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STRONG CONVERGENCE THEOREMS FOR COMMON FIXED POINT OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to give some strong convergence theorems for the problem of finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces by the combination of the regularization method and the viscosity approximation method.

KEYWORDS: Accretive operators; weak sequential continuous mapping; and contraction mapping.
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1. INTRODUCTION

Numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. For instance, if the nonexpansive mappings are projections onto some closed and convex sets, then the fixed point problem becomes the famous convex feasibility problem. Due to the practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be flourishing topic of interest in fixed point theory.
The problem of finding a common fixed point of nonexpansive mappings has been investigated by many researchers: see, for instance, Bauschke [4], O' Hara et al. [23], Jung [16], Chang et al. [8], Ceng et al. [9], Chidume et al. [11], Kang et al. [18], N. Buong et al. [6] and others.

In 2000, Moudafi [22] proposed a viscosity approximation method which was considered by many authors [7, 10, 24, 27, 30, 32, 33] of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If H is a Hilbert space, $T: C \to C$ is nonexpansive self-mapping on a nonempty closed convex C of H and $f: C \to C$ is a contraction mapping, then he proved the following results:

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(1) The sequence $\{x_n\}$ in C generated by the iterative scheme:

$$x_0 \in C,\ x_n = \frac{1}{1+\varepsilon_n}T(x_n) + \frac{\varepsilon_n}{1+\varepsilon_n}f(x_n),\ \forall n \geq 0,$$

converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \leq 0,\ \forall x \in F(T),$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers tending to zero.

(2) With a initial $z_0 \in C$, define the sequence $\{z_n\}$ in C by

$$z_{n+1} = \frac{1}{1+\varepsilon_n}T(z_n) + \frac{\varepsilon_n}{1+\varepsilon_n}f(z_n),\ \forall n \geq 0.$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=1}^{\infty} \varepsilon_n = +\infty$, and $\lim_{n\to\infty}\left|\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}\right| = 0.$ Then $\{z_n\}$ converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \leq 0,\ \forall x \in F(T).$$

On the other hand, the problem of finding a fixed point of a nonexpansive mapping is equivalent to the problem of finding a zero of the operator equation $0 \in A(x)$, involving the accretive mapping A.

One popular method of solving the problem of finding a zero of a maximal monotone operator is the proximal point algorithm, this algorithm is proposed by Rockafellar. In 1976, Rockafellar [25] proved the weak convergence of his algorithm, if the regularization sequence is bounded away from zero and if the sequence of the errors satisfies the suitable condition. In 1991, Güler [14] gave an example showing that Rockafellar's proximal point algorithm did not converge strongly in an infinite-dimensional Hilbert space. So, to have strong convergence, one has modify this algorithm. Recently, several authors proposed modifications of Rockafellar's proximal point algorithm to have strong convergence. Solodov and Svaiter [26] initiated such investigation followed by Kamimura and Takahashi [17] (in which the work of [26] is extended to the framework of uniformly convex and uniformly smooth Banach spaces). Lehdili and Moudafi [20] combined the technique of the proximal map and the Tikhonov regularization to introduce the prox-Tikhonov method. In 2006, Xu [31]; in 2009, Song and Yang [28] combined the regularization proximal point algorithm and a modification of iterative algorithms of Hapern's type [19] to obtain strong convergence theorems for the problem of finding a zero of maximal monotone operator in Hilbert space.

In 2011, by using the regularization proximal point algorithm of Xu [31], J. K. Kim and T. M. Tuyen [19] introduced an implicit iterative method in the form

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n u + (1-t_n)x_n,\ n \geq 0, \tag{1.1}$$

where $u, x_0 \in E$, and $A_i = I - T_i$ to find a common fixed point of a finite family of nonexpansive mappings $T_i:\ E \longrightarrow E,\ i = 1,2,...,N$ in Banach spaces. With this algorithm they are obtained the strong convergence of iterative $\{x_n\}$ generated by (1.1) to a common fixed point of T_i, when the sequences $\{r_n\}$ and $\{t_n\}$ are chosen suitable.

In this paper, we combine the regularization method and the viscosity approximation method, and use the tecnique of accretive operators to get convergence theorems for the problem of finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces. And also, we consider the stability of algorithms and we give an application for the convex feasibility problem in Banach spaces.

2. PRELIMINARIES AND NOTATIONS

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the same symbol $\|.\|$. We write $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ and $\longrightarrow$ to denote the weak convergence, weak* convergence and strong convergence, respectively.

Definition 2.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon$, imply there exists a $\delta = \delta(\varepsilon) \ge 0$ such that

$$\frac{\|x + y\|}{2} \le 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| :\ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\} \tag{2.1}$$

is called the modulus of convexity of the space E. The function $\delta_E(\varepsilon)$ defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0, 2]$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x + y\| + \|x - y\|\big) - 1 :\ \|x\| = 1,\ \|y\| = \tau\}, \tag{2.2}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive. In what follows, we denote

$$h_E(\tau) := \frac{\rho_E(\tau)}{\tau}. \tag{2.4}$$

The function $h_E(\tau)$ is nondecreasing. In addition, we have the following estimate

$$h_E(K\tau) \le LKh_E(\tau),\ \forall K > 1,\ \tau > 0, \tag{2.5}$$

where L is the Figiel's constant [1, 2, 13], $1 < L < 1.7$.

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.6}$$

is called the normalized duality mapping of E.

In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$ and, if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J be single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x,\ J(x_n) \overset{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

Definition 2.4. An operator $A :\ D(A) \subseteq E \rightrightarrows E$ is called accretive, if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \ge 0,\ \forall u \in A(x),\ v \in A(y). \tag{2.7}$$

Definition 2.5. An operator $A: \ D(A) \subseteq E \rightrightarrows E$ is called $m-$accretive, if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$.

If A is a $m-$accretive operator, then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightharpoonup x$ and $A(x_n) \longrightarrow f$, then $A(x) = f$ [2].

Definition 2.6. A mapping $T: \ C \longrightarrow E$ is said to be nonexpansive on a closed and convex subset C of Banach space E if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall x, y \in C. \tag{2.8}$$

If $T: \ C \longrightarrow E$ is a nonexpansive then $I - T$ is accretive operator.

Definition 2.7. A mapping $f: \ C \longrightarrow E$ is said to be contraction on a closed and convex subset C of Banach space E, if there exists $c \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq c\|x - y\| \ \forall x, y \in C. \tag{2.9}$$

Definition 2.8. Let G be a nonempty closed and convex subset of E. A mapping $Q_G: \ E \longrightarrow G$ is said to be

i) a retraction onto G if $Q_G^2 = Q_G$;

ii) a nonexpansive retraction, if it also satisfies the inequality

$$\|Q_G x - Q_G y\| \leq \|x - y\|, \ \forall x, y \in E; \tag{2.10}$$

iii) a sunny retraction, if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{2.11}$$

A closed and convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Definition 2.9. Let $C_1, \ C_2$ be convex subsets of E. The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| = \sup_{u \in C_1} d(u, C_2)$$

is said to be semideviation of the set C_1 from the set C_2. The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \ \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2.

In what follows, we shall make use of the following lemmas:

Lemma 2.10. [3] *If E is a uniformly smooth Banach space, C_1 and C_2 are closed and convex subsets of E such that the Hausdorff $\mathcal{H}(C_1, C_2) \leq \delta$, and Q_{C_1} and Q_{C_2} are the sunny nonexpansive retractions onto the subsets C_1 and C_2, respectively, then*

$$\|Q_{C_1} x - Q_{C_2} x\|^2 \leq 16R(2r + d)h_E\left(\frac{16L\delta}{R}\right), \tag{2.12}$$

where L is Figiel's constant, $r = \|x\|, \ d = \max\{d_1, d_2\}$, and $R = 2(2r + d) + \delta$. Here $d_i = dist(\theta, C_i)$, $i = 1, 2$, and θ is the origin of the space E.

Lemma 2.11. [1] *Let E be an uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping $T: \ D(A) \subseteq E \longrightarrow E$, then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \geq L^{-1}R^2\delta_E\left(\frac{\|Ax - Ay\|}{4R}\right), \tag{2.13}$$

where $\|x\| \leq R, \ \|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 2.12. [1] *In an uniformly smooth Banach space E, for all $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + c\rho_E(\|y\|), \tag{2.14}$$

where $c = 48\max(L, \|x\|, \|y\|)$.

Lemma 2.13. [12] *Let A be a continuous and accretive operator on the real Banach space E with $D(A) = E$. Then A is $m-$accretive.*

Lemma 2.14. [5, 29] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \sigma_n, \ \forall n \ge 0,$$

where $\{\alpha_n\} \subset (0,1)$ for each $n \ge 0$ such that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$, or (c) $\limsup \dfrac{\sigma_n}{\alpha_n} \le 0$. Then $a_n \to 0$ as $n \to \infty$.

3. MAIN RESULTS

Firstly, we consider the following problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N Fix(T_i), \tag{3.1}$$

where $Fix(T_i)$ is the set of fixed points of the nonexpansive mapping $T_i : E \longrightarrow E$, $i = 1, 2, ..., N$.

Let $x_0 \in E$ and let $f : E \longrightarrow E$ be contraction mapping on E with the contractive coefficient $k \in [0,1)$, we define the sequence $\{x_n\}$ as follow:

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n, \ n \ge 0, \tag{3.2}$$

where $\{r_n\}$ and $\{t_n\}$ are sequences of positive real numbers.

Remark 3.1. The algorithm (1.1) is a special case of the algorithm (3.2), when $f(x) = u$ for all $x \in E$.

Remark 3.2. In this paper, we use the symbol f to denote the contraction mapping on E with the contractive coefficient $k \in [0,1)$.

Theorem 3.3. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i : E \longrightarrow E, \ i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^N Fix(T_i) \ne \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0,1)$ satisfy*

- i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.2) converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I-f)(q), j(q-p)\rangle \le 0, \ \forall p \in S. \tag{3.3}$$

Proof. Firstly, we show that equation (3.2) defines a unique sequence $\{x_n\} \subset E$. Indeed, since the operator $\sum_{i=1}^N A_i$ is Lipschitz continuous and accretive on E, it is $m-$accretive (Lemma 2.13). Therefore equation (3.2) has a unique solution $x_{n+1} \in E$.

For every $x^* \in S$, we have

$$\langle \sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - x^*)\rangle \ge 0, \ \forall n \ge 0. \tag{3.4}$$

Therefore,

$$\langle t_n f(x_n) + (1-t_n)x_n - x_{n+1}, j(x_{n+1} - x^*)\rangle \geq 0, \ \forall n \geq 0. \tag{3.5}$$

It gives the inequality

$$\|x_{n+1} - x^*\|^2 \leq [t_n\|f(x_n) - x^*\| + (1-t_n)\|x_n - x^*\|].\|x_{n+1} - x^*\|.$$

Consequently, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq t_n\|f(x^*) - x^*\| + t_n\|f(x_n) - f(x^*)\| + (1-t_n)\|x_n - x^*\|\\ &\leq t_n\|f(x^*) - x^*\| + [1 - t_n(1-k)]\|x_n - x^*\|\\ &\leq \max\big(\frac{\|f(x^*) - x^*\|}{1-k}, \|x_n - x^*\|\big)\\ &\vdots\\ &\leq \max\big(\frac{\|f(x^*) - x^*\|}{1-k}, \|x_0 - x^*\|\big), \ \forall n \geq 0.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$, which converges weakly to a limit point $\overline{x} \in E$.

Suppose $\|x_n\| \leq R$ and $\|x^*\| \leq R$ with $R > 0$. By Lemma 2.11, we have

$$\begin{aligned}\delta_E\left(\frac{\|A_i(x_{n+1})\|}{4R}\right) &\leq \frac{L}{R^2}\langle A_i(x_{n+1}), j(x_{n+1} - x^*)\rangle\\ &\leq \frac{L}{R^2}\langle\sum_{k=1}^{N} A_k(x_{n+1}), j(x_{n+1} - x^*)\rangle\\ &\leq \frac{L}{R^2 r_n}\|t_n f(x_n) + (1-t_n)x_n - x_{n+1}\|.\|x_{n+1} - x^*\|\\ &\longrightarrow 0, \ n \longrightarrow \infty,\end{aligned}$$

for every $i = 1, 2, ..., N$.

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_{n+1}) \longrightarrow 0$, $i = 1, 2, ..., N$. It is clear that $\overline{x} \in S$ from the demiclosedness of A_i.

Let q be unique solution of the variational inequality (3.3). Then, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)(q), j(q - x_n)\rangle &= \lim_{k\to\infty}\langle (I-f)(q), j(q - x_{n_k})\rangle\\ &= \langle (I-f)(q), j(q - \overline{x})\rangle \leq 0.\end{aligned} \tag{3.6}$$

Next, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \langle -r_n\sum_{i=1}^{N} A_i(x_{n+1}) + t_n f(x_n) + (1-t_n)x_n - q, j(x_{n+1} - Q_S u)\rangle\\ &= -r_n\langle\sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - q)\rangle + \langle t_n f(x_n) + (1-t_n)x_n - q, j(x_{n+1} - q)\rangle\\ &\leq \frac{1}{2}[\|t_n f(x_n) + (1-t_n)x_n - q\|^2 + \|x_{n+1} - q\|^2].\end{aligned}$$

By Lemma 2.12 and the estimate above, we conclude that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \|t_n f(x_n)+(1-t_n)x_n-q\|^2\\
&= \|t_n(f(x_n)-f(q))+(1-t_n)(x_n-q)+t_n(f(q)-q)\|^2\\
&\le (1-t_n)^2\|x_n-q\|^2+2t_n(1-t_n)\langle (f(x_n)-f(q))+(f(q)-q), j(x_n-q)\rangle\\
&+ c\rho_E(t_n\|f(x_n)-q\|)\\
&\le [(1-t_n)^2+2kt_n(1-t_n)]\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\
&+ c\rho_E(t_n\|f(x_n)-q\|)\\
&\le \begin{cases} (1-t_n)\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\ +c\rho_E(t_n\|f(x_n)-q\|), \text{ if } k\in[0,\frac{1}{2}],\\ [1-2(1-k)t_n]\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\ +c\rho_E(t_n\|f(x_n)-q\|), \text{ if } k\in(\frac{1}{2},1).\end{cases}
\end{aligned}$$

Consequently, we have

$$\|x_{n+1}-q\|^2 \le \begin{cases} (1-t_n)\|x_n-q\|^2+\sigma_n, \text{ if } k\in[0,\frac{1}{2}],\\ [1-2(1-k)t_n]\|x_n-q\|^2+\sigma_n, \text{ if } k\in(\frac{1}{2},1),\end{cases} \tag{3.7}$$

where

$$\sigma_n = t_n\big[2(1-t_n)\langle f(q)-q, j(x_n-q)\rangle + c\frac{\rho_E(t_n\|f(x_n)-q\|)}{t_n}\big].$$

Since E is the uniformly smooth Banach space, the property of function $\rho_E(t)$ and the boundedness of $\{f(x_n)\}$, we get that $\dfrac{\rho_E(t_n\|f(x_n)-q\|)}{t_n} \longrightarrow 0,\ n\longrightarrow\infty$. By (3.6), we obtain $\limsup_{n\to\infty}\dfrac{\sigma_n}{t_n}\le 0$. So, an application of Lemma 2.14 onto (3.7) yields the desired result. □

Theorem 3.4. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i: E\longrightarrow E,\ i=1,2,...,N$ be nonexpansive mappings with $S=\cap_{i=1}^N Fix(T_i)\neq\emptyset$. If the sequences $\{r_n\}\subset(0,+\infty)$ and $\{t_n\}\subset(0,1)$ satisfy*

- i) $\lim_{n\to\infty} t_n=0$; $\sum_{n=0}^\infty t_n=+\infty,\ \sum_{n=0}^\infty |t_{n+1}-t_n|<+\infty$;
- ii) $\inf_n r_n=r>0,\ \sum_{n=0}^\infty \big|1-\dfrac{r_n}{r_{n+1}}\big|<+\infty$,

then the sequence $\{x_n\}$ generated by (3.2) converges strongly to a common fixed point q, which is unique solution of the variational inequality (3.3).

Proof. From the proof of Theorem 3.3, we obtain the sequence $\{x_n\}$ is bounded and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\overline{x}\in E$. Now, we show that $\overline{x}\in S$.

In (3.2), replacing n by $n+1$, we get

$$r_{n+1}\sum_{i=1}^N A_i(x_{n+2})+x_{n+2}=t_{n+1}f(x_{n+1})+(1-t_{n+1})x_{n+1}. \tag{3.8}$$

From (3.2) and (3.8) and by the accretiveness of $\sum_{i=1}^{N} A_i$, we have

$$\begin{aligned} r_{n+1}\langle x_{n+2}-x_{n+1}, j(x_{n+2}-x_{n+1})\rangle - (r_{n+1}-r_n)\langle x_{n+2}, j(x_{n+2}-x_{n+1})\rangle \\ \leq \langle r_n[t_{n+1}f(x_{n+1}) + (1-t_{n+1})x_{n+1}] \\ - r_{n+1}[t_n f(x_n) + (1-t_n)x_n], j(x_{n+2}-x_{n+1})\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} r_{n+1}\|x_{n+2}-x_{n+1}\| &\leq \|r_n[t_{n+1}f(x_{n+1}) + (1-t_{n+1})x_{n+1}] - r_{n+1}[t_n f(x_n) + (1-t_n)x_n]\| \\ &+ |r_{n+1}-r_n|.\|x_{n+2}\| \\ &\leq r_{n+1}(1-t_{n+1})\|x_{n+1}-x_n\| + (1-t_{n+1})|r_n - r_{n+1}|.\|x_{n+1}\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|x_n\| + |r_{n+1}-r_n|.\|x_{n+2}\| \\ &+ r_{n+1}t_{n+1}\|f(x_{n+1}-f(x_n))\| + t_{n+1}|r_{n+1}-r_n|.\|f(x_{n+1})\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|f(x_n)\| \\ &\leq r_{n+1}[1-(1-k)t_{n+1}]\|x_{n+1}-x_n\| + (1-t_{n+1})|r_n - r_{n+1}|.\|x_{n+1}\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|x_n\| + |r_{n+1}-r_n|.\|x_{n+2}\| \\ &+ t_{n+1}|r_{n+1}-r_n|.\|f(x_{n+1})\| + r_{n+1}|t_{n+1}-t_n|.\|f(x_n)\|. \end{aligned}$$

By $\{t_n\} \subset (0,1)$ and $r_n > 0$ for all n, we deduce

$$\|x_{n+2}-x_{n+1}\| \leq [1-(1-k)t_{n+1}]\|x_{n+1}-x_n\| + \left(2\big|t_{n+1}-t_n\big| + 3\big|1-\frac{r_n}{r_{n+1}}\big|\right)K, \quad (3.9)$$

where $K = \max\{\sup\|f(x_n)\|, \sup\|x_n\|\} < +\infty$. By Lemma 2.14, $\|x_{n+1}-x_n\| \longrightarrow 0$, as $n \longrightarrow \infty$.

Suppose $R > \max\{K, \|x^*\|\}$. By Lemma 2.11, we have

$$\begin{aligned} \delta_E\left(\frac{\|A_i(x_{n+1})\|}{4R}\right) &\leq \frac{L}{R^2}\langle A_i(x_{n+1}), j(x_{n+1}-x^*)\rangle \\ &\leq \frac{L}{R^2}\langle\sum_{k=1}^{N} A_k(x_{n+1}), j(x_{n+1}-x^*)\rangle \\ &\leq \frac{L}{R^2 r_n}\|t_n f(x_n) + (1-t_n)x_n - x_{n+1}\|.\|x_{n+1}-x^*\| \\ &\leq \frac{2L}{Rr}(2Rt_n + \|x_{n+1}-x_n\|) \\ &\longrightarrow 0, \ n \longrightarrow \infty, \end{aligned}$$

for every $i = 1, 2, ..., N$.

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_{n+1}) \longrightarrow 0$, $i = 1, 2, ..., N$. It is clear that $\overline{x} \in S$ from the demiclosedness of A_i.

The rest of the proof follows the pattern of Theorem 3.3. □

Now, we will give a method to solve the following more general problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} Fix(T_i), \quad (3.10)$$

where $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ is nonexpansive mapping and C_i is a closed, convex and nonexpansive retract of E.

Obviously, we have the following lemma:

Lemma 3.5. *Let E be a Banach space and let C be a closed, convex and retract of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $Fix(T) = Fix(TQ_C)$, where $Q_C : E \longrightarrow C$ is a retraction of E.*

We consider the iterative sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^{N} B_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n, \; x_0 \in E, \;\; n \geq 0, \tag{3.11}$$

where $B_i = I - T_i Q_{C_i}, \; i = 1, 2, ..., N$ and $Q_{C_i} : \; E \longrightarrow C_i$ is a nonexpansive retraction from E onto $C_i, \; i = 1, 2, ..., N$.

Theorem 3.6. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i, \; i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.11) converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I - f)(q), j(q - p) \rangle \leq 0, \; \forall p \in S. \tag{3.12}$$

Proof. By Lemma 3.5, we have $S = \cap_{i=1}^{N} Fix(T_i Q_{C_i})$ and apply Theorem 3.3 we obtain the proof of this theorem. □

Theorem 3.7. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i, \; i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty, \; \sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;
ii) $\inf_n r_n = r > 0, \; \sum_{n=0}^{\infty} \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{x_n\}$ generated by (3.11) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Proof. By Lemma 3.5, we have $S = \cap_{i=1}^{N} Fix(T_i Q_{C_i})$ and apply Theorem 3.4 we obtain the proof of this theorem. □

Next, we study stability of regularization algorithm (3.11) in the case that each C_i is closed, convex and sunny nonexpansive retract of E with respect to pertubations of operators T_i and constraints $C_i, \; i = 1, 2, ..., N$ satisfying conditions:

(P1) Instead of C_i, there is a sequence of closed, convex and sunny nonexpansive retracts $C_i^n \subset E$, $n = 1, 2, 3, ...$ such that

$$\mathcal{H}(C_i^n, C_i) \leq \delta_n, \; i = 1, 2, ..., N,$$

where $\{\delta_n\}$ is a sequence of positive numbers.

(P2) On the each set C_i^n, there is a nonexpansive self-mapping $T_i^n : C_i^n \longrightarrow C_i^n$, $i = 1, 2, ..., N$ satisfying the conditions: if for all $t > 0$, there exists the increasing positive functions $g(t)$ and $\xi(t)$ such that $g(0) \geq 0, \; \xi(0) = 0$ and $x \in C_i, \, y \in C_i^m, \; \|x - y\| \leq \delta$, then

$$\|T_i x - T_i^m y\| \leq g(\max\{\|x\|, \|y\|\})\xi(\delta). \tag{3.13}$$

Remark 3.8. Note that, the conditions (P1) and (P2) are considered in [1] by Y. Alber.

We establish the convergence and stability of regularization method (3.11) in the form

$$r_n \sum_{i=1}^{N} B_i^n(z_{n+1}) + z_{n+1} = t_n f(x_n) + (1-t_n)z_n,\ z_0 \in E,\ \ n \geq 0, \tag{3.14}$$

where $B_i^n = I - T_i^n Q_{C_i^n},\ i = 1,2,...,N,\ f:\ E \longrightarrow E$ is a contraction and $Q_{C_i^n}: E \longrightarrow C_i^n$ is a sunny nonexpansive retraction from E onto $C_i^n,\ i = 1,2,...,N$.

Theorem 3.9. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E and let $T_i:\ C_i \longrightarrow C_i,\ i = 1,2,...,N$ be nonexpansive mappings with $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{r_n\},\ \{\delta_n\}$ and $\{t_n\}$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$;
iii) $\sum_{n=0}^{\infty} r_n \xi(a\sqrt{h_E(\delta_n)}) < +\infty$ *or* $\lim_{n\to\infty} \dfrac{r_n\xi(a\sqrt{h_E(\delta_n)})}{t_n} = 0$ *for each* $a > 0$,

then the sequence $\{z_n\}$ generated by (3.14) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Proof. For each n, $\sum_{i=1}^N B_i^n$ is $m-$accretive operator on E, so the equation (3.14) define unique element $z_{n+1} \in E$.
From the equation (3.11) and (3.14) we have

$$\begin{aligned} & r_n\langle \sum_{i=1}^{N} B_i^n(z_{n+1}) - B_i^n(x_{n+1}), j(z_{n+1} - x_{n+1})\rangle \\ & + r_n\langle \sum_{i=1}^{N} B_i^n(x_{n+1}) - B_i(x_{n+1}), j(z_{n+1} - x_{n+1})\rangle + \|z_{n+1} - x_{n+1}\|^2 \\ & = (1-t_n)\langle z_n - x_n, j(z_{n+1} - x_{n+1})\rangle + t_n\langle f(z_n) - f(x_n), j(z_{n+1} - x_{n+1})\rangle. \end{aligned} \tag{3.15}$$

By the accretiveness of $\sum_{i=1}^N B_i^n$ and the equation (3.15), we deduce

$$\|z_{n+1} - x_{n+1}\| \leq [1-(1-k)t_n]\|z_n - x_n\| + r_n \sum_{i=1}^{N} \|B_i^n(x_{n+1}) - B_i(x_{n+1})\|. \tag{3.16}$$

For each $i \in \{1,2,...,N\}$,

$$\|B_i^n(x_{n+1}) - B_i(x_{n+1})\| = \|T_i^n Q_{C_i^n} x_{n+1} - T_i Q_{C_i} x_{n+1}\|. \tag{3.17}$$

Since $\{x_n\}$ is bounded and $\mathcal{H}(C_i, C_i^n) \leq \delta_n$, there exist constants $K_{1i} > 0$ and $K_{2i} > 1$ such that inequalities

$$\|Q_{C_i^n} x_{n+1} - Q_{C_i} x_{n+1}\| \leq K_{1i}\sqrt{h_E(K_{2i}\delta_n)} \leq K_{1i}\sqrt{K_{2i}L}\sqrt{h_E(\delta_n)} \tag{3.18}$$

hold.
By the condition (P2),

$$\|T_i^n Q_{C_i^n} x_{n+1} - T_i Q_{C_i} x_{n+1}\| \leq g(M_i)\xi(K_{1i}\sqrt{K_{2i}L}\sqrt{h_E(\delta_n)}), \tag{3.19}$$

where $M_i = \max\{\sup \|Q_{C_i^n} x_{n+1}\|, \sup \|Q_{C_i} x_{n+1}\|\} < +\infty$.
From (3.16), (3.17) and (3.19), we obtain

$$\|z_{n+1} - x_{n+1}\| \leq [1 - (1-k)t_n]\|z_n - x_n\| + Ng(M)r_n\xi(\gamma_{12}\sqrt{h_E(\delta_n)}), \tag{3.20}$$

where $M = \max\{M_1, M_2, ..., M_N\} < +\infty$ and $\gamma_{12} = \max\limits_{i=1,2,...,N}\{K_{1i}\sqrt{K_{2i}L}\}$.
By the assumption and Lemma 2.14, we conclude that $\|z_n - x_n\| \longrightarrow 0$. In addition, by Theorem 3.6,

$$\|z_n - q\| \leq \|z_n - x_n\| + \|x_n - q\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \tag{3.21}$$

which implies that z_n converges strongly to q. □

By a proof similar to the proof of Theorem 3.9 we have the following result:

Theorem 3.10. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{r_n\}$, $\{\delta_n\}$ and $\{t_n\}$ satisfy*

i) $\lim_{n\to\infty} t_n = 0; \sum_{n=0}^{\infty} t_n = +\infty, \sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty;$
ii) $\inf\limits_n r_n = r > 0, \sum_{n=0}^{\infty} \left|1 - \frac{r_n}{r_{n+1}}\right| < +\infty,$
iii) $\sum_{n=0}^{\infty} r_n\xi(a\sqrt{h_E(\delta_n)}) < +\infty$ *or* $\lim_{n\to\infty} \frac{r_n\xi(a\sqrt{h_E(\delta_n)})}{t_n} = 0$ *for each* $a > 0$,

then the sequence $\{z_n\}$ generated by (3.14) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Finally, in this section we give a method to solve the following problem:

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N Fix(T_i), \tag{3.22}$$

where $T_i : C_i \longrightarrow E$, $i = 1, 2, ..., N$ is nonexpansive nonself-mapping and C_i is a closed, convex and sunny nonexpansive retract of E.

Lemma 3.11. [21] *Let C be a closed and convex subset of a strictly convex Banach space E and let $T : C \longrightarrow E$ be a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $Fix(T) \neq \emptyset$, then $Fix(T) = Fix(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C.*
We have the following results:

Theorem 3.12. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E, let $T_i : C_i \longrightarrow E$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0; \sum_{n=0}^{\infty} t_n = +\infty;$
ii) $\lim_{n\to\infty} r_n = +\infty,$

then the sequence $\{u_n\}$ defined by

$$r_n \sum_{i=1}^{N} f_i(u_{n+1}) + u_{n+1} = t_n f(x_n) + (1 - t_n)u_n, \; u_0 \in E, \;\; n \geq 0, \tag{3.23}$$

converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I - f)(q), j(q - p) \rangle \leq 0, \; \forall p \in S, \tag{3.24}$$

where $f_i = I - Q_{C_i} T_i Q_{C_i},\ i = 1, 2, ..., N$.

Proof. By Lemma 3.5 and Lemma 3.11, $S = \cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N Fix(f_i)$. Apply Theorem 3.3 we obtain the proof of this theorem. □

Theorem 3.13. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E, let $T_i : C_i \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty,\ \sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;
ii) $\inf_n r_n = r > 0,\ \sum_{n=0}^{\infty} \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{u_n\}$ defined by

$$r_n \sum_{i=1}^{N} f_i(u_{n+1}) + u_{n+1} = t_n f(x_n) + (1 - t_n)u_n,\ u_0 \in E,\quad n \geq 0, \tag{3.25}$$

converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.24).

Proof. By Lemma 3.5 and Lemma 3.11, $S = \cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N Fix(f_i)$. Apply Theorem 3.4 we obtain the proof of this theorem. □

4. An Application

Consider the following convex feasibility problem:

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N S_i \neq \emptyset, \tag{4.1}$$

where $S_i,\ i = 1, 2, ..., N$ are closed, convex and nonexpansive retracts of a uniformly convex and uniformly smooth Banach space E.

In this section, we give an application of regularization algorithms (3.2) to find a solution of (4.1).

Let Q_{S_i} denote the nonexpansive retraction from E onto $S_i,\ i = 1, 2, ..., N$. It is clear that $F(Q_{S_i}) = S_i,\ i = 1, 2, ..., N$. Thus, the problem (4.1) is equivalent to the problem of finding a common fixed point of finite family of nonexpansive mappings $T_i = Q_{S_i},\ i = 1, 2, ..., N$.

By Theorem 3.3 and Theorem 3.4, we have the following results:

Theorem 4.1. *If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1 - t_n)x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.2}$$

converges strongly to a solution of (4.1)*, where* $A_i = I - Q_{S_i},\ i = 1, 2, ..., N$

Theorem 4.2. *If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty,\ \sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;
ii) $\inf_n r_n = r > 0,\ \sum_{n=0}^{\infty} \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{x_n\}$ *defined by*

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.3}$$

converges strongly to a solution of (4.1), *where* $A_i = I - Q_{S_i},\ i = 1, 2, ..., N$

Now, we consider a special case of problem (4.1), it is the problem of finding a solution of a general system of linear equations.

Let S denote the set of solutions of the general system of linear equations

$$\sum_{j=1}^{k} a_{ij}x_j = b_i,\ i = 1, 2, ..., N, \tag{4.4}$$

and we suppose $S \neq \emptyset$, and $\sum_{j=1}^{k} a_{ij}^2 > 0,\ \forall i = 1, 2, ..., N.$

Let

$$S_i = \{(x_1, x_2, ..., x_k) \mid \sum_{j=1}^{k} a_{ij}x_j = b_i\},\ i = 1, 2, ..., N. \tag{4.5}$$

Then, S_i is a hyperplane in $\mathbb{R}^k$.

It is well - known that, the orthogonal projection P_i from $\mathbb{R}^k$ onto S_i is also the sunny nonexpansive retraction from $\mathbb{R}^k$ onto $S_i,\ i = 1, 2, ..., N$. Moreover,

$$P_i(x) = \left(x_l - a_{il} \frac{\sum\limits_{j=1}^{k} a_{ij}x_j - b_l}{\sum\limits_{j=1}^{n} a_{ij}^2} \right)_{l=1}^{k},\ i = 1, 2, ..., N, \tag{4.6}$$

for all $x = (x_1, ..., x_k) \in \mathbb{R}^k$.

We have a corollary of Theorem 4.1 and Theorem 4.2:

Corollary 4.3. *If the sequences* $\{r_n\} \subset (0, +\infty)$ *and* $\{t_n\} \subset (0, 1)$ *satisfy the conditions i) and ii) in Theorem 4.1 or the conditions i) and ii) in Theorem 4.2, then the sequence* $\{x_n\}$ *defined by*

$$r_n \sum_{i=1}^{N} B_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.7}$$

converges strongly to a solution x^* *of system* (4.4), *where* $B_i = I - P_i,\ i = 1, 2, ..., N.$

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