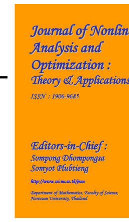
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 4, No. 2, (2013), 127-130
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



AN EMBEDDING THEOREM FOR A CLASS OF CONVEX SETS IN NONARCHIMEDEAN NORMED SPACES

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ABSTRACT. In this article we show that the class of all compact convex sets of a real nonarchimedean normed space can be embedded in a real nonarchimedean normed space.

KEYWORDS : Embedding theorem; Nonarchimedean normed space.
AMS Subject Classification: Primary: 46A55 , Secondary: 46S10

1. INTRODUCTION AND PRELIMINARIES

In [1], Rådström showed that the class of all compact convex sets of a real normed space can be embedded in a real normed space. In this article we give a nonarchimedean counterpart for this fact. We start by recalling a few essential concepts from [2].

Let K be a field. A nonarchimedean absolute value on K is a function $|\cdot| : K \to \mathbb{R}$ such that, for any $a, b \in K$ we have

1. $|a| \geq 0$,
2. $|a| = 0$ if and only if $a = 0$,
3. $|ab| = |a| \cdot |b|$,
4. $|a + b| \leq \max(|a|, |b|)$.

The field K is called nonarchimedean if it is equipped with a nonarchimedean absolute value such that the corresponding metric is complete.

Let X be a vector space over field K which is equipped with a nonarchimedean absolute value (nonarchimedean vector space, for short). A nonarchimedean norm $\| \cdot \|$ on X is a function $\| \cdot \| : X \to \mathbb{R}$ such that

1. $\|x\| = 0$ implies that $x = 0$;
2. $\|ax\| = |a| \cdot \|x\|$, for any $a \in K$ and $x \in X$;
3. $\|x + y\| \leq \max(\|x\|, \|y\|)$, for any $x, y \in X$.

Moreover, a nonarchimedean vector space X equipped with a nonarchimedean norm is called a nonarchimedean normed space. Nonarchimedean normed spaces

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Article history : Received 23 November 2012. Accepted 12 August 2013.

over the nonarchimedean field $\mathbb{R}$, will be called real nonarchimedean normed spaces.

Throughout this paper we assume that K is a nonarchimedean field and X is a nonarchimedean normed space over K. We set $\mathcal{B} =: \{a \in K : |a| \leq 1\}$.

A subset $A \subseteq X$ is called convex if either A is empty or is of the form $A = x + A_0$ for some vector $x \in X$ and some $\mathcal{B}$-submodule $A_0 \subseteq X$. A lattice L in X is an $\mathcal{B}$-submodule which satisfies the condition that for any vector $x \in X$ there is a nonzero scalar $a \in K$ such that $ax \in L$. For more basic facts see [2].

2. MAIN RESULTS

For nonempty convex subsets A and B of X and scalar $\lambda \in K$, let $A + B =: \{x + y : x \in A, y \in B\}$ and $\lambda A =: \{\lambda x : x \in A\}$. Addition and scalar multiplication satisfy $(A + B) + C = A + (B + C)$, $A + B = B + A$, and $\lambda(A + B) = \lambda A + \lambda B$.

Lemma 2.1. *Let A, B, and C be subsets of a nonarchimedean normed space X, where C is closed and B is nonempty convex and bounded. Then $A + B \subseteq C + B$ implies $A \subseteq C$.*

Proof. Since B is convex, there exist a $\mathcal{B}$-submodule B_0 and $b \in X$ such that $B = b + B_0$. By assumption we have $A + B_0 \subseteq C + B_0$. Let $a \in A \backslash C$. There is a lattice L such that $(a + L) \cap (C) = \emptyset$. Since L is $\mathcal{B}$-submodule of X, $(a + L) \cap (C + L) = \emptyset$. Boundedness of B implies that B_0 is bounded and so there is $\alpha \in K$ such that $B_0 \subseteq \alpha L$. If $|\alpha| \leq 1$, then $(a + B_0) \cap (C + B_0) = \emptyset$ which is a contradiction. If $|\alpha| > 1$, then $a = z + b$, for some $z \in C$ and $b \in B_0$. This implies that $(z + b + \alpha^{-1}B_0) \cap (C + \alpha^{-1}B_0) = \emptyset$, which is a contradiction since $(b + \alpha^{-1}B_0) \cap \alpha^{-1}B_0 \neq \emptyset$. □

Lemma 2.1 implies that:

Corollary 2.2. *Let A, B, and C be subsets of a nonarchimedean normed space X, where A and C are closed and B is nonempty convex and bounded. Then $A + B = C + B$ implies $A = C$.*

For subsets A and C of X, define

$$\mathfrak{h}(A, C) =: \inf\{\varepsilon > 0 : C \subseteq N_\varepsilon(A), A \subseteq N_\varepsilon(C)\},$$

where $N_\varepsilon(A) =: \{z \in X : d(z, A) < \varepsilon\}$ and $d(z, A)$ denotes distance of z from A. By convention $\inf \emptyset = \infty$. The extended real valued function $\mathfrak{h}$ has the following properties for each subset A, B, and C:

(i) $\mathfrak{h}(A, B) \geq 0$ and $\mathfrak{h}(A, A) = 0$;
(ii) $\mathfrak{h}(A, B) = \mathfrak{h}(B, A)$;
(iii) $\mathfrak{h}(A, B) \leq \max(\mathfrak{h}(A, C), \mathfrak{h}(C, B))$;
(iv) $\mathfrak{h}(A, B) = 0$ if and only if $\overline{A} = \overline{B}$, where $\overline{A}$ denotes the closure of A in X.

The Proof of Properties 1 and 2 are easy and we just give the proof of Properties 3 and 4. By contradiction, let $\mathfrak{h}(A, B) > \max(\mathfrak{h}(A, C), \mathfrak{h}(C, B))$ for some subsets A, B, C. Then there would be positive numbers λ_1 and λ_2 where $\lambda_1 < \mathfrak{h}(A, B)$, $\lambda_2 < \mathfrak{h}(A, B)$, $A \subseteq N_{\lambda_1}(C)$, $C \subseteq N_{\lambda_1}(A)$ and $C \subseteq N_{\lambda_2}(B)$, $B \subseteq N_{\lambda_2}(C)$. Therefore $B \subseteq N_\lambda(A)$ and $A \subseteq N_\lambda(B)$ where $\lambda = \max(\lambda_1, \lambda_2)$. This is a contradiction since $\lambda < \mathfrak{h}(A, B)$. To prove 4, let $\mathfrak{h}(A, B) = 0$ and $x \in \overline{A}$. For each $\gamma > 0$ there exists nonzero $\lambda > 0$ such that $\lambda \leq \gamma$ with $B \subseteq N_\lambda(A)$, $A \subseteq N_\lambda(B)$ and $N_\lambda(x) \cap A \neq \emptyset$. Since $A \subseteq N_\lambda(B)$ so $N_\lambda(x) \cap B \neq \emptyset$ and consequently $N_\gamma(x) \cap B \neq \emptyset$, that is $x \in \overline{B}$. By a similar way we have $\overline{B} \subseteq \overline{A}$. Conversely, if $\overline{A} = \overline{B}$ and $\mathfrak{h}(A, B) > 0$, then there exists $\lambda > 0$ such that either $B \nsubseteq N_\lambda(A)$ or $A \nsubseteq N_\lambda(B)$. If $x \in A\backslash N_\lambda(B)$, then $N_\lambda(x) \cap B = \emptyset$. That is to say x is not an element of $\overline{B}$, which is a contradiction.

Lemma 2.3. *If A and C are convex sets in a nonarchimedean normed space, then for each nonempty convex and bounded set B we have*

$$\mathfrak{h}(A, C) = \mathfrak{h}(A + B, C + B).$$

Proof. If $C \subseteq N_\lambda(A)$ and $A \subseteq N_\lambda(C)$, for some $\lambda \geq 0$, then $C + B \subseteq N_\lambda(A + B) = B + N_\lambda(A)$, $A + B \subseteq N_\lambda(C + B) = B + N_\lambda(C)$. Therefore $\mathfrak{h}(A + B, C + B) \leq \mathfrak{h}(A, C)$. The inverse inequality is obtained by Lemma 2.1. □

By part A of Theorem 1 in [1], if M is a commutative semigroup with the law of cancellation, then M can be embedded in a group N. Also, if G is a group in which M is embedded, then N is isomorphic to a subgroup of G containing M. Therefore, by Corollary 2.2, the semigroup of all nonempty compact convex subsets of a nonarchimedean normed space can be embedded in a minimal group N as a semigroup.

Hereafter let $\mathbb{R}$ be equipped with a nonarchimedean absolute value $|\cdot|$.

Theorem 2.4. *Let M be an additive commutative semigroup with the law of cancellation. If a multiplication by real scalars is defined on M which satisfies*

$$\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A, \quad \lambda_1(\lambda_2 A) = \lambda_1 \cdot \lambda_2 A, \quad 1A = A,$$

for every $A, B \in M$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, then M can be embedded in a minimal real nonarchimedean vector space N.

Moreover, if a metric d is given on M with

$$d(A + C, B + C) = d(A, B), \quad d(\lambda A, \lambda B) = \lambda d(A, B),$$

for every $A, B \in M$ and $\lambda \in \mathbb{R}$ and the operations addition and scalar multiplication are continuous in the topology induced by d, then a nonarchimedean norm can be defined on N which makes it as a real nonarchimedean normed space.

Proof. Following to the proof of Theorem 1 in [1], consider the equivalence relation $\sim$ defined as $(A, B) \sim (C, D)$ if and only if $A + D = B + C$, for $A, B, C, D \in M$. By $[A, B]$ denote the equivalence class containing the pair (A, B). The set N shall consist of equivalence classes $[A, B]$, where A and B are elements of M. Addition and scalar multiplication in N are defined by $[A, B] + [C, D] = [A + C, B + D]$ and $\lambda[A, B] = [\lambda A, \lambda B]$ for $\lambda \in [0, +\infty)$, otherwise $\lambda[A, B] = [-\lambda B, -\lambda A]$. Obviously the given operations are well defined and N constitutes a nonarchimedean vector space. For some $B \in M$ define $f : M \to N$ by $f(A) = [A + B, B]$ for each $A \in M$. The mapping f is well defined and embeds M in the nonarchimedean vector space N. Clearly, for $\lambda \in \mathbb{R}$ and $A \in M$ the scalar product λA coincides with the one given on M.

Let d be a nonarchimedean metric on M satisfying the assumptions of theorem. Define d_0 on $N \times N$ as

$$d_0([A, B], [C, D]) = d(A + D, B + C).$$

Let $[A, B], [C, D] \in N$ and $d_0([A, B], [C, D]) = 0$. So $d(A + D, B + C) = 0$ which implies that $A + D = B + C$, that is $(A, B) \sim (C, D)$. Conversely, if $(A, B) \sim (C, D)$, then $d_0([A, B], [C, D]) = 0$. Obviously

$$d_0([A, B], [C, D]) = d_0([C, D], [A, B]).$$

Also

$$\begin{aligned} d_0([A,B],[C,D]) &= d(A+D,B+C) \\ &\leq \max(d(A+F+E+D,B+E+E+D),d(B+E+F \\ &\quad +C,B+E+E+D)) \\ &= \max(d(A+F,B+E),d(E+D,F+C)) \\ &= \max(d_0([A,B],[E,F]),d_0([E,F],[C,D])). \end{aligned}$$

Since nonarchimedean metric d_0 is invariant under translation, so the function $\|\cdot\| : N \to \mathbb{R}$, where $\|[A,B]-[C,D]\| =: d_0([A,B],[C,D])$ is a nonarchimedean norm on N. Therefore addition and scalar multiplication are continuous operations, and if $A, B \in M$, the distance between A and B equals $d(A,B)$. □

By Corollaries 2.2 and 2.3 and Theorem 2.4, we have the following.

Theorem 2.5. *Let M be a class of nonempty closed, bounded convex subsets of X which is closed under addition and scalar multiplication and equipped with a nonarchimedean metric. Then M can be isometrically embedded in a real nonarchimedean normed space N. In particular the operations addition and scalar multiplication of M are induced by the operations of N.*

Moreover, if H is a nonarchimedean normed space in which M is embedded in the above way, then H contains a subspace containing M and isometric to N.

It is worth mentioning that the class of all nonempty compact convex sets of a real nonarchimedean normed space satisfies Theorem 2.5.

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