ON THE MEANS OF PROJECTIONS ON CAT(0) SPACES

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ABSTRACT. We improve a result on approximation a common element of two closed convex subsets of a complete CAT(0) space appeared as Theorem 4.1 in [2]. New practical iterative scheme is presented and conditions on two given sets are relaxed.

1. INTRODUCTION

von Neumann introduced the alternating projection method and proved the following strong convergence in Hilbert spaces [cf. 2]:

Theorem 1.1 (von Neumann). *Let H be a Hilbert space and $A, B \subset H$ its closed subspaces. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by*

$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}), \ n \in \mathbb{N}, \tag{1.1}$$

where P_A, P_B are projection mappings from H to A and B respectively. Then $\{x_n\}$ converges in norm to a point from $A \cap B$.

When "subspaces" are replaced by "convex subsets", we only have "weak convergence" for the alternating projections:

Theorem 1.2. [3] *Let H be a Hilbert space and $A, B \subset H$ closed convex sets with $A \cap B \neq \emptyset$. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by (1.1). Then $\{x_n\}$ weakly converges to a point from $A \cap B$.*

It took 39 years since 1965 until Hundal [7] in 2004 could provide a counter example:

Example 1.3. [7] *There exist a hyperplane $A \subset \ell_2$, a convex cone $B \subset \ell_2$ and a point $x_0 \in \ell_2$ such that the sequence generated by (1.1) from the starting point x_0 converges weakly to a point in $A \cap B$ but not in norm.*

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In 2011, Bačák, Searston, Sims [2] extend the result of Bregman for CAT(0) spaces.

Theorem 1.4. [2, Theorem 4.1] *Let X be a complete CAT(0) space and $A, B \subset X$ convex closed subsets such that $A \cap B \neq \emptyset$. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by (1.1). Then:*

(i) *$\{x_n\}$ weakly converges to a point $x \in A \cap B$.*

(ii) *If A and B are boundedly regular, then $x_n \longrightarrow x$.*

(iii) *If A and B are boundedly linearly regular, then $x_n \longrightarrow x$ linearly.*

(iv) *If A and B are linearly regular, then $x_n \longrightarrow x$ linearly with a rate independent of the starting point.*

It is the aim of this paper to present an iterative sequence which strongly converges to a common point of the sets A and B. We do not impose any requirements on A and B as stated in (ii).

2. PRELIMINARIES

Let (X, d) be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Obviously, c is an isometry and $d(x, y) = l$. We call the image of c a *geodesic segment* joining x and y. If it is unique this geodesic is denoted $[x, y]$. Write $c(\alpha\, 0 + (1 - \alpha)l) = \alpha x \oplus (1 - \alpha)y$ for $\alpha \in (0, 1)$. We also write the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ of a segment $[x, y]$ as $\frac{x \oplus y}{2}$. The space X is said to be a *geodesic space* if every two points of X are joined by a geodesic. It is said to be of *hyperbolic type* [6] if it satisfies the following inequality:

$$d(p, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(p, x) + (1 - \alpha)d(p, y) \tag{2.1}$$

for all $p \in X$. Following [5], let $\{v_1, v_2, ..., v_n\} \subset X$ and $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and write, by induction,

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1 - \lambda_n)\left(\frac{\lambda_1}{1 - \lambda_n}v_1 \oplus \frac{\lambda_2}{1 - \lambda_n}v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n}v_{n-1}\right) \oplus \lambda_n v_n. \tag{2.2}$$

Note for an example that $\frac{1}{3}v_1 \oplus \frac{1}{3}v_2 \oplus \frac{1}{3}v_3$ and $\frac{1}{3}v_2 \oplus \frac{1}{3}v_1 \oplus \frac{1}{3}v_3$ are not necessary coincide. Under (2.1) we can see that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i v_i, x\right) \leq \sum_{n=1}^{n} \lambda_i d(v_i, x) \tag{2.3}$$

for each $x \in X$.

A metric space X is said to be a *CAT(0) space* (cf.[4] p.163) if it is a geodesic space satisfying one of the following equivalent conditions.

(i) **(CN) inequality**: If $x_0, x_1 \in X$, then

$$d^2\left(y, \frac{x_0 \oplus x_1}{2}\right) \leq \frac{1}{2}d^2(y, x_0) + \frac{1}{2}d^2(y, x_1) - \frac{1}{4}d^2(x_0, x_1), \text{ for all } y \in X.$$

(ii) **Law of cosine**: If $a = d(p, q), b = d(p, r), c = d(q, r)$ and ξ is the Alexandrov angle at p between $[p, q]$ and $[p, r]$, then $c^2 \geq a^2 + b^2 - 2ab\cos\xi$.

Lemma 2.1. [4, Proposition 2.2] *Let X be a CAT(0) space. Then for each $p, q, r, s \in X$ and $\alpha \in [0, 1]$,*

$$d(\alpha p \oplus (1 - \alpha)q, \alpha r \oplus (1 - \alpha)s) \leq \alpha d(p, r) + (1 - \alpha)d(q, s). \tag{2.4}$$

In particular, (2.1) holds in CAT(0) spaces.

Let C be a nonempty subset of X. We will denote the family of nonempty bounded closed subsets of C by $BC(C)$ and the family of nonempty compact subsets of C by $K(C)$. Let $H(\cdot, \cdot)$ be the *Hausdorff distance* on $BC(X)$, that is,

$$H(A, B) = \max \left\{ \sup_{a \in A} dist(a, B), \sup_{b \in B} dist(b, A) \right\}, \quad A, B \in BC(X),$$

where $dist(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B.

A mapping $t : C \longrightarrow C$ and a multivalued mapping $T : C \longrightarrow BC(C)$ are said to be *nonexpansive* if for each $x, y \in C$,

$$d(tx, ty) \leq d(x, y), \text{ and}$$

$$H(Tx, Ty) \leq d(x, y),$$

respectively. If $tx = x$, we call x a fixed point of a single valued mapping t. And if $x \in Tx$, we call x a fixed point of a multivalued mapping T. We use the notation $Fix(S)$ to stand for the set of all fixed points of a mapping S. Thus $Fix(t) \cap Fix(T)$ is the set of common fixed points of t and T, i.e., $x \in Fix(t) \cap Fix(T)$ if and only if $x = tx \in Tx$.

Let $\{\lambda_n\}$ be a given sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, let $\{v_n\}$ be a bounded sequence in X and let v_0 be an arbitrary point in X. Let $\lambda_n' = \sum_{i=n+1}^{\infty} \lambda_i$ and assume that $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$. In [5] the element $\bigoplus_{n=1}^{\infty} \lambda_n v_n$ has been defined. Here is its description. Set

$$s_n := \lambda_1 v_1 \oplus \lambda_2 v_2 \oplus \cdots \oplus \lambda_n v_n \oplus \lambda_n' v_0.$$

Thus, by (2.2),

$$s_n = \left(\sum_{i=1}^{n} \lambda_i \right) w_n \oplus \lambda_n' v_0, \tag{2.5}$$

where $w_1 = v_1$ and for each $n \geq 2$,

$$w_n = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} v_n.$$

We know that $\{s_n\}$ is a Cauchy sequence. Thus $s_n \longrightarrow x$ as $n \longrightarrow \infty$ for some $x \in X$. Write

$$x = \bigoplus_{n=1}^{\infty} \lambda_n v_n.$$

By (2.5), $d(s_n, w_n) \leq \lambda_n' d(w_n, v_0)$, it is seen that $\lim_{n \longrightarrow \infty} s_n = \lim_{n \longrightarrow \infty} w_n$. Thus the limit x is independent of the choice of v_0. Moreover, it had been shown in [5] that

(A): if y_0 and v_n belong to X, $d(v_n, y_0) = d(x, y_0)$ for all n where $x = \bigoplus_{n=1}^{\infty} \lambda_n v_n$, then $v_n = x$ for all n.

Lemma 2.2. [5, Lemma 3.8] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $\{t_n : n \in \mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} Fix(t_n)$ is nonempty. Define $t : C \longrightarrow C$ by*

$$t(x) = \bigoplus_{n=1}^{\infty} \lambda_n t_n(x)$$

for all $x \in C$ where $\{\lambda_n\} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$. Then t is nonexpansive and $Fix(t) = \bigcap_{n=1}^{\infty} Fix(t_n)$.

Theorem 2.3. [8, Lemma 2.2] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $t : C \to C$ be nonexpansive, fix $u \in C$, and for each $s \in (0,1)$ let x_s be the point of $[u, t(x_s)]$ satisfying*

$$d(u, x_s) = sd(u, t(x_s)).$$

Then $Fix(t) \neq \emptyset$ if and only if $\{x_s\}$ remains bounded as $s \longrightarrow 1$. In this case, the following statements hold:

(1) *$\{x_s\}$ converges to the unique fixed point z of t which is nearest to u;*
(2) *$d^2(u, z) \leq \mu_n d^2(u, u_n)$ for all Banach limits μ and all bounded sequences $\{u_n\}$ with $d(u_n, t(u_n)) \longrightarrow 0$.*

We will follow the proof of the following theorem to prove our main result (Theorem 3.1).

Theorem 2.4. [5, Theorem 3.7] *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{t_n : C \longrightarrow C\}$ be a countable family of nonexpansive mappings and $T : C \longrightarrow K(C)$ be a nonexpansive mapping with $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T) \neq \emptyset$. Suppose that $T(p) = \{p\}$ for all $p \in \bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$. Let t and $\{\lambda_n\}$ be as in Lemma 2.2. Suppose that $u, z_1 \in C$ are arbitrarily chosen and $\{z_n\}$ is defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \left(\frac{1}{2} w_n(z_n) \oplus \frac{1}{2} y_n \right), \quad n \in \mathbb{N}, \tag{2.6}$$

such that $d(y_n, y_{n+1}) \leq d(z_n, z_{n+1})$ for all $n \in \mathbb{N}$, where $y_n \in T(z_n)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(C1) $\lim_{n \to \infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n \longrightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$ which is nearest to u.

In the course of the proof of Theorem 2.4, the following results play important role.

Lemma 2.5. [9, Proposition 2] *Let a be a real number and let $(a_1, a_2, ...) \in \ell^{\infty}$ be such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq a$.*

Lemma 2.6. [1, Lemma 2.3] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\eta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n \longrightarrow \infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \eta_n \text{ for all } n \in \mathbb{N}.$$

Then $\lim_{n \longrightarrow \infty} s_n = 0$.

3. MAIN RESULTS

We first consider a convergence result.

Theorem 3.1. *Let C be a closed convex subset of a complete CAT(0) space X, $t : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(t) \neq \emptyset$ and M a positive real number. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen and $\{z_n\}$ be defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n, \quad u_n \in C$$

such that

$$d(u_n, tz_n) \leq \varepsilon_n M \tag{3.1}$$

for all $n \in \mathbb{N}$. If $\{z_n\}$ is bounded, then the sequence $\{z_n\}$ converges to the unique point of $Fix(t)$ which is nearest to u.

Proof. We follow the proof of Theorem 2.4. By (3.1), we see that

$$\begin{aligned}
d(u_n, u_{n+1}) &\leq d(u_n, tz_n) + d(tz_n, tz_{n+1}) + d(tz_{n+1}, u_{n+1}) \\
&\leq d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}).
\end{aligned}$$

From the definition of z_n, we have

$$\begin{aligned}
d(z_{n+1}, z_n) &= d(\alpha_n u \oplus (1 - \alpha_n)u_n, \alpha_{n-1}u \oplus (1 - \alpha_{n-1})u_{n-1}) \\
&\leq d(\alpha_n u \oplus (1 - \alpha_n)u_n, \alpha_n u \oplus (1 - \alpha_n)u_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1 - \alpha_n)u_{n-1}, \alpha_{n-1}u \oplus (1 - \alpha_{n-1})u_{n-1}) \\
&\leq (1 - \alpha_n)d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) \\
&\leq (1 - \alpha_n)d(z_n, z_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) \\
&\quad + (1 - \alpha_n)M(\varepsilon_n + \varepsilon_{n-1}).
\end{aligned}$$

Putting in Lemma 2.6, $[s_n = d(z_n, z_{n-1}), \gamma_n = 0$ and $\eta_n = |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) + (1-\alpha_n)M(\varepsilon_n+\varepsilon_{n-1})]$ or $[s_n = d(z_n, z_{n-1}), \gamma_n = \left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|d(u, u_{n-1})$ and $\eta_n = (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ according to $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \to \infty}(\alpha_n/\alpha_{n+1}) = 1$, respectively. Thus, using (C3) and $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, we obtain

$$\lim_{n \to \infty} d(z_{n+1}, z_n) = 0.$$

It follows from (C1) that

$$\begin{aligned}
d(z_n, u_n) &\leq d(z_n, z_{n+1}) + d(z_{n+1}, u_n) \\
&= d(z_n, z_{n+1}) + d(\alpha_n u \oplus (1 - \alpha_n)u_n, u_n) \\
&\leq d(z_n, z_{n+1}) + \alpha_n d(u, u_n) \longrightarrow 0.
\end{aligned}$$

This implies

$$\begin{aligned}
d(u_n, tu_n) &\leq d(u_n, tz_n) + d(tz_n, tu_n) \\
&\leq \varepsilon_n M + d(z_n, u_n) \longrightarrow 0.
\end{aligned}$$

Let $x_s \in [u, tx_s]$ satisfying $d(u, x_s) = sd(u, tx_s)$ for all $s \in (0, 1)$. By Theorem 2.3, we have $z =: \lim_{s \to 1} x_s$ which is the unique point of $Fix(t)$ nearest to u and $\mu_n(d^2(u, z) - d^2(u, u_n)) \leq 0$ for all Banach limits μ. Moreover, since $d(u_n, u_{n+1}) \leq d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}) \longrightarrow 0$,

$$\limsup_{n \to \infty} \left(d^2(u, z) - d^2(u, u_n)\right) - \left(d^2(u, z) - d^2(u, u_{n+1})\right) = 0.$$

Therefore Lemma 2.5 implies

$$\limsup_{n \to \infty} \left(d^2(u, z) - (1 - \alpha_n)d^2(u, u_n)\right) = \limsup_{n \to \infty} \left(d^2(u, z) - d^2(u, u_n)\right) \leq 0.$$

Consider the following estimates:

$$\begin{aligned}
d^2(z_{n+1}, z) &= d^2(\alpha_n u \oplus (1 - \alpha_n)u_n, z) \\
&\leq \alpha_n d^2(u, z) + (1 - \alpha_n)d^2(u_n, z) - \alpha_n(1 - \alpha_n)d^2(u, u_n) \\
&= (1 - \alpha_n)d^2(u_n, z) + \alpha_n \left(d^2(u, z) - (1 - \alpha_n)d^2(u, u_n)\right) \\
&\leq (1 - \alpha_n)(d(u_n, tz_n) + d(tz_n, z))^2 + \alpha_n \left(d^2(u, z) - (1 - \alpha_n)d^2(u, u_n)\right) \\
&\leq (1 - \alpha_n)(d^2(z_n, z) + 2\varepsilon_n M d(z_n, z) + \varepsilon_n^2 M^2) \\
&\quad + \alpha_n \left(d^2(u, z) - (1 - \alpha_n)d^2(u, u_n)\right)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)d^2(z_n,z) + \alpha_n\left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right)\\
&\quad + (1-\alpha_n)(2\varepsilon_n M d(z_n,z) + \varepsilon_n^2 M^2)\\
&\leq (1-\alpha_n)d^2(z_n,z) + \alpha_n\left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right)\\
&\quad + (1-\alpha_n)(2\varepsilon_n M N + \varepsilon_n^2 M^2),
\end{aligned}
$$

where $N = \sup\{d(z_n,z) : n \in \mathbb{N}\}$. We can now use Lemma 2.6 to conclude the proof. $\qquad\square$

Here is our first main result.

Theorem 3.2. *Let X be a complete CAT(0) space and $\{A_i : i \in \mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty} \lambda_j$. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in X$ be arbitrarily chosen and set*

$$
r_n = \sup_{i\in\mathbb{N}}\{dist(z_n, A_i)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),
$$

$$
z_{n+1} = \alpha_n u \oplus (1-\alpha_n)u_n, \ where
$$

$$
u_n = \bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \ \ u_n^{A_i} \in A_i \cap B(z_n : dist(z_n, A_i) + \beta_n^2)
$$

for all $n \in \mathbb{N}$. Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty} A_i$ which is nearest to u.

Proof. For each $i \in \mathbb{N}$, let $p_i : X \longrightarrow A_i$ be the projection mapping. Using the law of cosine and the definition of β_n, we have

$$
\begin{aligned}
d^2(u_n^{A_i}, p_i z_n) &\leq d^2(z_n, u_n^{A_i}) - d^2(z_n, p_i z_n)\\
&\leq (d(z_n, p_i z_n) + \beta_n)^2 - d^2(z_n, p_i z_n)\\
&= 2\beta_n d(z_n, p_i z_n) + \beta_n^2 \leq \beta_n(2r_n + \beta_n)\\
&< \left(\frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right)\left(\frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} + r_n\right) = \varepsilon_n^2.
\end{aligned}
$$

Hence $d(u_n^{A_i}, p_i z_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. Let $p : X \longrightarrow X$ be defined by

$$
px = \bigoplus_{i=1}^{\infty} \lambda_i p_i x
$$

for each $x \in X$. From Lemma 2.2, p is nonexpansive and $Fix(p) = \bigcap_{i=1}^{\infty} Fix(p_i) = \bigcap_{i=1}^{\infty} A_i$. For each n, we can choose $m_n \in \mathbb{N}$ such that

$$
d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}\right) + d\left(\bigoplus_{i=1}^{\infty} \lambda_i p_i z_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right) < \varepsilon_n.
$$

Thus

$$
\begin{aligned}
d(u_n, p z_n) &\leq d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}\right) + d\left(\bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right)\\
&\quad + d\left(\bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n, \bigoplus_{i=1}^{\infty} \lambda_i p_i z_n\right)\\
&< \sum_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} d(u_n^{A_i}, p_i z_n) + \varepsilon_n < 2\varepsilon_n.
\end{aligned}
$$

Let $q \in \bigcap_{i=1}^{\infty} A_i$. Then

$$
\begin{aligned}
d(z_{n+1}, q) \;=\;\; & d(\alpha_n u \oplus (1 - \alpha_n) u_n, q) \\[2mm]
\leq\;\; & \alpha_n d(u, q) + (1 - \alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, q \right) \\[2mm]
\leq\;\; & \alpha_n d(u, q) + (1 - \alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} u_n^{A_i} \right) \\[2mm]
& + (1 - \alpha_n) d\left(\bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} u_n^{A_i}, q \right) \\[2mm]
\leq\;\; & \alpha_n d(u, q) + (1 - \alpha_n) \left(\varepsilon_n + \sum_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} \left(d(u_n^{A_i}, p_i z_n) + d(p_i z_n, q) \right) \right) \\[2mm]
\leq\;\; & \alpha_n d(u, q) + (1 - \alpha_n) d(z_n, q) + 2(1 - \alpha_n) \varepsilon_n \\[2mm]
\leq\;\; & \max\{ d(u, q), d(z_n, q) \} + 2(1 - \alpha_n) \varepsilon_n.
\end{aligned}
$$

By induction we have

$$
d(z_{n+1}, q) \leq \max\{ d(u, q), d(z_1, q) \} + 2 \sum_{n=1}^{\infty} (1 - \alpha_n) \varepsilon_n < \infty \text{ for all } n \in \mathbb{N}.
$$

This implies the sequence $\{z_n\}$ is bounded. The result now follows from Theorem 3.1. $\qquad\square$

When the domain is bounded, we have the following result where the sequence $\{z_n\}$ is computable.

Theorem 3.3. *Let X be a complete CAT(0) space and $\{A_i : i \in \mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ and $\bigcup_{i=1}^{\infty} A_i$ is bounded. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty} \lambda_j$. Let $\{\varepsilon_n\}$ be a sequence in $(0, \frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $\lambda_i' < \varepsilon_n$ for all $i \geq k_n$ and set*

$$
r_n = \sup_{i \in \mathbb{N}} \{ dist(z_n, A_i) \}, \quad \beta_n \in \left(0, \frac{1}{2} \sqrt{4 r_n^2 + 4 \varepsilon_n^2} - r_n \right),
$$

$$
z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n', \text{ where}
$$

$$
u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} u_n^{A_i}, \quad u_n^{A_i} \in A_i \cap B(z_n : dist(z_n, A_i) + \beta_n^2).
$$

Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty} A_i$ which is nearest to u.

Proof. Let p_i and p be as in the proof of Theorem 3.2. Thus we have

$$
d(u_n^{A_i}, p_i z_n) < \varepsilon_n
$$

for all $n \in \mathbb{N}$. For each n, we can choose $m_n > k_n$ such that

$$
d\left(\bigoplus_{i=1}^{\infty} \lambda_i p_i z_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} p_i z_n \right) < \varepsilon_n.
$$

Since $\lambda_i' < \varepsilon_n < \frac{1}{2}$, we have

$$
d\left(\bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} p_i z_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} p_i z_n \right)
$$

$$\leq \quad d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_i z_n, \bigoplus_{i=1}^{k_n+1}\frac{\lambda_i}{\sum_{j=1}^{k_n+1}\lambda_j}p_i z_n\right) + \cdots + d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_i z_n, \bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_i z_n\right)$$

$$\leq \quad \frac{\lambda_{k_n+1}}{\sum_{j=1}^{k_n+1}\lambda_j}d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_i z_n, p_{k_n+1}z_n\right) + \cdots + \frac{\lambda_{m_n}}{\sum_{j=1}^{m_n}\lambda_j}d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_i z_n, p_{m_n}z_n\right)$$

$$\leq \quad K\sum_{i=k_n+1}^{m_n}\frac{\lambda_i}{1-\lambda_i'} < 2K\sum_{i=k_n+1}^{m_n}\lambda_i < 2K\lambda_{k_n+1}' < 2K\varepsilon_n,$$

where $K = \sup_{n\in\mathbb{N}}\left\{\sup_{l\in\mathbb{N}}\left\{d\left(\bigoplus_{i=1}^{l}\frac{\lambda_i}{\sum_{j=1}^{l}\lambda_j}p_i z_n, p_{l+1}z_n\right)\right\}\right\} < \infty.$
Thus

$$d(u_n', pz_n) \leq \varepsilon_n(2K+2).$$

The result now follows from Theorem 3.1. □

As corollaries, with the same lines of proofs, the corresponding results hold for a finite family $\{t_i : i = 1, 2, ..., N\}$ of mappings.

Applications

Let X be a complete CAT(0) space. For a function $h : X \longrightarrow (-\infty, \infty]$, the $\alpha-$sublevel set is defined by

$$A_h^\alpha = \{x \in X : h(x) \leq \alpha\}.$$

Let $\{h_i : i \in \mathbb{N}\}$ be a family of lower semi-continuous and convex functions from X into $(-\infty, \infty]$. Bačák, Searston and Sims [2] introduced the method for approximating a minimizer of the functional $H : X \longrightarrow (-\infty, \infty]$, where $H = \sup_{i\in\mathbb{N}} h_i$ as the following:

Proposition 3.4. [2, Proposition 5.2] *Let X be a complete CAT(0) space and a mapping $F : X \longrightarrow (-\infty, \infty]$ be of the form $F = \max\{f, g\}$, where $f, g : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions. Let $\alpha > \inf_{x\in X} F(x) > -\infty$, and A_F^α be nonempty. Assume that f is both uniformly convex and uniformly continuous on bounded sets of X. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by*

$$x_{2n-1} = P_f(x_{2n-1}), \quad x_{2n} = P_g(x_{2n-1}), \ n \in \mathbb{N},$$

where P_f and P_g are projection mappings from X to A_f^α and A_g^α respectively. Then $\{x_n\}$ converges to $z \in A_F^\alpha$.

We now show Propositions providing the strong convergence of the sequence $\{z_n\}$ to an (approximative) minimizer of the functional H.

Proposition 3.5. *Let X be a complete CAT(0) space and a mapping $H : X \longrightarrow (-\infty, \infty]$ be of the form $H = \sup_{i\in\mathbb{N}} h_i$, where $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions for all $i \in \mathbb{N}$. Let $\alpha > \inf_{x\in X} H(x) > -\infty$. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, $\sum_{i=n}^{\infty}\lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty}\lambda_j$. Let $\{\varepsilon_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in X$ are arbitrarily chosen and set*

$$r_n = \sup_{i\in\mathbb{N}}\{dist(z_n, A_{h_i}^\alpha)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)u_n,$$

where

$$u_n = \bigoplus_{i=1}^{\infty} \lambda_i u_n^i, \ \ u_n^i \in A_{h_i}^{\alpha} \cap B(z_n : dist(z_n, A_{f_i}^{\alpha}) + \beta_n^2)$$

for all $n \in \mathbb{N}$. Then the sequence $\{z_n\}$ converges to the unique point of A_H^{α} which is nearest to u.

Proof. Since $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions, $A_{h_i}^{\alpha}$ is closed and convex for all $i \in \mathbb{N}$. The result then follows from Theorem 3.2. $\quad\square$

Proposition 3.6. *Let X be a complete CAT(0) space and a mapping $H : X \longrightarrow (-\infty, \infty]$ be of the form $H = \sup_{i \in \mathbb{N}} h_i$, where $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions for all $i \in \mathbb{N}$. Let $\alpha > \inf_{x \in X} H(x) > -\infty$. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty} \lambda_j$. Let $\{\varepsilon_n\}$ be a sequence in $(0, \frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $\lambda_i' < \varepsilon_n$ for all $i \geq k_n$ and set*

$$r_n = \sup_{i \in \mathbb{N}}\{dist(z_n, A_{h_i}^{\alpha})\}, \ \ \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)u_n',$$

where

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} u_n^i, \ \ u_n^i \in A_{h_i}^{\alpha} \cap B(z_n : dist(z_n, A_{h_i}^{\alpha}) + \beta_n^2).$$

If $\{z_n\}$ is bounded, then the sequence $\{z_n\}$ converges to the unique point of A_H^{α} which is nearest to u.

Proof. Here we apply Theorem 3.3. $\quad\square$

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