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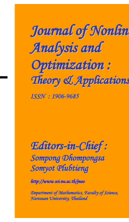

REFINEMENTS OF ε-DUALITY THEOREMS FOR A NONCONVEX PROBLEM WITH AN INFINITE NUMBER OF CONSTRAINTS

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ABSTRACT. Some remarks on approximate optimality conditions of a nonconvex optimization problem which has an infinite number of constraints are given. Results on ϵ-duality theorems of the problem are refined by using a mixed type dual problem of Wolfe and Mond-Weir type.

KEYWORDS: Almost ε-quasi solutions; ε-quasi solutions; ε-duality theorems.
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1. INTRODUCTION

It was known that one of the first results dealing with approximate optimality solutions of a nonconvex programming problem was the paper "Necessary condition for ε-optimality" published on 1982 by P. Loridan [8]. A bit earlier, the such results can be found in the book of P.-J. Laurent [7] and in the paper of S.S Kutateladze [6]. Since the appearance of these results, there were many papers concerning in approximate necessary/sufficient optimality conditions of nonconvex problems such as [2, 4, 5, 9, 11, 12, 14, 15]. Besides concept of ε-solutions of an optimization problem which have global character, there were concepts of approximate solutions which have local one such as ε-quasi solutions, almost ε-quasi solutions. If the concept of global solutions is suitable for convex problems, the concept of local solutions is crucial for nonconvex problems.

Recently, in [12], some sufficient ε-optimality conditions and ε-duality theorems of a nonconvex optimization problem which has an infinite number of constraints have been established without assuming any constraint qualification condition. These results can be improved. Let us reconsider the problem:

$$\begin{array}{lll}(\mathrm{P}) & \text{Minimize} & f(x) \\ & \text{s.t} & f_t(x) \leq 0, t \in T, \\ & & x \in C,\end{array}$$

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where $f, f_t : X \longrightarrow \mathbb{R}$, $t \in T$ are locally Lipschitz functions on a Banach space X, T is an arbitrary index set (not necessarily finite), C is a closed convex subset of X. In that paper, approximate optimality conditions are established based on a generalized Karush-Kuhn-Tucker condition up to ε and the properties of regularity or ε-semiconvexity applied for locally involved Lipschitz functions. Results on approximate duality theorems of Wolfe type are also presented.

The aim of this paper is to give some remarks on approximate sufficient conditions presented in [12]. Concretely, the approximate sufficient conditions can rebuilt with relaxed assumptions. Moreover, results on ϵ-duality theorems in the paper will be refined. Besides, relations between (P) and its dual problems via approximate dual theorems will be clarified. To improve results on ε-optimality conditions, we use the properties of ε-regularity and ε-semiconvexity for the locally involved Lipschitz functions instead of the properties of regularity and semiconvexity. To refine the results of ϵ-duality theorems given in the paper, we use a mixed type dual problem for (P). Then results on ϵ-duality theorems of Wolfe type and Mond-Weir type are derived. We also note that, the mixed type dual scheme was applied recently for a nonconvex multiobjective programming problem which has an infinite number of constraints [13].

The paper is organized as follows. The next section is devoted to preliminaries including basic concepts and definitions such as ϵ-semiconvex functions and locally approximate solutions. The main results are in the last two sections. In Section 3, some versions of approximate sufficient optimality conditions for (P) are given by using properties of ε-regularity or ε-semiconvexity applied for the functions involved of (P). In the last section, by formulating the dual problem of (P) in a mixed type, some new results on ϵ-duality theorems are proposed. Then some results on ϵ-duality theorems presented in [12] can be covered. Finally, evaluations between the approximate optimal values of primal-dual problems are studied.

2. PRELIMINARIES AND NOTATIONS

Throughout the paper, X is a Banach space, T is a compact topological space, C is a closed convex subset of X, and $f : X \longrightarrow \mathbb{R}$ is a locally Lipschitz function. We also assume that the constraint functions $f_t : X \longrightarrow \mathbb{R}$, $t \in T$, are locally Lipschitz with respect to x uniformly in t, i.e., that for each $x \in X$, there exists a neighbourhood U of x and a constant $K > 0$ such that

$$|f_t(z) - f_t(z')| \leq K \|z - z'\| \quad \forall\, z,\, z' \in U \text{ and } \forall\, t \in T.$$

Let $g : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$ is

$$g'(z; d) = \lim_{t \longrightarrow 0^+} \frac{g(z + td) - g(z)}{t}$$

if the limit exists.

The Clarke generalized directional derivative at $z \in X$ in direction $d \in X$ and the Clarke subdifferential of g at z are defined by

$$g^c(z; d) := \lim_{\substack{x \longrightarrow z \\ t \longrightarrow 0^+}} \sup \frac{g(x + td) - g(x)}{t},$$

$$\partial^c g(z) := \{v \in X^* \mid v(d) \leq g^c(z; d), \forall d \in X\},$$

respectively, where X^* is a dual space of X.

A locally Lipschitz function g is said to be quasidifferentiable (or regular in the sense of Clarke) at $z \in X$ if $g'(z;d)$ exists and

$$g^c(z;d) = g'(z;d), \forall d \in X.$$

For a closed subset D of X, the tangent cone to D is defined by

$$T_D(x) = \{v \in X \mid d_D^\circ(x;v) = 0\},$$

where d_D denotes the distance function to D, and the normal cone to D at x is defined by

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, v\rangle \leq 0, \forall v \in T_D(x)\}.$$

If D is convex, the normal cone to D at x coincides with the one in the sense of convex analysis, i.e,

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0, \forall y \in D\}.$$

Definition 2.1. [12] Let C be a subset of X and let $\alpha \geq 0$. A locally Lipschitz function $g : X \longrightarrow \mathbb{R}$ is said to be α-semiconvex at $z \in C$ if g is regular at z, and the following condition is satisfied

$$g'(z; x - z) + \sqrt{\alpha}\|x - z\| \geq 0 \implies g(x) + \sqrt{\alpha}\|x - z\| \geq g(z), \ \forall x \in C. \tag{2.1}$$

The function g is said to be α-semiconvex on C if g is α-semiconvex at every $z \in C$.

As $\alpha = 0$, we obtain the definition of semiconvex function proposed in [10].

Lemma 2.2. [10] *If $g : X \longrightarrow \mathbb{R}$ is a semiconvex function on a convex set $C \subset X$, $z \in C, z + d \in C$ then $g(z + d) \leq g(z)$ implies that $g'(z;d) \leq 0$.*

Definition 2.3. [8] Let $\varepsilon \geq 0$. A locally Lipschitz function $g : X \longrightarrow \mathbb{R}$ is said to be ε-regular at $z \in X$, provided that

$$0 \leq g^c(z;d) - g'(z;d) \leq \sqrt{\varepsilon}\|d\|, \forall d \in X.$$

We use the following linear space:

$$\mathbb{R}^{(T)} := \{(\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda = (\lambda_t) \in \mathbb{R}^{(T)}$, the supporting set according to λ is

$$T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}.$$

Obviously, it is a finite subset of T. We also denote by $\mathbb{R}_+^{(T)}$ the non-negative cone of $\mathbb{R}^{(T)}$,

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}.$$

It is easy to see that this cone is convex. For every $\lambda \in \mathbb{R}^{(T)}$, we define

$$\|\lambda\|_1 := \sum_{t\in T} |\lambda_t| = \sum_{t\in T(\lambda)} |\lambda_t|.$$

For $\alpha \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}^{(T)}$, $\lambda = (\lambda_t)_{t\in T}, \mu = (\mu_t)_{t\in T}$, we understand that

$$\begin{array}{rcl} \lambda + \mu & := & (\lambda_t + \mu_t)_{t\in T}, \\ \alpha.\lambda & := & (\alpha\lambda_t)_{t\in T}. \end{array}$$

With $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space, we define

$$\sum_{t\in T} \lambda_t z_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $\lambda \in \mathbb{R}^{(T)}, f_t, t \in T$, and $\{Y_t\}_{t\in T}$, a family of non-empty subsets of X, we understand that

$$\sum_{t\in T}\lambda_t f_t = \begin{cases} \sum_{t\in T(\lambda)}\lambda_t f_t & \text{if} \quad T(\lambda)\neq\emptyset, \\ 0 & \text{if} \quad T(\lambda)=\emptyset, \end{cases}$$

and

$$\sum_{t\in T}\lambda_t Y_t = \begin{cases} \sum_{t\in T(\lambda)}\lambda_t Y_t & \text{if} \quad T(\lambda)\neq\emptyset, \\ 0 & \text{if} \quad T(\lambda)=\emptyset. \end{cases}$$

We denote by A the feasible set of (P). Let $\epsilon > 0$, the ε-feasible set of (P) is defined by

$$A_\varepsilon := \{x \in C \mid f_t(x) \leq \sqrt{\varepsilon}, \forall t \in T\}.$$

Definition 2.4. Let $\varepsilon \geq 0$. A point $z_\varepsilon \in X$ is said to be

(i) an almost ε-solution of (P) if

$$z_\varepsilon \in A_\varepsilon \text{ and } f(z_\varepsilon) \leq f(x) + \varepsilon, \forall x \in A;$$

(ii) an almost ε-quasisolution of (P) if

$$z_\varepsilon \in A_\varepsilon \text{ and } f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\|, \forall x \in A;$$

(iii) an almost regular ε-solution of (P) if z_ε is an almost ε-solution and is an almost ε-quasisolution of (P).

As $z_\varepsilon \in A$, we obtain the definitions of ε-solution, ε-quasisolution, and regular ε-solution of (P), respectively.

3. ε-OPTIMALITY CONDITIONS

To give some remarks and to improve results in [12], some theorems are recalled for the sake of convenience. Firstly, we need the following conditions:

$(\mathcal{A})$ (a1) X is separable, or
(a2) T is metrizable and $\partial^c f_t(x)$ is upper continuous (w^*) in t for each $x \in X$.

$(\mathcal{B})$ $\exists d \in T_C(z), f_t^c(z;d) < 0, \forall t \in I(z)$, where $z \in A$, $I(z) = \{t \in T \mid f_t(z) = 0\}$.

Theorem 3.1. [12] *Let $\varepsilon \geq 0$ and z be an ε-quasisolution for* (P). *If the conditions $(\mathcal{A})$ and $(\mathcal{B})$ are satisfied and the convex hull of $\{\cup\partial^c f_t(z), t \in I(z)\}$ is $weak^*$-closed then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f(z) + \sum_{t\in T}\lambda_t\partial^c f_t(z) + N_C(z) + \sqrt{\varepsilon}B^*, f_t(z) = 0, \forall t \in T(\lambda), \tag{3.1}$$

where B^* is a closed unit ball in X^*.

A pair (z,λ) satisfies (3.1) is called a Karush-Kuhn-Tucker (KKT) pair up to ε. From the theorem above, a generalized KKT condition up to ε was proposed as follows.

Definition 3.1. [12] Let $\varepsilon \geq 0$. A pair $(z_\varepsilon,\lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ is said to be satisfied generalized KKT condition up to ε corresponding to (P) if

$$\begin{cases} 0 \in \partial^c f(z_\varepsilon) + \sum\limits_{t\in T(\lambda)}\lambda_t\partial^c f_t(z_\varepsilon) + N(C,z_\varepsilon) + \sqrt{\varepsilon}B^* \\ f_t(z_\varepsilon) \geq 0, \forall t \in T(\lambda). \end{cases}$$

The pair (z_ε,λ) is called a generalized KKT pair up to ε. It is called strict if $f_t(z_\varepsilon) > 0$ for all $t \in T(\lambda)$, which is equivalent to $\lambda_t = 0$ if $f_t(z_\varepsilon) \leq 0$.

Then, a sufficient condition for a strict generalized KKT pair up to ε was given.

Theorem 3.2. [12] *Let $\varepsilon > 0$ and the condition $(\mathcal{A})$ be satisfied. For every $x \in A_\varepsilon$, let the strong closure of the subset $co\{\cup\partial^c f_t(x), t \in I(x)\}$ be $weak^*$-closed. Then there exists an almost regular ε-solution z for* (P) *and $\lambda \in \mathbb{R}_+^{(T)}$ such that (z, λ) is a strict generalized KKT pair up to ε.*

The such generalized KKT pair condition was used as a hypothesis to survey almost ε-quasisolutions of (P).

Theorem 3.3. [12] *For the problem* (P)*, assume that C is convex and that the functions f_t, $t \in T$, are convex. Let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If f is ε-semiconvex at z_ε with respect to C, then*

$$f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \text{ for all } x \in C \text{ such that}$$
$$f_t(x) \leq f_t(z_\varepsilon),\ \forall t \in T(\lambda).$$

In particular, z_ε is an almost ε-quasisolution for (P).

By modifying the assumptions applied for the involved functions of (P), we extend the theorem above to the ones as follows. Firstly, assumptions posed on the involved functions of (P) are relaxed.

Theorem 3.4. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. Suppose that the function f is ε-regular at z_ε and the functions $f_t, t \in T$, are semiconvex on C. If the condition (2.1) of Definition 2.1 holds for f at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$ then*

$$f(z_\varepsilon) \leq f(x) + 2\sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that}$$
$$f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda).$$

In particular, z_ε is an almost 4ε-quasisolution for (P).

Proof. Let $\varepsilon \geq 0$. Assume that $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$is a generalized KKT pair up to ε. If $T(\lambda) \neq \emptyset$, we obtain $u \in \partial^c f(z_\varepsilon), u_t \in \partial^c f_t(z_\varepsilon), \forall t \in T(\lambda), w \in N(C, z_\varepsilon), v \in B^*$ and $f_t(z_\varepsilon) \geq 0$ for all $t \in T(\lambda)$ such that

$$u(x - z_\varepsilon) + \sum_{t\in T(\lambda)} \lambda_t u_t(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| = -w(x - z) \geq 0, \forall x \in C, \tag{3.2}$$

Note that $f_t, t \in T$, are semiconvex at z_ε. If $f_t(x) \leq f_t(z_\varepsilon)$ for all $t \in T(\lambda)$ then

$$u_t(x - z_\varepsilon) \leq f_t^c(z_\varepsilon; x - z_\varepsilon) = f_t'(z_\varepsilon; x - z_\varepsilon) \leq 0, \forall t \in T(\lambda), \forall x \in C.$$

Then, from (3.2), we obtain $u(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \geq 0$ for all $x \in C$. Since $u \in \partial^c f(z_\varepsilon)$ and f is ε-regular at z_ε, $f^c(z_\varepsilon; x - z_\varepsilon) \leq f'(z_\varepsilon; x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\|$. We get

$$f'(z_\varepsilon; x - z_\varepsilon) + \sqrt{4\varepsilon}\,\|x - z_\varepsilon\| \geq 0, \forall x \in C.$$

Since the condition (2.1) of Definition 2.1 holds for f at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$, from the inequality above, we deduce the desired result. As $T(\lambda) = \emptyset$, we get

$$u(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| = -w(x - z) \geq 0, \forall x \in C.$$

It is easy to see that the conclusion can be derived. $\square$

Corollary 3.2. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If $f_t, t \in T$, are semiconvex at z_ε and f is ε-semiconvex at z_ε then*

$$f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that}$$
$$f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda).$$

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. Arguing as in the proof of the theorem above with noticing that f is regular and $\alpha = \epsilon$, we can obtain the desired result. □

Remark 3.3. Since a convex function is a semiconvex function (see [8], [12]), we can see that Theorem 3.3 is a corollary of the one above.

Frequently, the Lagrangian function corresponding to (P) is formulated by

$$L(y, \lambda) = f(y) + \sum_{t \in T} \lambda_t f_t(y), \text{ for all } (y, \lambda) \in X \times \mathbb{R}_+^{(T)}.$$

It is obvious that, for every $\lambda \in \mathbb{R}_+^{(T)}$, the function $L(\cdot, \lambda)$ is locally Lipschitz on X. Note that if the functions f and $f_t, t \in T$, are semiconvex or ε- semiconvex at z_ε then $L(\cdot, \lambda)$ may not achieve the same property. We propose another version of theorem above.

Theorem 3.5. *For the problem* (P), *let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. Assume that $f_t, t \in T$, are regular at z_ε and f is ε-regular at z_ε. If the condition (2.1) of Definition 2.1 holds for $L(\cdot, \lambda)$ at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$ then*

$$\begin{aligned} &f(z_\varepsilon) \leq f(x) + 2\sqrt{\varepsilon}\, \|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that} \\ &f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda). \end{aligned}$$

In particular, z_ε is an almost 4ε-quasisolution for (P).

Proof. Let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If $T(\lambda) = \emptyset$, the proof is similar to the case in the proof of Theorem 3.4. When $T(\lambda) \neq \emptyset$, we get $f_t(z_\varepsilon) \geq 0$, for all $t \in T(\lambda)$. Using an argument similar to the one of the proof of Theorem 3.4, we obtain $u \in \partial^c f(z_\varepsilon), u_t \in \partial^c f_t(z_\varepsilon), \forall t \in T(\lambda), w \in N(C, z_\varepsilon), v \in B^*$ such that

$$u(x - z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t u_t(x - z_\varepsilon) + \sqrt{\varepsilon} v(x - z_\varepsilon) = -w(x - z_\varepsilon) \geq 0 \geq 0, \forall x \in C.$$

Since $f_t, t \in T$, are regular at z_ε and f is ε-regular at z_ε, we derive

$$f'(z_\varepsilon; x - z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t f_t'(z_\varepsilon; x - z_\varepsilon) + \sqrt{4\varepsilon}\, \|x - z_\varepsilon\| \geq 0, \forall x \in C,$$

i.e.,

$$L'(\cdot, \lambda)(z_\varepsilon; x - z_\varepsilon) + \sqrt{4}\varepsilon\, \|x - z_\varepsilon\| \geq 0, \forall x \in C.$$

Since the condition (2.1) of Definition 2.1 holds for $L(\cdot, \lambda)$ at z_ε with $\mu \geq 4\varepsilon$, it follows

$$f(x) + \sum_{t \in T(\lambda)} \lambda_t f_t(x) + \sqrt{4\varepsilon}\, \|x - z_\varepsilon\| \geq f(z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t f_t(z_\varepsilon), \forall x \in C.$$

On the other hand, we have $f_t(x) \leq f_t(z_\varepsilon)$, for all $t \in T(\lambda)$. Then,

$$f(x) + \sqrt{4\varepsilon}\, \|x - z_\varepsilon\| \geq f(z_\varepsilon), \forall x \in C.$$

Since $A \subset C$, it is easy to deduce that z_ε is an almost 4ε-quasisolution for (P). □

Corollary 3.4. *For the problem* (P), *let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a KKT pair up to ε. If $f, f_t, t \in T$, are regular at z_ε and $L(\cdot, \lambda)$ is ε-semiconvex at z_ε then*

$$\begin{aligned} &f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\, \|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that} \\ &f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda). \end{aligned}$$

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. If $L(\cdot,\lambda)$ is ε-semiconvex then the condition (2.1) of Definition 2.1 holds for for $L(\cdot,\lambda)$ with $\alpha=\varepsilon$. On the other hand, if f is regular at z_ε then $u(x-z_\varepsilon)\le f^c(z_\varepsilon;x-z_\varepsilon)=f'(z_\varepsilon;x-z_\varepsilon), u\in\partial^c(z_\varepsilon)$. Using a similar argument as in the proof of theorem above, we can deduce the desired result. □

4. ε-DUALITY THEOREMS

In [12], the dual problem of (P) was formulated in Wolfe type and some results on ε-duality theorems was established. In this part, we are interested in a dual problem of (P) in a mixed type of Wolfe and Mond-Weir type. With this approach, we can cover some results established before. In addition, ε-duality theorems in Mond-Weir are also derived. Besides ε-duality theorems, our results attempt to evaluate the relations between the approximate optimal values of (P) and its dual problems.

Let us consider the mixed type dual problem of (P):

$$\begin{array}{lll} \text{(D)} & \text{Maximize} & L(x,\lambda):=f(y)+\sum_{t\in T}\lambda_t f_t(y)\\ & \text{s.t} & 0\in\partial^c f(y)+\sum_{t\in T}(\lambda_t+\mu_t)\partial^c f_t(y)+N(C,x)+\sqrt{\varepsilon}B^*,\\ & & \mu_t f_t(y)\ge 0, t\in T,\\ & & (y,\lambda,\mu)\in C\times\mathbb{R}_+^{(T)}\times\mathbb{R}_+^{(T)}. \end{array}$$

Denote by F the feasible set of (D).

Based on the definition of ε-quasisolutions of the dual problem of (P) in Wolfe type presented in [12], we propose the definition of ε-quasisolutions of (D) as follows.

Definition 4.1. A point $(y_\varepsilon,\bar\lambda,\bar\mu)\in F$ is called an ε-quasisolution of (D) if

$$L(y_\varepsilon,\bar\lambda)\ge L(y,\lambda)-\sqrt{\varepsilon}\|y-y_\varepsilon\|-\sqrt{\varepsilon}\|\lambda-\bar\lambda\|_1,\forall(y,\lambda,\mu)\in F.$$

Theorem 4.1. *If $f,f_t,t\in T$, are regular on C and $L(\cdot,\zeta)$ is ε-semiconvex on C for every $\zeta\in\mathbb{R}_+^{(T)}$ then ε-weak duality between* (P) *and* (D) *holds, i.e.,*

$$f(x)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda),\forall x\in A,\forall(y,\lambda,\mu)\in F.$$

Proof. Let x and (y,λ,μ) be the feasible solutions of (P) and (D), respectively. We have

$$0\in\partial^c f(y)+\sum_{t\in T}(\lambda_t+\mu_t)\partial^c f_t(y)+N(C,y)+\sqrt{\varepsilon}B^*,\mu_t f_t(y)\ge 0,t\in T.$$

Using an argument as in the proofs of theorem above, we deduce that

$$L(x,\lambda+\mu)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda+\mu),\forall x\in C.$$

As $x\in A$, we get $f_t(x)\le 0$ for all $t\in T$. From this and $\mu_t f_t(y)\ge 0, t\in T$, the inequality above implies

$$f(x)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda).$$

□

Theorem 4.2. *Let $(z,\bar\lambda)\in A_\epsilon\times\mathbb{R}_+^{(T)}$ be a strict generalized KKT pair up to ε. If $f,f_t,t\in T$, are regular at z and $L(\cdot,\zeta)$ is ε-semiconvex at z for every $\zeta\in\mathbb{R}_+^{(T)}$ then $(z,\bar\lambda,0)$ is an ε-quasisolution of* (D).

Proof. Let $(y,\lambda,\mu)\in F$. Using an argument similar to the one in the proof the theorem above, we can deduce that

$$L(x,\lambda+\mu)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda+\mu)\ge L(y,\lambda),\forall x\in C.$$

Hence,

$$L(z,\lambda+\mu) \geq L(y,\lambda) - \sqrt{\varepsilon}\|z-y\|. \tag{4.1}$$

Note that

$$L(z,\bar{\lambda}) - L(z,\lambda+\mu) = \sum_{t\in T}(\bar{\lambda}_t - \lambda_t - \mu_t)f_t(z) \tag{4.2}$$

and $(z,\bar{\lambda})$ is a strict KKT pair up to ε. Hence, $\bar{\lambda}_t = 0$ if $f_t(z) \leq 0$.
So, if $f_t(z) \leq 0$ then we get

$$L(z,\bar{\lambda}) - L(z,\lambda+\mu) = -\sum_{t\in T}(\lambda_t+\mu_t)f_t(z) \geq 0. \tag{4.3}$$

If $0 < f_t(z) \leq \sqrt{\varepsilon}$ then

$$(\bar{\lambda}_t - \lambda_t - \mu_t)f_t(z) \geq -|\bar{\lambda}_t - \lambda_t - \mu_t|f_t(z) \geq -|\bar{\lambda}_t - \lambda_t|f_t(z).$$

Combining this, (4.2), and (4.3), we obtain $L(z,\bar{\lambda}) - L(z,\lambda+\mu) \geq -\sqrt{\varepsilon}\|\bar{\lambda}-\lambda\|_1$.
This and (4.1) imply that

$$L(z,\bar{\lambda}) - L(y,\lambda) = L(z,\bar{\lambda}) - L(z,\lambda+\mu) + L(z,\lambda+\mu) - L(y,\lambda) \geq -\sqrt{\varepsilon}\|\bar{\lambda}-\lambda\|_1 - \sqrt{\varepsilon}\|z-y\|.$$

Furthermore, since $(z,\bar{\lambda},0) \in F$, the desired conclusion follows. □

When $\mu = 0$, Problem (D) becomes the dual problem of (P) in Wolfe type and corresponding theorems can be derived. As $\lambda = 0$ we obtain the dual problem of (P) in Mond-Weir type as follows.

$$\begin{array}{lll} (\mathrm{D_M}) & \text{Maximize} & f(y) \\ & \text{s.t} & 0 \in \partial^c f(y) + \sum_{t\in T}\mu_t\partial^c f_t(y) + N(C,x) + \sqrt{\varepsilon}B^*, \\ & & \mu_t f_t(y) \geq 0, t\in T, \\ & & (y,\mu) \in C \times \mathbb{R}_+^{(T)}. \end{array}$$

The feasible set of $(\mathrm{D_M})$ is denoted by F_M.

Definition 4.2. A point $(z,\bar{\mu}) \in F_M$ is called an ε-quasisolution of $(\mathrm{D_M})$ if

$$f(z) + \sqrt{\varepsilon}\|z-y\| \geq f(y), \forall (y,\mu) \in F_M.$$

Remark 4.3. When $\lambda = 0$, from Theorem 4.1, we get $f(x) + \sqrt{\varepsilon}\|x-y\| \geq f(y), x \in A, (y,0,\mu) \in F$. Combining this and the problem $(\mathrm{D_M})$ we obtain an ε-weak duality theorem in Mond-Weir type. As $\mu = 0$, Theorem 4.1 becomes the ε-weak duality theorem presented in [12], and Theorem 4.2 reduces to Corollary 5.2 in [12].

Theorem 4.3. *Let $(z,\bar{\mu})$ be a KKT pair up to ϵ. Suppose that $f, f_t, t \in T$, are regular at z and $L(\cdot,\zeta)$ is ε-semiconvex at z for every $\zeta \in \mathbb{R}_+^{(T)}$. Then z is an ε-quasisolution of $(\mathrm{D_M})$.*

Proof. Let $(y,\mu) \in F_M$. By using an argument as in the proof of Theorem 4.1, we can deduce

$$L(x,\mu) + \sqrt{\epsilon}\|x-y\| \geq L(y,\mu) \geq f(y), \forall x \in C.$$

Since $(z,\bar{\mu})$ is a KKT pair up to ϵ, $(z,\bar{\mu})$ is a point of F_M. Furthermore, since $z \in A$, $f_t(z) \leq 0$ for all $t \in T$. Consequently, from the inequality above,

$$f(z) + \sqrt{\epsilon}\|z-y\| \geq L(x,\bar{\mu}) + \sqrt{\epsilon}\|z-y\| \geq L(y,\bar{\mu}) \geq f(y).$$

The desired result follows. □

Relations between (P) and its mixed type dual problem will be clarified some more by Theorem 4.4 and 4.5 below.

Theorem 4.4. *Suppose that $f, f_t, t \in T$, are regular at z_ε and $L(\cdot,\zeta)$ is ε-semiconvex at z_ε for every $\zeta \in \mathbb{R}_+^{(T)}$. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of* (D) *such that $\bar{\lambda}_t f_t(z_\varepsilon) \geq 0$ for all $t \in T$. If $z_\varepsilon \in A_\varepsilon$ then it is an almost ε-quasisolution for* (P).

Proof. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of (D). Using argument as above, we can deduce that

$$L(x, \bar{\lambda}, \bar{\mu}) + \sqrt{\epsilon}\|x - z\| \geq L(z_\varepsilon, \bar{\lambda}, \bar{\mu}) \geq f(z), \forall x \in C.$$

If $z_\varepsilon \in A_\varepsilon$ then for all $x \in A$ we obtain

$$f(x) + \sqrt{\epsilon}\|x - z\| \geq L(z_\varepsilon, \bar{\lambda}, \bar{\mu}) \geq f(z).$$

□

Remark 4.4. As $\mu = 0$, the theorem above becomes Proposition 5.2 in [12].

The following theorem is a small modification of the one above. The proof is omitted.

Theorem 4.5. *Suppose that $f_t, t \in T$, are semiconvex at z_ε and f is ε-semiconvex at z_ε. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of* (D) *such that $\bar{\lambda}_t f_t(z_\varepsilon) \geq 0$ for all $t \in T$. If $z_\varepsilon \in A_\varepsilon$ then it is an almost ε-quasisolution for* (P).

Remark 4.5. When $\mu = 0$, Theorems 4.4 and 4.5 are the ε-converse dual theorems in Wolfe type.

It is well known that, for an optimization problem, if strong duality between the problem and its dual problem appears then they have the same optimal value. In case approximate duality, it may need to know the error estimation between the optimal values of primal and dual problems. The next part is devoted to propose some results on error estimation between the value of (P) and the value of its dual problem at their ε-quasisolutions, respectively.

Theorem 4.6. *Given $\varepsilon > 0$, suppose that z_ε is an ε-quasisolution of* (P) *and there exist $\lambda^*, \mu^* \in \mathbb{R}_+^{(T)}$ such that $(z_\varepsilon, \lambda^* + \mu^*)$ is a KKT pair up to ε. Let $(y_\varepsilon, \bar{\lambda}, \bar{\mu})$ is an ε-quasisolution of* (D)*. If $L(\cdot,\zeta)$ is ε-semiconvex on C for every $\zeta \in \mathbb{R}_+^{(T)}$ then*

$$-\sqrt{\varepsilon}\|\bar{\lambda} - \lambda^*\|_1 - \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\| \leq L(y_\varepsilon, \bar{\lambda}) - f(z_\varepsilon) \leq \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\| \tag{4.4}$$

Proof. Let $(y_\varepsilon, \bar{\lambda}, \bar{\mu})$ be an ε-quasisolution of (D). We get

$$L(y_\varepsilon, \bar{\lambda}) \geq L(y, \lambda) - \sqrt{\varepsilon}\|y - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda - \bar{\lambda}\|_1, \forall (y, \lambda, \mu) \in F. \tag{4.5}$$

Let z_ε be an ε-quasisolution of (P) and $(z_\varepsilon, \lambda^* + \mu^*)$ be a KKT pair up to ε. We obtain $f_t(z_\varepsilon) = 0$ for all $t \in T(\lambda^* + \mu^*)$. Note that $T(\lambda^*), T(\mu^*) \subset T(\lambda^* + \mu^*)$. It implies that $f_t(z_\varepsilon) = 0$ for all $t \in T(\mu^*) \cup T(\lambda^*)$. Hence, $\mu_t^* f_t(z_\varepsilon) = 0$ for all $t \in T$. This deduces that $(z_\varepsilon, \lambda^* + \mu^*)$ is also a feasible point of (D). From (4.5), we obtain

$$L(y_\varepsilon, \bar{\lambda}) \geq L(z_\varepsilon, \lambda^*) - \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda^* - \bar{\lambda}\|_1.$$

Note that $f_t(z_\varepsilon) = 0$ for all $t \in T(\lambda^*)$. Hence, $L(z_\varepsilon, \lambda^*) = f(z_\varepsilon)$. So,

$$L(y_\varepsilon, \bar{\lambda}) \geq f(z_\varepsilon) - \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda^* - \bar{\lambda}\|_1. \tag{4.6}$$

On the other hand, by applying Theorem 4.1 with $L(\cdot, \bar{\lambda})$ to be ε-semiconvex on C, we obtain

$$f(z_\varepsilon) + \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| \geq L(y_\varepsilon, \bar{\lambda}).$$

This and (4.6) imply the conclusion. □

The following corollary can be obtained directly if the dual problem is formulated in Mond-Weir type.

Corollary 4.6. *Given $\varepsilon > 0$, suppose that z_ε is an ε-quasisolution of* (P) *and there exist $\mu^* \in \mathbb{R}_+^{(T)}$ such that (z_ε, μ^*) is a KKT pair up to ε. Let y_ε is an ε-quasisolution of the problem $(\mathrm{D_M})$. If f is ε-semiconvex on C then*

$$|f(y_\varepsilon) - f(z_\varepsilon)| \leq \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\|.$$

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