisfying $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm zero of $\{A_i : i = 1, 2, ..., N\}$, as $n \to \infty$.

Proof. The method of proof of Theorem 3.2 and using Theorem 3.3 provide the required assertion.

□

4. Numerical example

Now, we give an example of a finite family of λ-strictly pseudocontractive mappings satisfying Theorem 3.3 and some numerical experiment results to explain the conclusion of Theorem 3.3 as follows:

Example 4.1. Let $H = \mathbb{R}$ with absolute value norm. Let $K = [0,1]$ and $T_1, T_2 : K \longrightarrow K$ be defined by

$$T_1 x := \begin{cases} x + (x - \frac{1}{2})^2, x \in [0, \frac{1}{2}], \\ x, \qquad\qquad x \in (\frac{1}{2}, 1], \end{cases} \tag{4.1}$$

and

$$T_2 x := \begin{cases} x, \qquad\qquad x \in [0, \frac{2}{3}], \\ x - (x - \frac{2}{3})^2, x \in (\frac{2}{3}, 1], \end{cases} \tag{4.2}$$

Then we first show that T_1 is λ-strictly pseudocontractive mapping with $\lambda = \frac{1}{2}$. If $x, y \in [0, \frac{1}{2}]$ then

$$\begin{aligned} \langle (I - T_1)x - (I - T_1)y, x - y \rangle &= \langle -(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2, x - y \rangle \\ &= \big[(x - \frac{1}{2})^2 - (y - \frac{1}{2})^2\big](y - x) \\ &= \big[(x - \frac{1}{2})^2 - (y - \frac{1}{2})^2\big]\big[(y - \frac{1}{2}) - (x - \frac{1}{2})\big] \end{aligned}$$

$$\begin{aligned}
&= \big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]\frac{\big[(y-\frac{1}{2})^2-(x-\frac{1}{2})^2\big]}{(y-\frac{1}{2})+(x-\frac{1}{2})}\\
&= \big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]\frac{\big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]}{(\frac{1}{2}-x)+(\frac{1}{2}-y)}\\
&\geq \frac{1}{2}\big|(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big|^2\\
&= \frac{1}{2}\big|(I-T_1)x-(I-T_1)y|^2.
\end{aligned}$$

If $x\in[0,\frac{1}{2}]$ and $y\in(\frac{1}{2},1]$ we get that

$$\begin{aligned}
\langle (I-T_1)x-(I-T_1)y, x-y\rangle &= \langle -(x-\frac{1}{2})^2, x-y\rangle = (x-\frac{1}{2})^2(y-x)\\
&= (x-\frac{1}{2})^2\big[(y-\frac{1}{2})-(x-\frac{1}{2})\big]\\
&\geq (x-\frac{1}{2})^2(\frac{1}{2}-x)\\
&\geq (x-\frac{1}{2})^2\frac{(\frac{1}{2}-x)^2}{(\frac{1}{2}-x)} \geq \frac{1}{2}|(x-\frac{1}{2})^2|^2\\
&= \frac{1}{2}\big|(I-T_1)x-(I-T_1)y|^2.
\end{aligned}$$

If $x,y\in(\frac{1}{2},1]$ then we get that $|(I-T_1)x-(I-T_1)y|=0$ and hence

$$\langle (I-T_1)x-(I-T_1)y, x-y\rangle \geq \frac{1}{2}|(I-T_1)x-(I-T_1)y|^2.$$

Therefore, T_1 is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T_1)=[\frac{1}{2},1]$.

Similarly, we can show that T_2 is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T_2)=[0,\frac{2}{3}]$.

Thus, if $Tx:=\theta T_1x+(1-\theta)T_2x$, where $\theta=\frac{1}{2}$, then T is given by

$$Tx=\begin{cases} x+\frac{1}{2}(x-\frac{1}{2})^2, x\in[0,\frac{1}{2}],\\ x, \qquad\qquad x\in(\frac{1}{2},\frac{2}{3}],\\ x-\frac{1}{2}(x-\frac{2}{3})^2, x\in(\frac{2}{3},1],\end{cases} \tag{4.3}$$

which is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T)=[\frac{1}{2},\frac{2}{3}]=F(T_1)\cap F(T_2)$. Now, taking $\alpha_n=\frac{1}{n+1}$, $\beta_n=\frac{1}{100}+\frac{1}{n+1}$, scheme (3.11) reduces to

$$x_{n+1}=P_K\Big[(1-\frac{1}{n+1})\Big((1-(\frac{1}{100}+\frac{1}{n+1}))x_n+(\frac{1}{100}+\frac{1}{n+1})Tx_n\Big)\Big], n\geq 1. \tag{4.4}$$

Therefore, by Theorem 3.3, the sequence $\{x_n\}$ in (4.4) converges strongly to $\frac{1}{2}$, the common minimum norm fixed point of T_1 and T_2.

Next, we show the numerical experiment result tables using software Mathlab 7.5 for the iteration process of the sequence $\{x_n\}$ with initial point $x_1=0.2$ and $x_1=0.7$, respectively.

n	1	200	600	1000	1200	1400	1600	1800	2000
x_n	0.200	0.386	0.432	0.446	0.451	0.457	0.457	0.460	0.462

n	1	200	600	1000	1200	1400	1600	1800	2000
x_n	0.700	0.386	0.432	0.446	0.451	0.454	0.457	0.460	0.462

Remark 4.2. Theorem 3.1 improves Theorem 3.1 of Yang *et.al* [12] and Yao and Xu [13] to a more general class of finite family of $\lambda-$strictly pseudocontractive mappings. Moreover, Theorem 3.3 improves Theorem 3.2 of Yang *et.al* [12] and Yao and Xu [13] in the sense that our scheme provides a minimum-norm fixed point of finite family of $\lambda-$strict pseudocontractive mapping T.

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