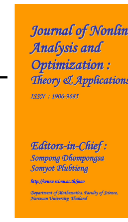
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 4, No. 2, (2013), 85-103
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



A UNIFYING SEMI-LOCAL ANALYSIS FOR ITERATIVE ALGORITHMS OF HIGH CONVERGENCE ORDER

IOANNIS K. ARGYROS[1,*] AND S.K. KHATTRI[2]

[1]Department of Mathematical Sciences, Cameron University, Lawton, Oklahoma 73505-6377, USA

[2] Department of Engineering, Stord Haugesund University College, Norway

ABSTRACT. We present a unifying semi-local convergence analysis of two-step Newton-type methods for solving nonlinear equations in a Banach space setting. Convergence order of these methods is higher than two. Our analysis expands the applicability of these methods by providing weaker convergence criteria and a convergence analysis – which is tighter than earlier studies [1–4, 24–34] – is also presented. Numerical examples illustrating the developed theoretical results are also given.

KEYWORDS: Two-point Newton type methods; Banach space; majorizing sequence; semi-local convergence; recurrent functions.
AMS Subject Classification: 65B05 65J15 65N30 65N35 65H10 47H17 49M15.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is a twice Fréchet differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Numerous problems in science and engineering can be reduced to solving the above equation [18, 32]. Consequently, solving these equations is an important scientific field of research. In many situations, finding a closed form solution for the non-linear equation (1.1) is not possible. Therefore, iterative solution techniques are employed for solving these equations. The study about convergence analysis of iterative methods is usually divided into two categories : semi-local and local convergence analysis. The semilocal convergence analysis is based upon the information around an initial point to give criteria ensuring the convergence of the iterative procedure. While the

* *Corresponding author.*
Email address : ioannisa@cameron.edu(Ioannis K. Argyros).
Article history : Received 1 March 2013. Accepted 7 July 2013.

local convergence analysis is based on the information around a solution to find estimates of the radii of convergence balls.

In the present paper, we study the semi-local convergence of the Two-step Newton-type method (**TSNTM**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\mathcal{T}_{\mathcal{F}}(x_n)\mathcal{F}(y_n) \end{aligned}\right\} \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $x_0 \in \mathbf{D}$ is an initial point, the operator $\mathcal{T}_{\mathcal{F}}(x) : \mathbf{D} \to \mathbf{Y}$ is given as

$$\mathcal{T}_{\mathcal{F}}(x) = \mathcal{I} + \mathcal{V}_{\mathcal{F}}(x) + \mathcal{V}_{\mathcal{F}}(x)^2\mathcal{G}_{\mathcal{F}}(x),$$

where the operator $\mathcal{V}_{\mathcal{F}}(x) : \mathbf{D} \to \mathbf{Y}$ is defined by

$$\mathcal{V}_{\mathcal{F}}(x) = \mathcal{F}'(x)^{-1}\mathcal{F}''(x)\mathcal{F}'(x)^{-1}\mathcal{F}(x)$$

and $\mathcal{G}_{\mathcal{F}} : \mathbf{D} \to \mathbf{L}(\mathbf{X}, \mathbf{X})$ is a given linear operator for each $x \in \mathbf{D}$. Some special cases of (**TSNTM**) are

Case $-$ 1. two-step Newton method of order three (**TSNM-O-3**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.3}$$

for each $n = 0, 1, 2, \ldots,$

Case $-$ 2. Two-step Newton method of order four (**TSNM-O-4**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}(\mathcal{I} + \mathcal{V}_{\mathcal{F}}(x_n))\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.4}$$

for each $n = 0, 1, 2, \ldots,$

Case $-$ 3. Two-step Newton method of order five (**TSNM-O-5**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\Big(\mathcal{I} + \mathcal{V}_{\mathcal{F}}(x_n) \\ &\qquad + \frac{\mathcal{V}_{\mathcal{F}}(x_n)^2}{2}\Big(\frac{5}{2}\mathcal{I} - \mathcal{V}_{\mathcal{F}'}(x_n)\Big)\Big)\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.5}$$

for each $n = 0, 1, 2, \ldots.$

Many other choices of operator $\mathcal{T}_{\mathcal{F}}$ lead to other popular iterative methods such as Halley's-type or Chebyshev-type methods []. Concerning the order of convergence of such methods - in the case when $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ - a theorem by Traub [33] states that for sufficiently smooth $\mathcal{G}_{\mathcal{F}}(x)$ (**TSNTM**) has order four.

The following set of conditions (**C**) have been used to perform semi-local convergence analysis of these method [1–29]

$\mathbf{C}_1$. there exists $x_0 \in \mathbf{D}$ such that $\mathcal{F}'(x_0)^{-1} \in \mathbf{L}(\mathbf{Y}, \mathbf{X})$,

$\mathbf{C}_2$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta$,

$\mathbf{C}_3$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x)\right\| \leq \mathcal{L}$ for each $x \in \mathbf{D}$ or $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(y))\right\| \leq \mathcal{L}\left\|x - y\right\|$ for each $x, y \in \mathbf{D}$,

$\mathbf{C}_4$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}''(x) - \mathcal{F}''(y))\right\| \leq \mathcal{M}\left\|x - y\right\|$ for each $x, y \in \mathbf{D}$,

$\mathbf{C}_5$. $\eta \leq \dfrac{\mathcal{L}^2 + 4\mathcal{M} - \mathcal{L}\sqrt{L^2 + 2\mathcal{M}}}{3\mathcal{M}(\mathcal{L} + \sqrt{\mathcal{L}^2 + 2\mathcal{M}})}$,

$\mathbf{C}_6$. $\overline{U}(x_0, R_0) \subseteq \mathbf{D}$ where R_0 is the small positive root of

$$p(t) = \frac{\mathcal{M}}{6}t^2 + \frac{\mathcal{L}}{2}t - t + \eta.$$

However, simple numerical examples can be used to show that even though the condition $(\mathbf{C}_5)$ is not satisfied but still (**TSNTM**) converges to the solution $x^\star$. As an example, let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $x_0 = 1$ and $\mathbf{D} = [\zeta, 2-\zeta]$ for $\zeta \in (0,1)$. Define function $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^5 - \zeta. \tag{1.6}$$

Then, through some simple calculations, the conditions (**C**) yield

$$\eta = \frac{(1-\zeta)}{5}, \quad \mathcal{L} = 4(2-\zeta)^3, \quad \mathcal{M} = 12(2-\zeta)^2.$$

Figure 1 plots the criterion ($\mathbf{C}_4$) for the problem (1.6). The curve (defined by the right

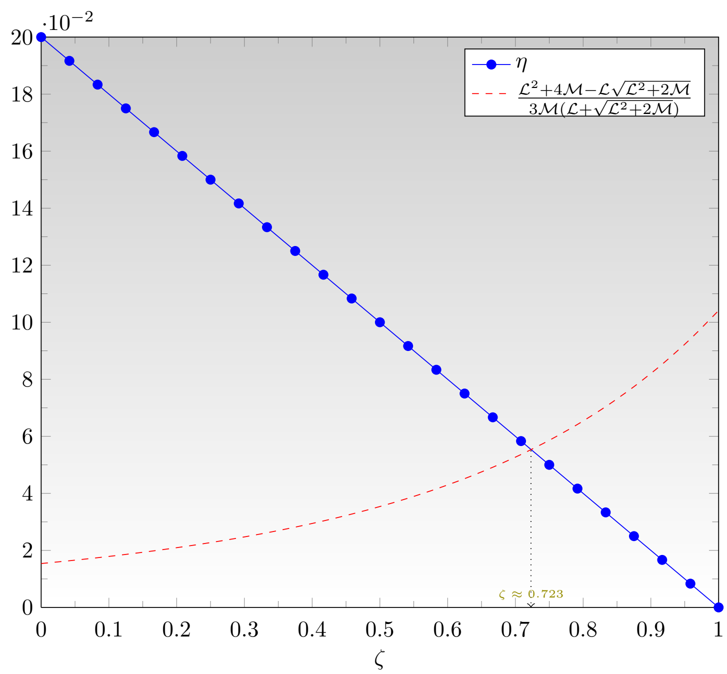


FIGURE 1. Convergence criterion ($\mathbf{C}_5$) for (1.6).

hand side of the inequality ($\mathbf{C}_4$)) intersect the line η (see Figure 1) at $\zeta \approx 0.72$. We notice in the Figure 1 that for $\zeta < 0.72$ the criterion ($\mathbf{C}_4$) is not satisfied. However, one may see that the method (1.2) is convergent. For additional examples, see the Section 4.

In this paper, we are concerned with expanding the applicability of (**TSNTM**) where the the condition $(\mathbf{C}_5)$ (or $(\mathbf{C}_6)$) fails. To achieve this, we introduce the center-Lipschitz conditions

$\mathbf{C}_7$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x_0))\right\| \leq \mathcal{L}_0 \left\|x - x_0\right\|$ for each $x \in \mathbf{D}$,
$\mathbf{C}_8$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x)\mathcal{F}'(x_0)\right\| \leq b$ for each $x \in \mathbf{D}$,
$\mathbf{C}_9$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x))\mathcal{F}'(x_0)\right\| \leq c$ for each $x \in \mathbf{D}$.

Here onwards, the conditions ($\mathbf{C}_1$), ($\mathbf{C}_2$), ($\mathbf{C}_3$), ($\mathbf{C}_4$), ($\mathbf{C}_7$), ($\mathbf{C}_8$) and ($\mathbf{C}_9$) are referred as the (**H**) conditions.

Several techniques are usually considered to study the convergence of iterative methods, as we can see in the studies [1–33]. Among these, the most popular techniques are based on majorizing sequences. In the studies that lead to convergence condition (**C**$_5$), the condition $(\mathbf{C}_3)$ was used to compute the upper bound

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \le \frac{1}{1-\mathcal{L}\,\|x_n-x_0\|}. \tag{1.7}$$

Instead of using (**C**$_3$), we use the more precise and less expensive condition (**C**$_4$) which leads to

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \le \frac{1}{1-\mathcal{L}_0\,\|x_n-x_0\|}. \tag{1.8}$$

Note that

$$\mathcal{L}_0 \le \mathcal{L} \tag{1.9}$$

holds in general and $\mathcal{L}/\mathcal{L}_0$ can be arbitrarily large [23]. This change - in the study of semi-local convergence of method - leads to tighter error estimates on the distances $\|y_n - x_n\|$, $\|x_{n+1} - y_n\|$, $\|x_{n+1} - y_n\|$, $\|y_n - x^\star\|$, $\|x_n - x^\star\|$ and weaker convergence criteria.

The rest of the paper is organized as follows. Section 2 develop results on majorizing sequences for (**TSNTM**) (1.2), where as in the Section 3 we develop the semilocal convergence of the (**TSNTM**). Section 4 presents a Lemma about the special case Two-point Newton method. Finally, numerical examples are given in the concluding Section 5.

2. Majorizing sequences

Here, we find sufficient conditions for the convergence of scalar sequences that will be shown - in the next section - to be majorizing for (**TSNTM**). Let $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$, $b \ge 0$, $c \ge 0$ and $\eta > 0$ be some positive constants. It is convenient for us to define functions γ, α and h_i for $i = 1, 2, 3$ by

$$\gamma(t) = \frac{b\mathcal{L}\,t}{2}, \quad \gamma = \gamma(\eta), \tag{2.1}$$

$$\alpha(t) = \frac{\Big[\dfrac{\mathcal{L}\gamma(t)^2}{2} + \mathcal{L}\gamma(t) + \dfrac{c\mathcal{L}}{2}\Big]t}{1-\mathcal{L}_0(1+\gamma(t))t}, \quad \alpha = \alpha(\eta), \tag{2.2}$$

$$h_1(t) = [a(t) + \mathcal{L}_0(1+\gamma(t))]t - 1, \tag{2.3}$$

$$h_2(t) = \frac{b\mathcal{L}}{2}\alpha(t)t + \mathcal{L}_0\gamma(t)(1+\gamma(t))t - \gamma(t) \tag{2.4}$$

and

$$h_3(t) = a(t)t + \mathcal{L}_0(1+\gamma(t))(1+\alpha(t))t - 1 \tag{2.5}$$

where

$$a(t) = \frac{\mathcal{L}}{2}\gamma(t)^2 + \mathcal{L}\gamma(t) + \frac{c\mathcal{L}}{2}, \quad a = a(\eta).$$

Let the minimum positive zeros of the functions h_1, h_2 and h_3 be η_1, η_2 and η_3, respectively. Note that - by the choice of η_1 - $\alpha(t)$ is well defined on $(0, \eta_1)$ and $\alpha \in (0, 1)$. We set

$$\eta_0 = \min\{\eta_1, \eta_2, \eta_3\}. \tag{2.6}$$

Then, for all $t \in (0, \eta_0)$ we have

$$\alpha \in (0, 1) \tag{2.7}$$

$$h_1(t) < 0 \tag{2.8}$$

$$h_2(t) \le 0 \tag{2.9}$$

and

$$h_3(t) \leq 0. \tag{2.10}$$

We can show the following result about the convergence of majorizing sequences.

Lemma 2.1. *Let the positive constants be $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$, $b \geq 0$, $c \geq 0$, $\mathcal{M} \geq 0$ and $\eta > 0$. Furthermore suppose that*

$$\eta \begin{cases} \leq \eta_0 & \text{if} \quad \eta_0 \neq \eta_1, \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.11}$$

Then, scalar sequence $\{t_n\}$ generated by

$$\begin{aligned} t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{b\mathcal{L}(s_n - t_n)^2}{2(1 - \mathcal{L}_0 t_n)}, \\ s_{n+1} = t_{n+1} + \frac{\frac{\mathcal{L}}{2}(t_{n+1} - s_n)^2 + \mathcal{L}(s_n - t_n)(t_{n+1} - s_n) + \frac{c\mathcal{L}}{2}(s_n - t_n)^2}{1 - \mathcal{L}_0 t_{n+1}} \end{aligned} \tag{2.12}$$

is increasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\gamma}{1-\alpha}\right)\eta \tag{2.13}$$

and converges to its unique least upper bound $t^\star$ which satisfies

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.14}$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$0 \leq t_{n+1} - s_n \leq \gamma(s_n - t_n) \leq \gamma\alpha^n\eta \tag{2.15}$$

and

$$0 < s_{n+1} - t_{n+1} \leq \alpha(s_n - t_n) \leq \alpha^{n+1}\eta. \tag{2.16}$$

Proof. We use mathematical induction to prove (2.15) and (2.16). By (2.1), (2.2) and (2.12), estimates (2.15) and (2.16) hold for $n = 0$ since

$$t_1 - s_0 = \frac{b\mathcal{L}}{2}(s_0 - t_0)(s_0 - t_0) = \gamma(s_0 - t_0) \tag{2.17}$$

and

$$\begin{aligned} s_1 - t_1 &= \frac{\frac{\mathcal{L}}{2}(t_1 - s_0)^2 + \mathcal{L}(s_0 - t_0)(t_1 - s_0) + \frac{c\mathcal{L}}{2}(s_0 - t_0)^2}{1 - \mathcal{L}_0 t_1}, \\ &\leq \frac{\frac{\mathcal{L}}{2}\gamma^2(s_0 - t_0)^2 + \mathcal{L}\gamma(s_0 - t_0)^2 + \frac{c\mathcal{L}}{2}(s_0 - t_0)^2}{1 - \mathcal{L}_0(1+\gamma)\eta}, \\ &\leq \frac{a(s_0 - t_0)}{1 - \mathcal{L}_0(1+\gamma)\eta}(s_0 - t_0) = \alpha(s_0 - t_0). \end{aligned} \tag{2.18}$$

Let us assume that (2.15) and (2.16) hold for all $k \leq n$. Then, we have

$$\begin{aligned} t_{k+1} - s_k \leq \gamma(s_k - t_k) \leq \gamma\alpha^k\eta, \\ s_{k+1} - t_{k+1} \leq \alpha(s_k - t_k) \leq \alpha^{k+1}\eta \end{aligned}$$

and

$$\begin{aligned} t_{k+1} &\leq s_k + \gamma\alpha^k\eta \leq t_k + \alpha^k\eta + \gamma\alpha^k\eta \\ &\leq t - k - 1 + \alpha^{k-1}\eta + \alpha^k\eta + \gamma\alpha^{k-1}\eta + \gamma\alpha^k\eta \end{aligned}$$

$$\begin{aligned}
&\le \cdots \le t_2 + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le s_1 + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le t_1 + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le \eta + \gamma\eta + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&= \frac{1-\alpha^{k+1}}{1-\alpha}(1+\gamma)\eta < \frac{1+\gamma}{1-\alpha}\eta = t^{\star\star}. \qquad (2.19)
\end{aligned}$$

Evidently, estimates (2.15) and (2.16) are true provided that

$$\frac{b\mathcal{L}(s_k - t_k)}{2(1-\mathcal{L}_0 t_k)} \le \gamma \tag{2.20}$$

and

$$\frac{a(s_k - t_k)}{(1-\mathcal{L}_0 t_{k+1})} \le \alpha. \tag{2.21}$$

The estimate (2.20) can be written as

$$\frac{b\mathcal{L}}{2}\alpha^k\eta + \gamma\mathcal{L}_0(1+\gamma)\frac{1-\alpha^k}{1-\alpha}\eta - \gamma \le 0. \tag{2.22}$$

Inequality (2.22) motivates us to define recurrent functions f_k on $[0,1)$ for each $k = 1, 2, 3, \ldots$ by

$$f_k(t) = \frac{b\mathcal{L}}{2}t^k\eta + \gamma\mathcal{L}_0(1+\gamma)\frac{1-t^k}{1-t}\eta - \gamma. \tag{2.23}$$

We need a relationship between two consecutive functions f_k. We have by (2.23) that

$$\begin{aligned}
f_{k+1}(t) &= f_k(t) + \frac{b\mathcal{L}}{2}t^{k+1}\eta - \frac{b\mathcal{L}}{2}t^k\eta + \gamma\mathcal{L}_0(t^k\eta - t^{k-1}\eta + \gamma t^k\eta - \gamma t^{k-1}\eta) \\
&= f_k(t)(t-1)\Big[\frac{b\mathcal{L}}{2}t + +\gamma\mathcal{L}_0(1+\gamma)\Big]t^{k-1}\eta. \qquad (2.24)
\end{aligned}$$

It follows from (2.24) that

$$f_{k+1}(t) \le f_k(t) \le \cdots \le f_1(t). \tag{2.25}$$

In view of (2.22) and (2.25) it suffices to show that

$$f_1(\alpha) \le 0 \tag{2.26}$$

which is true by the choice of η_2, (2.4) and (2.11). Similarly, estimate (2.21) can be written as

$$a\alpha^{k-1}\eta + \mathcal{L}_0(1+\gamma)\frac{1-\alpha^{k+1}}{1-\alpha}\eta - 1 \le 0. \tag{2.27}$$

Define recurrent functions g_k on $[0,1)$ for each $k = 1, 2, \ldots$ by

$$g_k(t) = at^{k-1}\eta + \mathcal{L}_0(1+\gamma)\frac{1-t^{k+1}}{1-t}\eta - 1. \tag{2.28}$$

Then, using (2.28) we get that

$$g_{k+1}(t) = g_k(t) + (t-1)\Big[a + \mathcal{L}_0(1+\gamma)(1+t)\Big]t^{k-1}\eta. \tag{2.29}$$

It follows from (2.29) that

$$g_{k+1}(t) \le g_k(t) \le \cdots \le g_1(t). \tag{2.30}$$

We can show instead of (2.27) that

$$g_1(\alpha) \le 0, \tag{2.31}$$

which is true by the choice of η_3, (2.5) and (2.11). The induction for (2.15) and (2.16) is complete. Hence, sequence $\{t_n\}$ is increasing, bounded from above by $t^{\star\star}$ (given by (2.13)) and converges to its unique least upper bound $t^\star$. The proof of the Lemma is complete. □

We have the following useful and obvious extension of Lemma 2.1.

Lemma 2.2. *Suppose there exists $N \geq 0$ such that*

$$t_0 < s_0 < t_1 < \cdots < t_N < s_N < t_{N+1} < \frac{1}{\mathcal{L}_0}. \tag{2.32}$$

and

$$s_N - t_N \begin{cases} \leq \eta_0 & \text{if} \quad \eta_0 \neq \eta_1 \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.33}$$

Then, the conclusions of the Lemma 2.1 hold for sequence $\{t_n\}$. Moreover, the following estimates hold for each $n = 0, 1, 2, 3, \ldots$

$$0 < t_{N+1+n} - s_{N+n} \leq \gamma_N (s_{N+n} - t_{N+n}) \tag{2.34}$$

and

$$0 < s_{N+1+n} - t_{N+1+n} \leq \alpha_N (s_{N+n} - t_{N+n}) \tag{2.35}$$

where $\gamma_N = \gamma(s_N - t_N)$, $\alpha_N = \alpha(s_N - t_N)$ and $t_N^{\star\star} = \dfrac{1+\gamma_N}{1-\alpha_N}(s_N - t_N)$.

Remark 2.3.
R1. Note that for $N = 0$, the Lemma 2.2 reduces to Lemma 2.1 with $\alpha_0 = \alpha$ and $\gamma_0 = \gamma$.

3. Semi-local convergence analysis

We need the following Ostrowski-type representation connecting $\mathcal{F}(x_{n+1})$ to the method [1–28].

Lemma 3.1. *Suppose that all iterates of the method* **(TSNTM)** (1.2) *are well defined. Then, the following identity holds for each $n = 0, 1, 2, \ldots$*

$$\mathcal{F}(x_{n+1}) = \int_0^1 \Big[\mathcal{F}'(y_n+\theta(x_{n+1}-y_n))-\mathcal{F}'(y_n)\Big](x_{n+1}-y_n)\mathrm{d}\theta+(\mathcal{F}'(y_n)-\mathcal{F}'(x_n))(x_{n+1}-y_n)$$
$$+ (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n)) \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n)\mathrm{d}\theta. \tag{3.1}$$

Proof. We have – by the definition of the method **(TSNTM)** (1.2) – that

$$\begin{aligned}\mathcal{F}(y_n) &= \mathcal{F}(y_n) - \mathcal{F}(x_n) - \mathcal{F}'(x_n)(y_n - x_n) \\ &= \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n)\mathrm{d}\theta.\end{aligned} \tag{3.2}$$

Moreover, we get in turn that

$$\begin{aligned}\mathcal{F}(x_{n+1}) &= \mathcal{F}(x_{n+1}) - \mathcal{F}(y_n) - \mathcal{F}'(y_n)(x_{n+1} - y_n) + \mathcal{F}(y_n) + \mathcal{F}'(y_n)(x_{n+1} - y_n) \\ &= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\ &\quad + \mathcal{F}(y_n) + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + \mathcal{F}'(x_n)(x_{n+1} - y_n)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + \mathcal{F}(y_n) + \mathcal{F}'(x_n)\mathcal{F}'(x_n)^{-1}\mathcal{T}_{\mathcal{F}}(x_n)\mathcal{F}(y_n) \\
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n))\mathcal{F}(y_n) \\
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n)) \\
&\qquad\qquad \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n)\mathrm{d}\theta.
\end{aligned}
$$

The proof of the Lemma is complete. □

We can show the main semi-local convergence result for the method (1.2) under the (**H**) conditions.

Theorem 3.2. *Suppose that the (**H**) conditions and the conditions of Lemma 2.1 hold. Moreover, suppose that*

$$\overline{U}(x_0, t^\star) \subseteq \mathbf{D}. \tag{3.3}$$

*Then, sequence $\{x_n\}$ generated by the (**TSNTM**) (1.2) is well defined, remain in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{3.4}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{3.5}$$

$$\|x_n - x^\star\| \leq t^\star - t_n \tag{3.6}$$

and

$$\|y_n - x^\star\| \leq t^\star - s_n. \tag{3.7}$$

Furthermore, if there exists $R \geq t^\star$ such that

$$U(x_0, R) \subseteq \mathbf{D} \tag{3.8}$$

and

$$\frac{\mathcal{L}_0}{2}(t^\star + R) = 1 \tag{3.9}$$

then, the solution $x^\star$ is unique in $U(x_0, R)$.

Proof. We shall prove that (3.4) and (3.5) hold using mathematical induction. Using ($\mathbf{C}_2$), (1.2) and (2.12) , we get that

$$\|y_0 - x_0\| = \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta = s_0 - t_0 \leq t^\star.$$

That is (3.4) holds for $n = 0$ and $y_0 \in \overline{U}(x_0, t^\star)$ (by (2.13)). In view of (1.2), (2.12), ($\mathbf{C}_3$) and (3.2), we obtain that

$$
\begin{aligned}
\|x_1 - y_0\| &\leq \left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0)\right\| \\
&\leq \frac{b\mathcal{L}}{2}(s_0 - t_0)^2 = t_1 - s_0,
\end{aligned} \tag{3.10}
$$

which shows that (3.5) hold for $n = 0$. We also get that

$$\|x_1 - x_0\| \le \|x_1 - y_0\| + \|y_0 - x_0\| \le t_1 - s_0 + s_0 - t_0 = t_1 \le t^\star,$$

which implies that $x_1 \in \overline{U}(x_0, t^\star)$. Let us assume that (3.4), (3.5), $y_k \in \overline{U}(x_0, t^\star)$ and $x_{k+1} \in \overline{U}(x_0, t^\star)$ hold for all $k \le n$. It follows from the proof of Lemma 2.1 and ($\mathbf{C}_5$) that

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x_{k+1}) - \mathcal{F}'(x_0))\right\| \le \mathcal{L}_0 \|x_{k+1} - x_0\| \le \mathcal{L}_0 t_{k+1} < 1. \tag{3.11}$$

Estimate (3.11) and the Banach Lemma on invertible operators [23] imply that

$$\begin{gathered}\mathcal{F}'(x_{k+1})^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X}),\\ \left\|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right\| \le \frac{1}{1 - \mathcal{L}_0 \|x_{k+1} - x_0\|} \le \frac{1}{1 - \mathcal{L}_0 t_{k+1}}.\end{gathered} \tag{3.12}$$

Then, we have by (1.2), ($\mathbf{C}_3$), (2.12) and (3.12) (for k replacing by $k+1$) and the induction hypotheses that

$$\begin{aligned}\|x_{k+1} - y_k\| &\le \left\|\mathcal{F}'(x_k)^{-1}\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_k)\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_k)\right\|\\ &\le \frac{b\mathcal{L}}{2(1 - \mathcal{L}_0 t_k)}(s_k - t_k)^2 = t_{k+1} - s_k. \end{aligned} \tag{3.13}$$

Using (1.2), ($\mathbf{C}_3$), ($\mathbf{C}_4$), (2.12), (3.1), (3.12), (3.13) and the induction hypotheses we obtain in turn that

$$\begin{aligned}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\| \le& \left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(y_k + \theta(x_{k+1} - y_k)) - \mathcal{F}'(y_k)]\mathrm{d}\theta\right\| \|x_{k+1} - y_k\|\\ &+ \left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(y_k) - \mathcal{F}'(x_k))\right\| \|x_{k+1} - y_k\| + \left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_k))\mathcal{F}'(x_0)\right\|\\ &\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_k + \theta(y_k - x_k)) - \mathcal{F}'(x_k)]\mathrm{d}\theta\right\| \|y_k - x_k\|\\ \le& \frac{\mathcal{L}}{2}\|x_{k+1} - y_k\|^2 + \mathcal{L}\|y_k - x_k\| \|x_{k+1} - y_k\| + \frac{c\mathcal{L}}{2}\|y_k - x_k\|^2\\ \le& \frac{\mathcal{L}}{2}(t_{k+1} - s_k)^2 + \mathcal{L}(s_k - t_k)(t_{k+1} - s_k) + \frac{c\mathcal{L}}{2}(s_k - t_k)^2. \end{aligned} \tag{3.14}$$

Then, by (1.2), (2.12), (3.13) and (3.14), we get that

$$\begin{aligned}\|y_{k+1} - x_{k+1}\| &\le \|\mathcal{F}'(x_{k+1})\mathcal{F}'(x_0)\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\|\\ &\le \frac{\frac{\mathcal{L}}{2}(t_{k+1} - s_k)^2 + \mathcal{L}(s_k - t_k)(t_{k+1} - s_k) + \frac{c\mathcal{L}}{2}(s_k - t_k)^2}{1 - \mathcal{L}_0 t_{k+1}}\\ &= s_{k+1} - t_{k+1}. \end{aligned} \tag{3.15}$$

We shall also have that

$$\|y_{k+1} - x_0\| \le \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \le s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \le t^\star$$

and

$$\|x_{k+2} - x_0\| \le \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_0\| \le t_{k+2} - s_{k+1} + s_{k+1} - t_0 = t_{k+2} \le t^\star$$

Hence, y_{k+1} and x_{k+2} belongs to $\overline{U}(x_0, t^\star)$. It follows from (3.7), (3.8) and Lemma 2.1 that sequence $\{x_n\}$ is complete in a Banach space $\mathbf{X}$ and a such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (3.14) we obtain $\mathcal{F}(x^\star) = 0$. Estimates (3.9) and (3.10) follows from (3.7) and (3.8) by using standard majorization techniques. Finally to the uniqueness part, $y^\star \in \overline{U}(x_0, R)$

be a solution of equation $\mathcal{F}(x)=0$. Let $Q=\int_0^1 \mathcal{F}'(x^\star+\theta(y^\star-x^\star))\mathrm{d}\theta$. Using (C_5), (3.11) and (3.12), we get that

$$\begin{aligned}\left\|\mathcal{F}'(x_0)^{-1}(Q-\mathcal{F}'(x_0))\right\| &\leq \int_0^1 \left\|\mathcal{F}'(x_0)^{-1}\left[\int_0^1 [\mathcal{F}'(x^\star+\theta(y^\star-x^\star))-\mathcal{F}'(x_0)]\mathrm{d}\theta\right]\right\| \\ &\leq \frac{\mathcal{L}_0}{2}(t^\star+R)=1. \end{aligned} \tag{3.16}$$

It follows from (3.16) and the Banach lemma on invertible operators that $Q^{-1}\in \mathbf{L}(\mathbf{Y},\mathbf{X})$. Then, using the identity

$$0=\mathcal{F}(y^\star)-\mathcal{F}(x^\star)=Q(y^\star-x^\star)$$

we deduce that $x^\star=y^\star$. The proof of the Theorem is complete. □

Remark 3.3.

R1. The limit point $t^\star$ can be replaced by $t^{\star\star}$ (given in closed from by (2.13)) in Theorem 3.2.

R2. The conclusions of Theorem 3.2 hold if hypotheses of Lemma 2.1 are replaced by those of Lemma 2.2.

R3. It follows from the **(H)** conditions that there exist b_0, c_0, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ satisfying

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}'(x_0)\right\|\leq b_0, \tag{3.17}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x_1)-\mathcal{F}'(x_0))\right\|\leq \mathcal{L}_1\left\|x_1-x_0\right\|, \tag{3.18}$$

$$\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(y_0+\theta(x_1-y_0))-\mathcal{F}'(y_0)]\mathrm{d}\theta\right\|\leq \mathcal{L}_2\theta\left\|x_1-y_0\right\|, \tag{3.19}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\right\|\leq \mathcal{L}_2\left\|y_0-x_0\right\|, \tag{3.20}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I}-\mathcal{T}_{\mathcal{F}}(x_0))\mathcal{F}'(x_0)\right\|\leq c_0, \tag{3.21}$$

and

$$\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_0+\theta(y_0-x_0))-\mathcal{F}'(x_0)]\mathrm{d}\theta\right\|\leq \mathcal{L}_3\theta\left\|y_0-x_0\right\|, \tag{3.22}$$

where

$$y_0=x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)$$

and

$$x_1=x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)-\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}(x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)).$$

Note that

$$b_0\leq b,\quad c_0\leq c,\quad \mathcal{L}_1\leq\mathcal{L}_0,\quad \mathcal{L}_2\leq\mathcal{L}\quad\text{and}\quad \mathcal{L}_3\leq\mathcal{L} \tag{3.23}$$

and b/b_0, c/c_0, $\mathcal{L}_0/\mathcal{L}_1$, $\mathcal{L}/\mathcal{L}_2$, $\mathcal{L}/\mathcal{L}_3$ can be arbitrarily large [23].

We may notice that estimates (3.17) - (3.21) are not additional to the **(H)** conditions, since in practice the verifications of $(\mathbf{C}_2)$-$(\mathbf{C}_5)$ require the computation of b_0, c_0, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. Note that finding these constants only involve computations at the initial data. We define

$$
\begin{aligned}
r_0 = 0, \quad q_0 = \eta, \quad r_1 = q_0 + \frac{b_0 \mathcal{L}_3 (q_0 - r_0)^2}{2},\\
q_1 = r_1 + \frac{\frac{\mathcal{L}_2}{2}(r_1 - q_0)^2 + \mathcal{L}_2 (q_0 - r_0)(r_1 - q_0) + \frac{c_0 \mathcal{L}_3}{2}(q_0 - r_0)^2}{(1 - \mathcal{L}_1 r_1)}\\
r_{n+1} = q_n + \frac{b \mathcal{L}(q_n - r_n)^2}{2(1 - \mathcal{L}_0 r_n)},\\
q_{n+1} = r_{n+1} + \frac{\frac{\mathcal{L}}{2}(r_{n+1} - q_n)^2 + \mathcal{L}(q_n - r_n)(r_{n+1} - q_n) + \frac{c\mathcal{L}}{2}(q_n - r_n)^2}{(1 - \mathcal{L}_0 r_{n+1})}
\end{aligned}
\tag{3.24}
$$

Furthermore, according to the proof of Theorem 3.2, $\{r_n\}$ is a majorizing sequence for $\{x_n\}$ (see also (3.4) – (3.6)) and the tables in the next section. Note that the majorizing sequence $\{v_n\}$ – for the method (1.2) – is given by

$$
\begin{aligned}
v_0 = 0, \quad v_{n+1} = u_n + \frac{b\mathcal{L}(u_n - v_n)^2}{2(1 - \mathcal{L} v_n)},\\
u_{n+1} = v_{n+1} + \frac{\frac{\mathcal{L}}{2}(v_{n+1} - u_n)^2 + \mathcal{L}(u_n - v_n)(v_{n+1} - u_n) + \frac{c\mathcal{L}}{2}(u_n - v_n)^2}{(1 - \mathcal{L} v_{n+1})}.
\end{aligned}
\tag{3.25}
$$

A simple inductive argument shows that

$$q_n \le s_n \le u_n \tag{3.26}$$

$$r_n \le t_n \le v_n \tag{3.27}$$

$$r_{n+1} - q_n \le t_{n+1} - s_n \le v_{n+1} - u_n \tag{3.28}$$

$$q_{n+1} - r_{n+1} \le s_{n+1} - t_{n+1} \le u_{n+1} - v_{n+1} \tag{3.29}$$

and

$$r^\star = \lim_{n \longrightarrow \infty} r_n \le t^\star \le v^\star = \lim_{n \longrightarrow \infty} v_n. \tag{3.30}$$

Left hand side in the estimates (3.26) – (3.30) hold as strict inequalities if any of the inequalities in (3.23) is strict. Moreover, right hand side in the estimates (3.26) – (3.30) also hold as strict inequalities for $n > 1$ if $\mathcal{L}_0 < \mathcal{L}$. Furthermore, $\{r_n\}$, $\{t_n\}$ can replace $\{v_n\}$ in the convergence results in the literature under the sufficient convergence conditions given there [1–4] (see also ($\mathbf{C}_5$)).

Finally note that the conditions of Lemma 2.1 or Lemma 2.2 can be weaker than those in the literature. In practice we shall use $\{r_n\}$ or $\{t_n\}$ to estimate error bounds on the distances $\|x_{n+1} - y_n\|$, $\|y_n - x_n\|$, $\|x_n - x^\star\|$, $\|y_n - x^\star\|$ and we shall test if conditions of Lemma 2.1 or Lemma 2.2 or those in the literature hold.

4. Special case I : Two-point Newton method

Let $\mathcal{T}_{\mathcal{F}}(x) = \mathcal{I}$. Then, we can choose $b = 1$ and $c = 0$. In this case method (1.2) reduces to the two-point Newton method. In this case, Lemma 2.1 reduces to the following Lemma.

Lemma 4.1. *Let the positive constants be $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$ and $\eta > 0$. Suppose that*

$$\eta \begin{cases} \le \eta_0 & \text{if} \quad \eta_0 \ne \eta_1 \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{4.1}$$

Then, scalar sequence $\{t_n\}$ *generated by*

$$\begin{aligned} t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{\mathcal{L}(s_n - t_n)^2}{2(1 - \mathcal{L}_0 t_n)} \\ s_{n+1} = t_{n+1} + \frac{\frac{\mathcal{L}}{2}(t_{n+1} - s_n)^2 + \mathcal{L}(t_{n+1} - s_n)(s_n - t_n)}{1 - \mathcal{L}_0 t_{n+1}} \end{aligned} \tag{4.2}$$

is increasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\gamma}{1-\alpha}\right)\eta \tag{4.3}$$

and converges to its unique least upper bound $t^\star$ *which satisfies*

$$0 \le t^\star \le t^{\star\star}. \tag{4.4}$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$0 < t_{n+1} - s_n \le \gamma(s_n - t_n) \le \gamma\alpha^n\eta \tag{4.5}$$

and

$$0 < s_{n+1} - t_{n+1} \le \alpha(s_n - t_n) \le \alpha^{n+1}\eta. \tag{4.6}$$

5. Numerical examples

Example 5.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ be equipped with the max-norm, $x_0 = \omega$, $\mathbf{D} = [-2, 2]$. Let us define $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^3 - 1. \tag{5.1}$$

Here, $w \in \mathbf{D}$. Through some algebraic manipulations, for the conditions (**H**), we obtain

$$\eta = \frac{|\omega^3 - 1|}{3\omega^2}, \quad \mathcal{L} = \frac{4}{\omega^2}, \quad \mathcal{M} = \frac{2}{\omega^2}, \quad \mathcal{L}_0 = \frac{2 + |\omega|}{\omega^2}, \quad b = \frac{179}{144}, \quad c = \frac{35}{144}.$$

For $\omega = 1.21$, the convergence criterion ($\mathbf{C}_5$) yields

$$0.1756621815 \le 0.1731485558.$$

Thus the criterion ($\mathbf{C}_5$) does not hold. Even though the criterion ($\mathbf{C}_5$) is not satisfied. We can see that the method (1.2) converges. For example, let us choose $\mathcal{G}_{\mathcal{F}}(x) = -\mathcal{I}$ and which will result in a fourth order convergent iterative procedure. The performance of this method for (5.1) is reported in the table 2.

Now let us validate the hypotheses of Lemma 2.1 and 2.2. From (2.1) – (2.5), we obtain

$$\eta_1 = 0.2196968398, \quad \eta_2 = 0.1803308682, \quad \eta_3 = 0.1803308682$$

and from the formulation (2.6), we obtain

$$\eta_0 = \eta_2 = 0.1803308682.$$

We notice that the condition (2.11) - of Lemma 2.1 - holds. That is : $0.1756621815 < 0.1803308682$. For the sequence (2.12), we obtain the Table 1. From (2.13), we get

$$t^{\star\star} = 0.4114076922.$$

Comparing the $t^{\star\star}$ with the values in the Table 1, we notice that the inequality (2.14) holds. Furthermore, we notice in the Table 1 the hypothesis of Lemma 2.2 also hold. Since the conditions of Lemma 2.1 - and also that of Lemma 2.2 - holds thus the Theorem 3.2 is applicable. Comparing tables 1 and 2, we see that the estimates (3.4) – (3.7) hold. Comparing Tables 1 and 2, we notice that the estimates of Theorem 3.2 hold.

Example 5.2. In this example, we provide an application of our results to a special nonlinear Hammerstein integral equation of the second kind. Consider the integral equation

$$x(s) = 1 + \frac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1], \tag{5.2}$$

where, G is the Green kernel on $[0,1]\times[0,1]$ defined by

$$G(s,t) = \begin{cases} t(1-s), & t \le s; \\ s(1-t), & s \le t. \end{cases} \tag{5.3}$$

Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0,1]$ and $\mathbf{D}$ be a suitable open convex subset of $\mathbf{X}_1 := \{x \in \mathbf{X} : x(s) > 0, s \in [0,1]\}$, which will be given below. Define $\mathcal{F} : \mathbf{D} \to \mathbf{Y}$ by

$$[\mathcal{F}(x)](s) = x(s) - 1 - \frac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1]. \tag{5.4}$$

The first and second derivatives of $\mathcal{F}$ are given by

$$[\mathcal{F}(x)'y](s) = y(s) - \frac{12}{5}\int_0^1 G(s,t)x(t)^2 y(t)\,dt, \quad s \in [0,1], \tag{5.5}$$

and

$$[\mathcal{F}(x)''yz](s) = \frac{24}{5}\int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1], \tag{5.6}$$

respectively. We use the max-norm. Let $x_0(s) = 1$ for all $s \in [0,1]$. Then, for any $y \in \mathbf{D}$, we have

$$[(I - \mathcal{F}'(x_0))(y)](s) = \frac{12}{5}\int_0^1 G(s,t)y(t)\,dt, \quad s \in [0,1], \tag{5.7}$$

which means

$$\|I - \mathcal{F}'(x_0)\| \le \frac{12}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \frac{12}{5\times 8} = \frac{3}{10} < 1. \tag{5.8}$$

It follows from the Banach theorem that $\mathcal{F}'(x_0)^{-1}$ exists and

$$\|\mathcal{F}'(x_0)^{-1}\| \le \frac{1}{1 - \frac{3}{10}} = \frac{10}{7}. \tag{5.9}$$

On the other hand, we have from (5.4) that

$$\|\mathcal{F}(x_0)\| = \frac{4}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \frac{1}{10}.$$

Then, we get $\eta = 1/7$. Note that $\mathcal{F}''(x)$ is not bounded in $\mathbf{X}$ or its subset $\mathbf{X}_1$. Take into account that a solution $x^\star$ of equation (1.1) with $\mathcal{F}$ given by (5.3) must satisfy

$$\|x^\star\| - 1 - \frac{1}{10}\|x^\star\|^3 \le 0, \tag{5.10}$$

i.e., $\|x^\star\| \le \rho_1 = 1.153467305$ and $\|x^\star\| \ge \rho_2 = 2.423622140$, where ρ_1 and ρ_2 are the positive roots of the real equation $z - 1 - z^3/10 = 0$. Consequently, if we look for a solution such that $x^\star < \rho_1 \in \mathbf{X}_1$, we can consider $\mathbf{D} := \{x : x \in \mathbf{X}_1 \quad \text{and} \quad \|x\| <$

$r\}$, with $r \in (\rho_1, \rho_2)$, as a nonempty open convex subset of $\mathbf{X}$. For example, choose $r = 1.7$. Using (3.7) and (3.8), we have that for any $x, y, z \in \mathbf{D}$

$$\begin{aligned}\|[(\mathcal{F}'(x) - \mathcal{F}'(x_0))y]\,(s)\| &= \frac{12}{5}\left\|\int_0^1 G(s,t)(x(t)^2 - x_0(t)^2)y(t)\,dt\right\| \\ &\le \frac{12}{5}\int_0^1 G(s,t)\|x(t) - x_0(t)\|\,\|x(t) + x_0(t)\|y(t)\,dt \\ &\le \frac{12}{5}\int_0^1 G(s,t)\,(r+1)\|x(t) - x_0(t)\|y(t)\,dt, \quad s \in [0,1]\end{aligned} \tag{5.11}$$

and

$$\|(F''(x)yz)(s)\| = \frac{24}{5}\int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1]. \tag{5.12}$$

Then, we get

$$\|\mathcal{F}'(x) - \mathcal{F}'(x_0)\| \le \frac{12}{5}\frac{1}{8}(r+1)\|x - x_0\| = \frac{81}{100}\|x - x_0\|, \tag{5.13}$$

$$\|F''(x)\| \le \frac{24}{5} \times \frac{r}{8} = \frac{51}{50} \tag{5.14}$$

and

$$\|[[F''(x) - \mathcal{F}''(\overline{x})]\,yz]\,(s)\| = \frac{24}{5}\left\|\int_0^1 G(s,t)\,(x(t) - \overline{x}(t)))y(t)z(t)\right\|dt \tag{5.15}$$

$$\le \frac{24}{5}\frac{1}{8}\|x - \overline{x}\| = \frac{3}{5}\|x - \overline{x}\|. \tag{5.16}$$

Now we can choose constants as follows:

$$\mathcal{M} = \frac{6}{7}, \quad \mathcal{L} = \frac{51}{35}, \quad \mathcal{L}_0 = \frac{81}{70}, \quad b = \frac{22}{15}, \quad c = \frac{7}{15},$$
$$b_0 = \frac{11}{15}, \quad c_0 = \frac{2}{15}, \quad \mathcal{L}_1 = \frac{11}{70}, \quad \mathcal{L}_2 = \frac{16}{35}, \quad \mathcal{L}_2 = \frac{16}{35}, \quad \text{and} \quad \eta = \frac{1}{7}.$$

We can verify that the condition ($\mathbf{C}_5$) holds. From equations (2.1) – (2.6), we obtain

$$\eta_1 = 0.5292437221, \quad \eta_2 = 0.4285556173, \quad \eta_3 = 0.4285556173.$$

From the formulation (2.7), we get

$$\eta_0 = \eta_2 = 0.4285556173.$$

We may see that the hypothesis (2.11) of Lemma 2.1 holds. Now let us compare the sequences (2.12), (3.24) and (3.25), with (3.7). Comparison – among sequences (2.12), (3.24) and (3.25) – is reported in Table 3. In the Table 3, we observe that the sequence $\{q_n\}$ is finer than the sequence $\{s_n\}$ and $\{s_n\}$ is finer than than $\{u_n\}$ – which is also true by the estimates (3.26) and (3.29).

Concerning the uniqueness balls, let us denote the radii [1, 3, 4, 7, 9, 18–21] by γ_1 and γ_2, respectively. These are given as the smallest positive roots of the polynomials

$$p_1(t) = \mathcal{L}_0\,t - 1 \quad (\text{for} \quad t^\star = \mathbb{R}) \tag{5.17}$$

and

$$p_2(t) = \frac{\mathcal{M}}{6}t^3 + \frac{\mathcal{L}}{2}t^2 - t + \eta \tag{5.18}$$

respectively. Using the values of $\mathcal{L}_0$, $\mathcal{L}$, $\mathcal{M}$ and η we get

$$\gamma_1 = 0.8641975309, \quad \gamma_2 = 0.1517444889. \tag{5.19}$$

Note that $\overline{U}(x_0, r-1) \subseteq \mathbf{D}$, $\mathcal{L}_0 < \mathcal{L}$ and $\gamma_2 < \gamma_1$. Therefore, the new approach provides the largest uniqueness ball and since $r-1 < \gamma_1$, we deduce that $x^\star$ is unique in $\overline{U}(x_0, r-1) = \overline{U}(1, 0.7) \subseteq \mathbf{D}$.

Example 5.3. We consider nonlinear Hammerstein integral equation

$$x(s) = 1 + \int_0^1 G(s,t)x(t)^2\,dt, \quad s,t \in [0,1] \tag{5.20}$$

where $s \in \mathcal{C}[0,1]$, and the kernel $G(s,t)$ is given as

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ (1-t)s, & s \le t. \end{cases}$$

Hammerstein integral equations are associated with boundary value problems for differential equations [1]. For these equations higher order methods – utilizing information about the second derivatives – may be advantageous [1].

To solve the nonlinear integral equation (4.1), we divide the interval ($s, t \in [0,1]$) into $n-$points and approximate the integral part through an $n-$point Gauss-Legendre quadrature. Let these $n-$points be ξ_i with $i = 1, 2, \ldots, n$. Thus we obtain

$$x(\xi_j) = 1 + \int_0^1 G(\xi_j, t)x(t)^2\,dt \approx 1 + \sum_{i=1}^{n} \omega_i\, G(\xi_j, \xi_i)x(\xi_i)^2 \tag{5.21}$$

where the nodes ξ_i and weights w_i are given as

$$\xi_i = \frac{1}{2}z_i + \frac{1}{2}, \quad \omega_i = \frac{2}{(1-z_i^2)(\mathcal{P}_n'(z_i))^2}$$

where z_i (also known as $i-$th Gauss-node) are the $i-$th zeros of the normalized Legendre, i.e. $\mathcal{P}_n(1) = 1$, polynomial $\mathcal{P}_n(z)$

$$\mathcal{P}_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{d\,x^n}[(x^2-1)^n].$$

From (5.21), we get the nonlinear-system $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}^n$

$$\mathcal{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - \mathcal{A}\,\mathbf{v}_x = 0 \tag{5.22}$$

where

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}, \quad \mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}}, \quad \mathcal{A} = [a_{i,j}]_{i,j=1}^n, \quad \mathbf{v}_x = [x_1^2, x_2^2, \ldots, x_n^2]^{\mathrm{T}}$$

where $a_{i,j} = \omega_i\, G(\xi_j, \xi_i)$. Moreover, $\mathcal{F}'(\mathbf{x}) = \mathbf{I} - 2\mathcal{A}\mathbf{D}(x)$ where $\mathbf{D}(\mathbf{x}) = \mathrm{diag}\{x_1, x_2, \ldots, x_n\}$ and $\mathcal{F}''(\mathbf{x}) = \mathcal{A}$. The discretized system of equations (5.22) satisfies the condition ($\mathbf{C}_5$) and it also satisfies the hypothesis – condition (2.11) – of Lemma 2.1.

To solve the nonlinear integral equation (4.1), we divide the interval through a $20-$point Gauss-Legendre quadrature rule which results in $20-$nonlinear equations with 20 unknowns. Solution is reported in the Table 4 when the residual is $\|x_{n+1} - x_n\|_{\mathrm{L}_2} \le 1 \times 10^{-50}$. For a second derivative $\mathcal{F}''(\mathbf{x})$ of size $m \times m$ the computational cost of order is $\mathrm{O}(m^2)$ [1]. As a result, for sufficiently large systems the computational cost during each iteration of the four methods (**NM-O2**, **TSNM-O3**, **TSNM-O4**, **TSNM-O5**) is of the same order [1]. Therefore, the fifth order method **TSNM-O5** is the most computationally efficient for solving such systems.

References

1. S. Amat, C. Bermúdez, S. Busquier, M. J. Legaz, and S. Plaza, On a Family of High-Order Iterative Methods under Kantorovich Conditions and Some Applications, Abstract and Applied Analysis, (2012). doi:10.1155/2012/782170.
2. S. Amat, C. Bermúdez, S. Busquier and S. Plaza, On a third-order Newton-type method free of bilinear operators, Numerical Linear Algebra with Applications, 17(4), (2010), 639–653.
3. S. Amat and S. Busquier, Third-order iterative methods under Kantorovich conditions, Journal of Mathematical Analysis and Applications, 336(1), (2007), 243–261.
4. S. Amat, S. Busquier and J.M. Gutiérrezb, Third-order iterative methods with applications to Hammerstein equations: A unified approach, Journal of Computational and Applied Mathematics, 235(9), (2011), 2936–2943.
5. I.K. Argyros, D. Chen, The Halley method for solving nonlinear operator equations in Banach space, Applied Mathematics Letters, 5, 7–9, (1992).
6. I.K. Argyros, D. Chen, The Halley method in Banach spaces and the Ptak error estimates, Appl. Math. Comp., 62, 1–15, (1994).
7. I.K. Argyros, D. Chen, On the Halley iterative method for solving nonlinear operator equations in Banach space and its applications in integral equations. Rev. Anal. Numér. Théor. Approx. 23, 139–152, (1994)
8. I.K. Argyros, Y.J. Cho, S. Hilout, On the Halley method for solving equations, Applied Mathematics and Computation, 216, 2321–2332, (2010).
9. I.K. Argyros, S. Hilout, On the semilocal convergence of a Newton-type method of order three, J. Korea S. M. A. Ser. B.Pure and Appl. Math., 17, 1–27, (2010).
10. I.K. Argyros, The convergence of a Halley-Chebysheff-type method under Newton-Kantorovich hypotheses, Applied Mathematics Letters, 6 (5), (1993), 71–74.
11. I.K. Argyros and D. Chen, Results on the Chebyshev method in Banach spaces, Proyecciones, 12, (1993), 119–128.
12. I.K. Argyros, On the convergence of an Euler-Chebysheff-type method under Newton-Kantorovich hypotheses, Pure Mathematics and Applications, 4(3), (1993), 369–373.
13. I.K. Argyros, D. Chen and Q.S. Qian, A note on the Halley method in Banach spaces, Applied Mathematics and Computation, 58, (1993), 215–224.
14. I.K. Argyros, D. Chen and Q.S. Qian, Error bound representations of Chebyshev-Halley type methods in Banach spaces, Rev. Academia de Ciencias Zaragoza, 49, (1994), 57–69.
15. I.K. Argyros, On the method of tangent hyperbolas, Approximation Theory and its Applications, 12, (1996), 78–95.
16. I.K. Argyros, On the method of tangent parabolas, Functiones et Approximatio Commentarii Mathematici, (1996), 3–15.
17. I.K. Argyros, The super-Halley method using divided differences, Applied Mathematics Letters, 10 (4), (1997) 91–95.
18. I.K. Argyros, Chebysheff-Halley-like methods in Banach spaces, Journal of Applied Mathematics and Computing, 4(1), (1997), 83–107.
19. I.K. Argyros, On the Monotone Convergence of an Euler-Chebysheff-type Method in Partially Ordered Topological Spaces, Rev. Anal. Numer. Theor. Approx., 27(1), (1998), 23–31.
20. I.K. Argyros, Error bounds for the Chebyshev method in Banach spaces, Bulletin of the Institute of Mathematics Academia Sinica, 26(4), (1998), 269–282.

21. I.K. Argyros, Improved Error Bounds for a Chebysheff-Halley-Type Method, Acta Mathematica Hungarica, 84(3), (1999), 209–219.
22. I.K. Argyros, J.A. Ezquerro, J.M. Gutiérrez, M.A. Hernández, S. Hilout, On the semilocal convergence of efficient Chebyshev-Secant-type methods, Journal of Computational and Applied Mathematics, 235(10),(2011), 3195–3206.
23. I.K. Argyros, Y.J. Cho, S. Hilout, Numerical Methods for Equations and Variational Inclusions, CRC Press/Taylor & Francis, 2012.
24. J.A. Ezquerro, A modification of the Chebyshev method, IMA Journal of Numerical Analysis, 17(4), (1997), 511–525.
25. J.A. Ezquerro, J. M. Gutiérrez, M. A. Hernández and M. A. Salanova, Chebyshev-like methods and quadratic equations, Revue d'Analyse Numerique et theorie de l'Approximation, 28,(1), 2000, 23–35.
26. J.A. Ezquerro, M.A. Hernández, New Kantorovich-type conditions for Halley's method, Appl. Numer. Anal. Comput. Math., 2 (1), (2005), 70–77.
27. E. Halley, A new, exact, and easy method of finding roots of any equations generally, and that without any previous reduction, Philos. Trans. Roy. Soc. London, 18, (1694), 136–145.
28. M.A. Hernández, Second-Derivative-Free Variant of the Chebyshev Method for Nonlinear Equations, Journal of Optimization Theory and Applications, 104(3), (2000), 501–515.
29. W. Werner, Some improvements of classical iterative methods for the solution of nonlinear equations, Numerical Solution of Nonlinear Equations Lecture Notes in Mathematics, 878/1981, (1981), 426–440.
30. V. Candela and A. Marquina, Recurrence relations for rational cubic methods II: The Chebyshev method, Computing, 45(4), (1990), 355–367.
31. S. Kanno, Convergence theorems for the method of tangent hyperbolas, Mathernatica Japonica, 87 (4), (1992), 711–722.
32. A.M. Ostrowski, Solution of equations in Euclidean and Banach spaces, Academic Press, New York, 3rd Ed. 1973.
33. J. F. Traub, Iterative Methods for the Solution of Equations, American Mathematical Soc., 1982.
34. T. Yamamoto, On the method of tangent hyperbolas in Banach spaces, Journal of Computational and Applied Mathematics, 21(1), (1988), 75–86

n	t_n	s_n	$s_n - t_n$	$t_{n+1} - s_n$	$t^\star - t_n$	$t^\star - s_n$
0	$0.00 \times 10^{+00}$	3.85×10^{-02}	3.85×10^{-02}	1.04×10^{-02}	6.39×10^{-02}	2.54×10^{-02}
1	4.89×10^{-02}	6.14×10^{-02}	1.25×10^{-02}	1.67×10^{-03}	1.49×10^{-02}	2.46×10^{-03}
2	6.31×10^{-02}	6.39×10^{-02}	7.79×10^{-04}	7.25×10^{-06}	7.86×10^{-04}	7.46×10^{-06}
3	6.39×10^{-02}	6.39×10^{-02}	2.05×10^{-07}	5.07×10^{-13}	2.05×10^{-07}	5.07×10^{-13}
4	6.39×10^{-02}	6.39×10^{-02}	3.77×10^{-18}	1.71×10^{-34}	3.77×10^{-18}	1.71×10^{-34}
5	6.39×10^{-02}	6.39×10^{-02}	2.34×10^{-50}	6.56×10^{-99}	2.34×10^{-50}	6.56×10^{-99}
6	6.39×10^{-02}	6.39×10^{-02}	5.55×10^{-147}	3.71×10^{-292}	5.55×10^{-147}	3.71×10^{-292}
7	6.39×10^{-02}	6.39×10^{-02}	7.47×10^{-437}	6.70×10^{-872}	7.47×10^{-437}	6.70×10^{-872}
8	6.39×10^{-02}	6.39×10^{-02}	1.81×10^{-1306}	$0.00 \times 10^{+00}$	1.81×10^{-1306}	$0.00 \times 10^{+00}$
9	6.39×10^{-02}	6.39×10^{-02}	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$

TABLE 1. Majorizing sequence (2.12) for (4.1).

n	$\|x_{n+1} - x_n\|$	$\|x_{n+1} - y_n\|$	$\|x_n - y_n\|$	$\|x_n - x^\star\|$	$\|y_n - x^\star\|$
0	4.00×10^{-02}	1.50×10^{-03}	3.85×10^{-02}	4.00×10^{-02}	1.52×10^{-03}
1	1.61×10^{-05}	2.58×10^{-10}	1.61×10^{-05}	1.61×10^{-05}	2.58×10^{-10}
2	5.35×10^{-19}	2.86×10^{-37}	5.35×10^{-19}	5.35×10^{-19}	2.86×10^{-37}
3	6.53×10^{-73}	4.27×10^{-145}	6.53×10^{-73}	6.53×10^{-73}	4.27×10^{-145}
4	1.46×10^{-288}	2.12×10^{-576}	1.46×10^{-288}	1.46×10^{-288}	2.12×10^{-576}
5	3.59×10^{-1151}	$0.00 \times 10^{+00}$	3.59×10^{-1151}	3.59×10^{-1151}	$0.00 \times 10^{+00}$
6	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
7	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
8	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
9	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$

TABLE 2. Method (1.2) applied to $\mathcal{F}(x) = x^3 - 1$.

n	q_n	s_n	u_n	$r_{n+1} - q_n$	$t_{n+1} - s_n$	$v_{n+1} - u_n$
0	1.43×10^{-01}	1.43×10^{-01}	1.43×10^{-01}	3.42×10^{-03}	2.18×10^{-02}	2.18×10^{-02}
1	1.47×10^{-01}	1.76×10^{-01}	1.80×10^{-01}	9.69×10^{-07}	1.85×10^{-04}	3.40×10^{-04}
2	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.24×10^{-13}	5.28×10^{-09}	2.17×10^{-08}
3	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	2.00×10^{-27}	3.79×10^{-18}	6.91×10^{-17}
4	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	5.23×10^{-55}	1.96×10^{-36}	7.02×10^{-34}
5	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.56×10^{-110}	5.20×10^{-73}	7.23×10^{-68}
6	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.65×10^{-220}	3.68×10^{-146}	7.68×10^{-136}
7	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.57×10^{-441}	1.84×10^{-292}	8.66×10^{-272}
8	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.66×10^{-882}	4.62×10^{-585}	1.10×10^{-543}
9	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.60×10^{-1765}	2.90×10^{-1170}	1.78×10^{-1087}

TABLE 3. Comparison among the sequences (2.12), (3.24) and (3.25). Estimates (3.26) – (3.30) hold.

n	$\|x_{n+1} - x_n\|_{L_2}$			
	NM-O2	**TSNM-O3**	**TSNM-O4**	**TSNM-O5**
1	9.869×10^{-2}	1.931×10^{-3}	1.074×10^{-4}	6.652×10^{-5}
2	4.275×10^{-4}	4.233×10^{-6}	2.139×10^{-16}	4.122×10^{-23}
3	3.957×10^{-8}	8.426×10^{-18}	4.275×10^{-63}	1.886×10^{-123}
4	1.931×10^{-16}	3.957×10^{-50}	$---$	$---$
5	2.224×10^{-33}	$---$	$---$	$---$
6	8.001×10^{-65}	$---$	$---$	$---$

TABLE 4. Errors for the Newton (**NM-O2**) and the methods (1.2) applied to (5.20).