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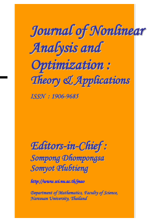

EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR NEUTRAL DIFFERENCE EQUATION WITH VARIABLE DELAY

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ABSTRACT. In this paper, we study the existence of positive periodic solutions of the nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))).$$

The main tool employed here is the Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is completely continuous. The results obtained here generalize the work of Raffoul and Yankson [7].

KEYWORDS : Positive periodic solutions, nonlinear neutral difference equations, fixed point theorem.
AMS Subject Classification: 39A10, 39A12, 39A23.

1. INTRODUCTION

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential and difference equations, see the references in this article and references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay difference equations. Motivated by the papers [1]-[5],[7],[8] and the references therein, we concentrate on the existence of positive periodic solutions for the nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))), \qquad (1.1)$$

where

$$g, f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(n) = x(n+1) - x(n)$ for any

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sequence $\{x(n), n \in \mathbb{Z}\}$. Also, we define the operator E by $Ex(n) = x(n+1)$. For more on the calculus of difference equations, we refer the reader to [6].

The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions for equation (1.1). To apply Krasnoselaskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is completely continuous. In the case $g(n,x) = cx$, Raffoul and Yankson in [7] to show that (1.1) has a positive periodic solutions by using Krasnoselskii's fixed point theorem.

The organization of this paper is as follows. In Section 2, we present the inversion of difference equation (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [9]. In Section 3, we present our main results on existence of positive periodic solutions of (1.1). The results presented in this paper generalize the main results in [7].

2. PRELIMINARIES

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(P_T, \|.\|)$ is a Banach space with the maximum norm

$$\|x\| = \sup_{n \in [0, T-1] \cap \mathbb{Z}} |x(n)|.$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$a(n+T) = a(n), \ \tau(n+T) = \tau(n), \tag{2.1}$$

with τ being scalar sequence and $\tau(n) \geq \tau^* > 0$. Also, we assume

$$0 < a(n) < 1. \tag{2.2}$$

We also assume that the functions $g(n,x)$ and $f(n,x)$ are continuous in x and periodic in n with period T, that is,

$$g(n+T, x) = g(n,x), \ f(n+T, x) = f(n,x). \tag{2.3}$$

The following lemma is fundamental to our results.

Lemma 2.1. *Suppose (2.1)-(2.3) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= g(n, x(n-\tau(n))) \\ &\quad + \sum_{u=n}^{n+T-1} G(n,u) \left[f(u, x(u-\tau(u))) - (1-a(u)) g(u, x(u-\tau(u))) \right], \end{aligned} \tag{2.4}$$

where

$$G(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}. \tag{2.5}$$

Proof. We consider two cases, $n \geq 1$ and $n \leq 0$. Let $x \in P_T$ be a solution of (1.1). For $n \geq 1$ equation (1.1) is equivalent to

$$\triangle \left[x(n) \prod_{s=0}^{n-1} a^{-1}(s) \right] = \left[\triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))) \right] \prod_{s=0}^{n} a^{-1}(s). \tag{2.6}$$

By summing (2.6) from n to $n+T-1$, we obtain

$$\sum_{u=n}^{n+T-1} \triangle \left[x(u) \prod_{s=0}^{u-1} a^{-1}(s) \right] = \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s).$$

As a consequence, we arrive at

$$x(n+T) \prod_{s=0}^{n+T-1} a^{-1}(s) - x(n) \prod_{s=0}^{n-1} a^{-1}(s) = \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s).$$

Since $x(n+T) = x(n)$, we obtain

$$x(n) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right] = \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s). \tag{2.7}$$

Rewrite

$$\sum_{u=n}^{n+T-1} \triangle g(u, x(u-\tau(u))) \prod_{s=0}^{u} a^{-1}(s) = \sum_{u=n}^{n+T-1} E\left[\prod_{s=0}^{u-1} a^{-1}(s) \right] \triangle g(u, x(u-\tau(u))).$$

Performing a summation by parts on the on the above equation, we get

$$\begin{aligned} \sum_{u=n}^{n+T-1} \triangle g(u, x(u-\tau(u))) \prod_{s=0}^{u} a^{-1}(s) &= g(n, x(n-\tau(n))) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right] \\ &- \sum_{u=n}^{n+T-1} g(u, x(u-\tau(u))) \triangle \left[\prod_{s=0}^{u-1} a^{-1}(s) \right] \\ &= g(n, x(n-\tau(n))) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right] \\ &- \sum_{u=n}^{n+T-1} g(u, x(u-\tau(u))) \left[1 - a(u)\right] \prod_{s=0}^{u} a^{-1}(s). \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$x(n) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right]$$

$$= g(n, x(n - \tau(n))) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right]$$
$$- \sum_{u=n}^{n+T-1} g(u, x(u - \tau(u)))[1 - a(u)] \prod_{s=0}^{u} a^{-1}(s)$$
$$+ \sum_{u=n}^{n+T-1} f(u, x(u - \tau(u))) \prod_{s=0}^{u} a^{-1}(s).$$

Dividing both sides of the above equation by $\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)$, we obtain (2.4).

Now for $n \leq 0$, equation (1.1) is equivalent to

$$\triangle \left[x(n) \prod_{s=n}^{0} a^{-1}(s)\right] = [\triangle g(n, x(n - \tau(n))) + f(n, x(n - \tau(n)))] \prod_{s=n+1}^{0} a^{-1}(s).$$

Summing the above expression from n to $n + T - 1$, we obtain (2.4) by a similar argument. This completes the proof. □

To simplify notation, we let

$$m = \min\{G(n, u) : n \geq 0,\ u \leq T\} = G(n, n) > 0, \tag{2.9}$$

and

$$M = \max\{G(n, u) : n \geq 0,\ u \leq T\} = G(n, n + T - 1) = G(0, T - 1) > 0. \tag{2.10}$$

It is easy to see that for all $n, u \in \mathbb{Z}$, we have

$$G(n + T, u + T) = G(n, u). \tag{2.11}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [9].

Theorem 2.1 (Krasnoselskii). *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is completely continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L is non-negative constant and K is positive constant. We express equation (2.4) as

$$\varphi(n) = (\mathcal{B}\varphi)(n) + (\mathcal{A}\varphi)(n) := (H\varphi)(n),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(n) = \sum_{u=n}^{n+T-1} G(n, u)[f(u, \varphi(u - \tau(u))) - (1 - a(u)) g(u, \varphi(u - \tau(u)))], \tag{3.1}$$

and

$$(\mathcal{B}\varphi)(n) = g(n, \varphi(n - \tau(n))). \tag{3.2}$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $g(n,x) \geq 0$ and $g(n,x) \leq 0$ for all $n \in \mathbb{Z}$, $x \in \mathbb{D}$. We assume that function $g(n,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(n,x) - g(n,y)| \leq k \|x - y\|, \text{ for all } n \in [0, T-1] \cap \mathbb{Z}, \ x, y \in \mathbb{D}. \tag{3.3}$$

In the case $g(n,x) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1 x \leq g(n,x) \leq k_2 x, \text{ for all } n \in [0, T-1] \cap \mathbb{Z}, \ x \in \mathbb{D}, \tag{3.4}$$

$$k_2 < 1, \tag{3.5}$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$\frac{L(1-k_1)}{mT} \leq f(n,x) - [1 - a(n)]\, g(n,x) \leq \frac{K(1-k_2)}{MT}, \tag{3.6}$$

where m and M are defined by (2.9) and (2.10), respectively.

Lemma 3.1. *Suppose that the conditions (2.1)-(2.3) and (3.4)-(3.6) hold. Then $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Proof. We first show that $(\mathcal{A}\varphi)(n+T) = (\mathcal{A}\varphi)(n)$.
Let $\varphi \in \mathbb{D}$. Then using (3.1) we arrive at

$$\begin{aligned}
&(\mathcal{A}\varphi)(n+T) \\
&= \sum_{u=n+T}^{n+2T-1} G(n+T,u)\left[f(u, \varphi(u - \tau(u))) - (1 - a(u))\, g(u, \varphi(u - \tau(u)))\right].
\end{aligned}$$

Let $j = u - T$, then

$$\begin{aligned}
&(\mathcal{A}\varphi)(n+T) \\
&= \sum_{j=n}^{n+T-1} G(n+T, j+T)\left[f(j+T, \varphi(j+T-\tau(j+T)))\right. \\
&\left. - (1 - a(j+T))\, g(j+T, \varphi(j+T-\tau(j+T)))\right] \\
&= \sum_{j=n}^{n+T-1} G(n,j)\left[f(j, \varphi(j - \tau(j))) - (1 - a(j))\, g(j, \varphi(j - \tau(j)))\right] \\
&= (\mathcal{A}\varphi)(n),
\end{aligned}$$

by (2.1), (2.3) and (2.11).

To see that $\mathcal{A}(\mathbb{D})$ is uniformly bounded, we let $n \in [0, T-1] \cap \mathbb{Z}$ and for $\varphi \in \mathbb{D}$, we have by (3.6) that

$$\begin{aligned}
&|(\mathcal{A}\varphi)(n)| \\
&\leq \left| \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u - \tau(u))) - (1 - a(u))\, g(u, \varphi(u - \tau(u)))\right] \right| \\
&\leq MT \frac{K(1-k_2)}{MT} = K(1-k_2).
\end{aligned}$$

From the estimation of $|(\mathcal{A}\varphi)(n)|$ it follows that

$$\|\mathcal{A}\varphi\| \leq K(1-k_2).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathcal{A}$ maps bounded subsets into compact sets. As $\mathcal{A}(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, then $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathcal{A}$ is completely continuous. This completes the proof. □

Lemma 3.2. *Suppose that (3.3) holds. If $\mathcal{B}$ is given by (3.2) with*

$$k < 1, \tag{3.7}$$

then $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (3.2). Obviously, $(\mathcal{B}\varphi)(n+T) = (\mathcal{B}\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned} |(\mathcal{B}\varphi)(n) - (\mathcal{B}\psi)(n)| &\leq |g(n, \varphi(n-\tau(n))) - g(n, \psi(n-\tau(n)))| \\ &\leq k \|\varphi - \psi\|. \end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq k\|\varphi - \psi\|$. Thus $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction by (3.7). □

Theorem 3.1. *Suppose (2.1)-(2.3) and (3.3)-(3.7) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is completely continuous. Also, from Lemma 3.2, the operator $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned} &(\mathcal{B}\psi)(n) + (\mathcal{A}\varphi)(n) \\ &= g(n, \psi(n-\tau(n))) \\ &+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\leq k_2 K + M \sum_{u=n}^{n+T-1} \left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\leq k_2 K + MT\frac{K(1-k_2)}{MT} = K. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\mathcal{B}\psi)(n) + (\mathcal{A}\varphi)(n) \\ &= g(n, \psi(n-\tau(n))) \\ &+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\geq k_1 L + m \sum_{u=n}^{n+T-1} \left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\geq k_1 L + mT\frac{L(1-k_1)}{mT} = L. \end{aligned}$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}x + \mathcal{B}x$. By Lemma 2.1 this fixed point is a solution of (1.1) and the proof is complete. □

Remark 3.3. When $g(n, x) = cx$, Theorem 3.1 reduces to Theorem 3.2 of [7].

In the case $g(n,x) \leq 0$, we substitute conditions (3.4)-(3.6) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq g(n,x) \leq k_4 x, \text{ for all } n \in [0, T-1] \cap \mathbb{Z},\ x \in \mathbb{D}, \tag{3.8}$$

$$-k_3 < 1, \tag{3.9}$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$\frac{L - k_3 K}{mT} \leq f(n,x) - [1 - a(n)]\, g(n,x) \leq \frac{K - k_4 L}{MT}. \tag{3.10}$$

Theorem 3.2. *Suppose (2.1)-(2.3), (3.3) and (3.7)-(3.10) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

The proof follows along the lines of Theorem 3.1, and hence we omit it.

Remark 3.4. When $g(n,x) = cx$, Theorem 3.2 reduces to Theorem 3.3 of [7].

Example 3.5. Consider the following nonlinear neutral difference equation

$$x(n+1) = a(n)\, x(n) + \triangle g(n, x(n - \tau(n))) + f(n, x(n - \tau(n))), \tag{3.11}$$

where

$$T = 4,\ \tau(n) = 5,\ a(n) = \frac{1}{5},\ g(n,x) = 0.8 \sin(x),$$

and

$$f(n,x) = \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.64 \sin(x) + 0.024.$$

Then Equation (3.11) has a positive 4-periodic solution x satisfying $0.004 \leq x \leq \frac{\pi}{2}$. To see this, we have $L = 0.004$, $K = \frac{\pi}{2}$. A simple calculation yields

$$k = 0.8,\ m = \frac{5}{224},\ M = \frac{225}{224},\ k_1 = \frac{2}{\pi},\ k_2 = 0.8.$$

Define the set $\mathbb{D} = \left\{ \varphi \in P_4 : 0.004 \leq \varphi(n) \leq \frac{\pi}{2},\ n \in [0,3] \cap \mathbb{Z} \right\}$. Then for $x \in \left[0.004, \frac{\pi}{2}\right]$ we have

$$\begin{aligned} f(n,x) - [1 - a(n)]\, g(n,x) &= \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.024 \\ &\leq 0.058 < 0.078 \simeq \frac{K(1 - k_2)}{MT}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(n,x) - [1 - a(n)]\, g(n,x) &= \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.024 \\ &\geq 0.024 > 0.016 \simeq \frac{L(1 - k_1)}{mT}. \end{aligned}$$

By Theorems 3.1, Equation (3.11) has a positive 4-periodic solution x such that $0.004 \leq x \leq \frac{\pi}{2}$.

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