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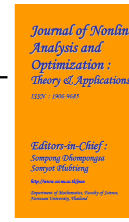

SUZUKI-TYPE FIXED POINT THEOREMS FOR TWO MAPS ON METRIC-TYPE SPACES

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ABSTRACT. In this paper, we generalize the Suzuki-type fixed point theorems in [N. Hussain, D. Dorić, Z. Kadelburg, and S. Radenović, *Suzuki-type fixed point results in metric type spaces*, Fixed Point Theory Appl **2012:126** (2012), 1 - 10] for two maps on metric-type spaces. Examples are given to validate the results.

KEYWORDS: Suzuki-type fixed point; metric-type space.
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1. INTRODUCTION AND PRELIMINARIES

In [2], Hussain, Dorić, Kadelburg and Radenović have proved the following theorems. These results are generalizations of Suzuki-type fixed point theorems in [8] and [9].

Theorem 1.1 ([2], Theorem 3). *Let (X, D, K) be a complete metric-type space, let $T : X \longrightarrow X$ be a map and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by*

$$\theta(r) = \theta_K(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r \le b_K \\ \frac{1}{K+r} & \text{if } b_K < r < 1 \end{cases}$$

where $b_K = \frac{1-K+\sqrt{1+6K+K^2}}{4}$ is the positive solution of $\frac{1-r}{r^2} = \frac{1}{K+r}$, satisfying the following conditions

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(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0,1)$ such that for each $x, y \in X$,*

$$\theta(r)D(x,Tx) \leq D(x,y) \text{ implies } D(Tx,Ty) \leq \frac{r}{K}M(x,y) \tag{1.1}$$

where

$$M(x,y) = \max\Big\{D(x,y), D(x,Tx), D(y,Ty), \frac{1}{2K}\big[D(x,Ty)+D(y,Tx)\big]\Big\}.$$

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{T^n x\}$ converges to z.*
(iii) *T has the property (P).*

Theorem 1.2 ([2], Theorem 4)**.** *Let (X, D, K) be a metric-type space and let $T : X \longrightarrow X$ be a map satisfying the following conditions*

(i) *X is compact.*
(ii) *D is continuous.*
(iii) *For all $x, y \in X$ and $x \neq y$,*

$$\frac{1}{1+K}D(x,Tx) < D(x,y) \text{ implies } D(Tx,Ty) < \frac{1}{K}D(x,y). \tag{1.2}$$

Then T has a unique fixed point in X.

In this paper, we extend the main results in [2] for two maps on metric-type spaces. Examples are given to validate the results.

First we recall some notions and lemmas which will be useful in what follows.

Definition 1.3 ([6], Definition 6)**.** Let X be a nonempty set, let $K \geq 1$ be a real number and let $D : X \times X \longrightarrow [0,\infty)$ satisfy the following properties

(i) $D(x,y) = 0$ if and only if $x = y$.
(ii) $D(x,y) = D(y,x)$ for all $x, y \in X$;
(iii) $D(x,z) \leq K\big[D(x,y) + D(y,z)\big]$ for all $x, y, z \in X$.

Then (X, D, K) is called a *metric-type space*.

Note that a metric-type space was introduced and studied under the name of a *b-metric space* by Czerwik in [1]. Moreover, in [5], Khamsi introduced another definition of a metric-type space with a bit difference, where the condition (3) in Definition 1.3 is replaced by

$D(x,z) \leq K\big[D(x,y_1) + \cdots + D(y_n,z)\big]$ for all $x, y_1, \cdots, y_n, z \in X$.

Definition 1.4 ([6], Definition 7)**.** Let (X, D, K) be a metric-type space.

(i) A sequence $\{x_n\}$ is called *convergent* to $x \in X$ if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.
(ii) A sequence $\{x_n\}$ is called *Cauchy* if $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$.
(iii) (X, D, K) is called *complete* if every Cauchy sequence is a convergent sequence.

Definition 1.5 ([3], page 2)**.** A map $T : X \to X$ is called to have the *property* (P) if $\mathcal{F}(T) = \mathcal{F}(T^n)$ for all $n \in \mathbb{N}$, where $\mathcal{F}(T) = \{x \in X : Tx = x\}$.

Definition 1.6 ([7], Definition 1.2)**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map. T is called *sequentially convergent* if $\{y_n\}$ is convergent provided $\{Ty_n\}$ is convergent.

Lemma 1.7 ([4], Lemma 3.1)**.** *Let $\{y_n\}$ be a sequence in a metric-type space (X, D, K) such that*

$$D(y_n, y_{n+1}) \le \lambda D(y_{n-1}, y_n) \tag{1.3}$$

for some $\lambda \in [0, \frac{1}{K})$ and all $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in (X, D, K).

2. MAIN RESULTS

The following result is a sufficient condition for a map on a metric-type space having the property (P). If $K = 1$, this result becomes [3, Theorem 1.1].

Lemma 2.1. *Let (X, D, K) be a metric-type space and $T : X \longrightarrow X$ be a map such that*

$$D(Tx, T^2x) \le \lambda D(x, Tx) \tag{2.1}$$

for some $0 \le \lambda < 1$ and all $x \in X$. Then T has property (P).

Proof. If $u \in \mathcal{F}(T^n)$, that is, $T^n u = u$, then from (2.1) we have

$$D(u, Tu) = D(TT^{n-1}u, T^2T^{n-1}u) \le \lambda D(T^{n-1}u, TT^{n-1}u) \le \cdots \le \lambda^n D(u, Tu).$$

Since $0 \le \lambda^n < 1$, we get $D(u, Tu) = 0$, that is, $u \in \mathcal{F}(T)$.

If $u \in \mathcal{F}(T)$, that is $Tu = u$, then

$$D(u, T^n u) = D(u, T^{n-1}u) = \ldots = D(u, Tu) = 0.$$

Then $T^n u = u$, that is $u \in \mathcal{F}(T^n)$. This proves that T has property (P). □

The first main result of the paper is as follows.

Theorem 2.2. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by*

$$\theta(r) = \theta_K(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r \le b_K \\ \frac{1}{K+r} & \text{if } b_K < r < 1 \end{cases} \tag{2.2}$$

where $b_K = \frac{1 - K + \sqrt{1 + 6K + K^2}}{4}$ is the positive solution of $\frac{1-r}{r^2} = \frac{1}{K+r}$, satisfying the following conditions

- (i) *D is continuous in each variable.*
- (ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$*

$$\theta(r)D(Fx, FTx) \le D(Fx, Fy) \text{ implies } D(FTx, FTy) \le \frac{r}{K}M(x, y) \tag{2.3}$$

where

$$M(x, y) = \max\left\{D(Fx, Fy), D(Fx, FTx), D(Fy, FTy), \frac{1}{2K}\big[D(Fx, FTy) + D(Fy, FTx)\big]\right\}.$$

- (iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

- (i) *T has a unique fixed point $a \in X$.*
- (ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fa.*
- (iii) *If $TF = FT$, then T has the property (P) and F, T have a unique common fixed point.*

Proof. (1). For each $x \in X$, since $\theta(r) \le 1$, we have $\theta(r)D(Fx, FTx) \le D(Fx, FTx)$. It follows from (2.3) that

$$
\begin{aligned}
& D(FTx, FT^2x) \qquad (2.4)\\
\le\ & \frac{r}{K}\max\Big\{D(Fx, FTx), D(Fx, FTx), D(FTx, FT^2x),\\
& \frac{1}{2K}\big[D(Fx, FT^2x) + D(FTx, FTx)\big]\Big\}\\
\le\ & \frac{r}{K}\max\Big\{D(Fx, FTx), D(FTx, FT^2x), \frac{1}{2K}K\big[D(Fx, FTx) + D(FTx, FT^2x)\big]\Big\}\\
=\ & \frac{r}{K}\max\Big\{D(Fx, FTx), D(FTx, FT^2x)\Big\}.
\end{aligned}
$$

We consider following two cases.

Case 1. $\max\Big\{D(Fx, FTx), D(FTx, FT^2x)\Big\} = D(FTx, FT^2x)$. Then (2.4) becomes $D(FTx, FT^2x) \le \frac{r}{K}D(FTx, FT^2x)$. Since $\frac{r}{K} < 1$, we have

$$D(FTx, FT^2x) = 0 \qquad (2.5)$$

that is $FTx = FT^2x$. Note that F is one-to-one, then $Tx = T^2x$. Therefore, $a = Tx$ is a fixed point of T.

Case 2. $\max\Big\{D(Fx, FTx), D(FTx, FT^2x)\Big\} = D(Fx, FTx)$. Then (2.4) becomes

$$D(FTx, FT^2x) \le \frac{r}{K}D(Fx, FTx). \qquad (2.6)$$

Put $x_{n+1} = Tx_n$ and $y_n = FTx_n$ for all $n \in \mathbb{N}$ where $x_0 = x$. We also have $x_n = T^n x$ and $y_n = Fx_{n+1}$. It follows from (2.6) that

$$D(y_n, y_{n+1}) = D(FTx_n, FT^2x_n) \le \frac{r}{K}D(Fx_n, FTx_n) = \frac{r}{K}D(y_{n-1}, y_n). \qquad (2.7)$$

Using Lemma 1.7, we conclude that $\{y_n\}$ is a Cauchy sequence in the compete metric-type space X. Then y_n converges to z for some $z \in X$. Since F is sequentially convergent, $\{x_n\}$ converges to some $a \in X$ and also from the continuity of F, $\{Fx_n\}$ converges to Fa. Note that $\{y_{n-1}\}$ converges to z, then

$$y_{n-1} = FTx_{n-1} = Fx_n \to Fa = z. \qquad (2.8)$$

Let us prove now that

$$D(FTx, z) \le \frac{r}{K}\max\Big\{D(Fx, z), D(Fx, FTx)\Big\} \qquad (2.9)$$

holds for each $x \ne a$. Indeed, since $Fx_n \to z$ and $FTx_n \to z$ and by the continuity of D, we have

$$D(Fx_n, FTx_n) \to 0 \text{ and } D(Fx_n, Fx) \to D(z, Fx) \ne 0. \qquad (2.10)$$

Then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\theta(r)D(Fx_n, FTx_n) < D(Fx_n, Fx). \qquad (2.11)$$

From (2.3) and (2.11), we have for such n

$$
\begin{aligned}
D(FTx_n, FTx) \le\ & \frac{r}{K}\max\Big\{D(Fx_n, Fx), D(Fx_n, FTx_n), D(Fx, FTx), \qquad (2.12)\\
& \frac{1}{2K}\big[D(Fx_n, FTx) + D(Fx, FTx_n)\big]\Big\}.
\end{aligned}
$$

Taking the limit as $n \to \infty$ in (2.12) and using (2.10) and the continuity of D, we get

$$\begin{aligned} & D(z, FTx) \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx), \frac{1}{2K}\big(D(z,FTx)+D(Fx,z)\big)\Big\} \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx), \frac{1}{2K}K\big(D(z,Fx)+D(Fx,FTx)\big)+\frac{1}{2K}D(Fx,z)\Big\} \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx)\Big\}. \end{aligned}$$

Hence, we have (2.9) .

For each $n \geq 1$, put $x = T^{n-1}a$. Therefore,

$$D(FT^n a, FT^{n+1}a) \leq \frac{r}{K} D(FT^{n-1}a, FT^n a)$$

holds for each $n \in \mathbb{N}$ where $FT^0 a = z$. By induction, we have

$$D(FT^n a, FT^{n+1}a) \leq \frac{r^n}{K^n} D(z, FTa). \tag{2.13}$$

Now we will prove that

$$D(FT^n a, z) \leq D(FTa, z) \tag{2.14}$$

holds for all $n \geq 1$ by induction. For $n = 1$ this relation is obvious. Suppose that it holds for some n. If $FT^n a = z$, note that $z = Fa$ and F is one-to-one, then $T^n a = a$. It implies that $FT^{n+1}a = FTa$ and $D(FT^{n+1}a, z) = D(FTa, z)$. If $FT^n a \neq z$, then from (2.9), (2.13) and the induction hypothesis, we get

$$\begin{aligned} D(FT^{n+1}a, z) &\leq \frac{r}{K}\max\Big\{D(FT^n a, z), D(FT^n a, FT^{n+1}a)\Big\} \\ &\leq \frac{r}{K}\max\Big\{D(FTa, z), \frac{r^n}{K^n}D(z, FTa)\Big\} \\ &\leq \frac{r}{K}D(FTa, z) \end{aligned}$$

and that (2.14) is proved.

Now we will prove that a is a fixed point of T. Suppose to the contrary that $Ta \neq a$, that is, $FTa \neq Fa$ or equivalently,

$$FTa \neq z. \tag{2.15}$$

We consider following two subcases.

Subcase 2.1. $0 \leq r < b_K$. That implies $\theta(r) \leq \dfrac{1-r}{r^2}$.
We will prove

$$D(FT^n a, FTa) \leq \frac{r}{K} D(FTa, z) \tag{2.16}$$

holds for all $n \geq 1$ by induction. For $n = 1$, (2.16) obvious and for $n = 2$, (2.16) follows from (2.13). Suppose that (2.16) holds for some $n > 2$. Then we have

$$D(z, FTa) \leq K\big[D(z, FT^n a) + D(FT^n a, FTa)\big] \leq K\big[D(z, FT^n a) + \frac{r}{K}D(FTa, z)\big].$$

Hence

$$D(z, FTa) \ \leq\ \frac{K}{1-r} D(z, FT^n a). \tag{2.17}$$

Since $\theta(r) \le \dfrac{1-r}{r^2}$ and by using (2.8), (2.13) and (2.17), we get

$$\begin{aligned}
\theta(r)D(FT^n a, FT^{n+1}a) &\le \frac{1-r}{r^2}D(FT^n a, FT^{n+1}a)\\
&\le \frac{1-r}{r^n}D(FT^n a, FT^{n+1}a)\\
&\le \frac{1-r}{K^n}D(z, FTa)\\
&\le \frac{1}{K^{n-1}}D(z, FT^n a)\\
&\le D(z, FT^n a)\\
&= D(Fa, FT^n a).
\end{aligned}$$

Assumption (2.3) implies that

$$\begin{aligned}
D(FTa, FT^{n+1}a) \le \frac{r}{K}\max\Big\{&D(Fa, FT^n a), D(Fa, FTa), D(FT^n a, FT^{n+1}a),\\
&\frac{1}{2K}\big(D(Fa, FT^{n+1}a) + D(FT^n a, FTa)\big)\Big\}.
\end{aligned}$$

Using (2.13), (2.14) and the induction hypothesis, we obtain the last maximum is equal to $D(FTa, z)$. That is $D(FTa, FT^{n+1}a) \le \dfrac{r}{K}D(FTa, z)$ and (2.16) is proved by induction.

From (2.15), we have $FT^n a \ne z$ for each $n \in \mathbb{N}$. If $FT^n a = z$ for some $n \in \mathbb{N}$, then from (2.16) we get $D(z, FTa) = 0$. It is a contradiction with (2.15). So $FT^n a \ne z$ for each $n \in \mathbb{N}$. Hence, (2.9) and (2.13) imply that

$$\begin{aligned}
D(FT^{n+1}a, z) &\le \frac{r}{K}\max\Big\{D(FT^n a, z), D(FT^n a, FT^{n+1}a)\Big\} \qquad (2.18)\\
&\le \frac{r}{K}\max\Big\{D(FT^n a, z), \frac{r^n}{K^n}D(z, FTa)\Big\}.
\end{aligned}$$

Since $D(FTa, z) \le K\big[D(FTa, FT^n a) + D(FT^n a, z)\big]$, it follows from (2.16) that

$$D(FT^n a, z) \ge \frac{1}{K}D(FTa, z) - D(FTa, FT^n a) \ge \frac{1-r}{K}D(FTa, z).$$

Note that there exists $n_1 \in \mathbb{N}$ such that $1 - r \ge r^n$ for all $n \ge n_1$ and $0 \le r \le b_K$. For $n \ge n_1$, we have

$$D(FT^n a, z) \ge \frac{r^n}{K}D(FTa, z) \ge \frac{r^n}{K^n}D(FTa, z).$$

Using (2.18), we have

$$0 \le D(FT^{n+1}a, z) \le \frac{r}{K}D(FT^n a, z) \le \cdots \le \left(\frac{r}{K}\right)^{n-n_1+1} D(FT^{n_1}a, z). \qquad (2.19)$$

Taking the limit as $n \to \infty$ in (2.19), we get $FT^n a \to z$ and let again $n \to \infty$ in (2.16), we get $D(FTa, z) \le \dfrac{r}{K}D(FTa, z)$ that means $D(FTa, z) = 0$. Therefore, $FTa = z$. It is a contradiction with (2.15).

Subcase 2.2. $b_K \le r < 1$. That implies $\theta(r) = \dfrac{1}{K+r}$. We will prove there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\theta(r)D(Fx_{n_j+1}, FTx_{n_j+1}) = \theta(r)D(y_{n_j}, y_{n_j+1}) \le D(y_{n_j}, z) \qquad (2.20)$$

holds for each $j \in \mathbb{N}$. If

$$\frac{1}{K+r}D(y_{n-1}, y_n) > D(y_{n-1}, z) \text{ and } \frac{1}{K+r}D(y_n, y_{n+1}) > D(y_n, z)$$

hold for some $n \in \mathbb{N}$, then (2.7) we have

$$\begin{aligned} D(y_{n-1}, y_n) &\leq K\big[D(y_{n-1}, z) + D(z, y_n)\big] \\ &< \frac{K}{K+r}\big[D(y_{n-1}, y_n) + D(y_n, y_{n+1})\big] \\ &\leq \frac{K}{K+r}\big[D(y_{n-1}, y_n) + \frac{r}{K}D(y_{n-1}, y_n)\big] \\ &= D(y_{n-1}, y_n). \end{aligned}$$

It is impossible. Hence

$$\theta(r)D(y_{n-1}, y_n) \leq D(y_{n-1}, z) \text{ or } \theta(r)D(y_n, y_{n+1}) \leq D(y_n, z)$$

holds for some $n \in \mathbb{N}$. In particular

$$\theta(r)D(y_{2n-1}, y_{2n}) \leq D(y_{2n-1}, z) \text{ or } \theta(r)D(y_{2n}, y_{2n+1}) \leq D(y_{2n}, z)$$

holds for all $n \in \mathbb{N}$. In other words there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that satisfies (2.20) for each $j \in \mathbb{N}$. But the assumption (2.3) implies that

$$\begin{aligned} & D(FTx_{n_j+1}, FTa) \\ \leq\ & \frac{r}{K}.\max\Big\{D(Fx_{n_j+1}, Fa), D(Fx_{n_j+1}, FTx_{n_j+1}), D(Fa, FTa), \\ & \frac{r}{2K}\big[D(Fx_{n_j+1}, FTa) + D(Fa, FTx_{n_j+1})\big]\Big\}. \end{aligned} \tag{2.21}$$

Taking the limit as $j \to \infty$ in (2.21), we obtain

$$D(z, FTa) \leq \frac{r}{K}.D(Fa, FTa) = \frac{r}{K}D(z, FTa).$$

It implies $D(z, FTa) = 0$, that is $z = FTa$. It is a contradiction with (2.15).

From two above subcases, we get $Ta = a$, that is a is a fixed point of T.

Finally, we prove that a is a unique fixed point of T. Indeed, if a and b are two fixed points of T, then (2.9) implies that

$$D(Fa, Fb) = D(FTa, Fb) \leq \frac{r}{K}\max\Big\{D(Fa, Fb), D(Fa, FTa)\Big\} = \frac{r}{K}D(Fa, Fb).$$

Since $\frac{r}{K} < 1$, we have $D(Fa, Fb) = 0$, that is $Fa = Fb$. Also since F is one-to-one, we get $a = b$.

(2). It is a direct consequence of (2.8).

(3). From (2.5) and (2.6), we have

$$D(FTx, FT^2x) \leq \frac{r}{K}D(Fx, FTx). \tag{2.22}$$

Note that the property (P) follows from (2.22) and Lemma 2.1. We need only prove T and F have a unique common fixed point. Let a be the unique fixed point of T. Suppose to the contrary that $Fa \neq a$. Since F is one-to-one, $F^2a \neq Fa$. Then

$$\theta(r)D(Fa, FTa) = 0 < D(Fa, F^2a).$$

It follows from (2.3) that

$$D(FTa, FTFa) = D(FTa, F^2Ta) = D(Fa, F^2a) \leq \frac{r}{K} M(a, Fa)$$

where

$$\begin{aligned} & M(a, Fa) \\ = \quad & \max\Big\{D(Fa, F^2a), D(Fa, FTa), D(F^2a, F^2Ta), \frac{1}{2K}\big[D(Fa, F^2Ta) + D(F^2a, FTa)\big]\Big\} \\ = \quad & D(Fa, F^2a). \end{aligned}$$

Therefore,

$$D(Fa, F^2a) \leq \frac{r}{K} D(Fa, F^2a) < D(Fa, F^2a).$$

It is a contradiction. This proves that a is a unique common fixed point of T and F. □

Remark 2.3. By choosing F is the identity in Theorem 2.2, we get Theorem 1.1.

From Theorem 2.2, we get following corollaries.

Corollary 2.4. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \Big(\dfrac{1}{K+1}, 1\Big]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy) \text{ implies } D(FTx, FTy) \leq \frac{r}{K} D(Fx, Fy). \tag{2.23}$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Corollary 2.5. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \Big(\dfrac{1}{K+1}, 1\Big]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy)$$

$$\text{implies } D(FTx, FTy) \leq \frac{r}{K} \max\Big\{D(Fx, FTx), D(Fy, FTy)\Big\}. \tag{2.24}$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Corollary 2.6. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy)$$

$$\text{implies } D(FTx, FTy) \leq \frac{r}{2K}\big[D(Fx, FTy) + D(Fy, FTx)\big]. \tag{2.25}$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Remark 2.7. Corollary 2.4 is a generalization of [2, Corollary 1], Corollary 2.5 is a generalization of [2, Corollary 2] and Corollary 2.6 is a generalization of [2, Corollary 3].

The second main result of the paper is as follows.

Theorem 2.8. *Let (X, D, K) be a metric-type space where D is continuous and let $T, F : X \longrightarrow X$ be two maps satisfying the conditions*

(i) *For all $x, y \in X$ and $x \neq y$,*

$$\frac{1}{1+K}D(Fx, FTx) < D(Fx, Fy) \text{ implies } D(FTx, FTy) < \frac{1}{K}D(Fx, Fy). \tag{2.26}$$

(ii) *$F(X)$ is compact.*
(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point in X.*
(ii) *If $TF = FT$ then F, T have a unique common fixed point.*

Proof. (1). First, denote $\beta = \inf\{D(Fx, FTx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X such that $D(Fx_n, FTx_n) \to \beta$. Since $F(X)$ is compact, so there exist $Fv, Fw \in F(X)$ such that $Fx_n \to Fv$ and $FTx_n \to Fw$. Since F is continuous, one-to-one and sequentially convergent, we get $x_n \to v$ and $Tx_n \to w$. Note that the continuity of D implies

$$\lim D(Fx_n, Fw) = \lim D(Fv, Fw) = \lim D(Fx_n, FTx_n) = \beta.$$

We will prove $\beta = 0$. Suppose to the contrary that $\beta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\frac{2+K}{2+2K}\beta < D(Fx_n, Fw) \text{ and } D(Fx_n, FTx_n) < \frac{2+K}{2}\beta.$$

Then $\frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, Fw)$ and the assumption (2.26) implies that

$$D(FTx_n, FTw) < \frac{1}{K}D(Fx_n, Fw). \tag{2.27}$$

Taking the limit as $n \to \infty$ in (2.27), we obtain $D(Fw, FTw) \leq \frac{1}{K}\beta$.

If $K > 1$, then $D(Fw, FTw) < \beta$. It is impossible by the definition of β.

If $K = 1$, then $D(Fw, FTw) = \beta$ and

$$\frac{1}{1+K}D(Fw, FTw) < D(Fw, FTw).$$

It follows from (2.26) that

$$D(FTw, FT^2w) < \frac{1}{K}D(Fw, FTw) = \beta.$$

It is also impossible by the definition of β.

Hence, in all cases we obtain a contradiction and it follows that $\beta = 0$ and so $Fv = Fw$. Since F is one-to-one, we have $v = w$.

Now we prove that T has a fixed point. Suppose to the contrary that $Tz \neq z$ for all $z \in X$. Since F is one-to-one, we have $FTz \neq Fz$ for all $z \in X$. In particular, we get

$$0 < \frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, FTx_n).$$

It follows from (2.26) that $D(FTx_n, FT^2x_n) < \frac{1}{K}D(Fx_n, FTx_n)$. Therefore,

$$\begin{aligned} D(Fv, FT^2x_n) &\leq K\big[D(Fv, FTx_n) + D(FTx_n, FT^2x_n)\big] \\ &< KD(Fv, FTx_n) + D(Fx_n, FTx_n). \end{aligned} \tag{2.28}$$

Taking the limit as $n \to \infty$ in (2.28), we get $D(Fv, FT^2x_n) \to 0$, that is, $FT^2x_n \to Fv$.

Suppose that

$$\frac{1}{1+K}D(Fx_n, FTx_n) \geq D(Fx_n, Fv)$$

and

$$\frac{1}{1+K}D(FTx_n, FT^2x_n) \geq D(FTx_n, Fv)$$

both hold for some $n \in \mathbb{N}$. Then

$$\begin{aligned} D(Fx_n, FTx_n) &\leq K\big[D(Fx_n, Fv) + D(FTx_n, Fv)\big] \\ &\leq \frac{K}{1+K}\big[D(Fx_n, FTx_n) + D(FTx_n, FT^2x_n)\big] \\ &< \frac{K}{1+K}\big[D(Fx_n, FTx_n) + \frac{1}{K}.D(Fx_n, FTx_n)\big] \\ &= D(Fx_n, FTx_n). \end{aligned}$$

That is impossible. Thus, for each $n \in \mathbb{N}$, either

$$\frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, Fv)$$

or

$$\frac{1}{1+K}D(FTx_n, FT^2x_n) < D(FTx_n, Fv)$$

holds. It follows from (2.26) that, for each $n \in \mathbb{N}$, either

$$D(FTx_n, FTv) < \frac{1}{K}D(Fx_n, Fv) \tag{2.29}$$

or

$$D(FT^2x_n, FTv) < \frac{1}{K}D(FTx_n, Fv) \tag{2.30}$$

holds. If (2.29) holds only for finitely many $n \in \mathbb{N}$, then (2.32) holds for infinitely many $n \in \mathbb{N}$. Thus, there exists a sequence $\{n_k\}$ such that

$$D(FT^2x_{n_k}, FTv) < \frac{1}{K}D(FTx_{n_k}, Fv) \tag{2.31}$$

holds for each $k \in \mathbb{N}$. If (2.29) holds for infinitely many $n \in \mathbb{N}$, then there exists a sequence $\{n_j\}$ such that

$$D(FTx_{n_j}, FTv) < \frac{1}{K}D(Fx_{n_j}, Fv) \tag{2.32}$$

holds for each $j \in \mathbb{N}$.

In both cases, taking the limit as $k \to \infty$ in (2.31) or $j \to \infty$ in (2.32), we obtain $D(Fv, FTv) = 0$, that is, $Fv = FTv$. Since F is one-to-one, we get $v = Tv$. This is a contradiction with the assumption that T has no any fixed point.

Finally, we prove the uniqueness of the fixed point. Suppose to the contrary that y, z are two fixed points of T and $z \neq y$. Then $Fz = FTz$ and $Fy \neq Fz$. Therefore,

$$\frac{1}{1+K}D(Fz, FTz) < D(Fz, Fy)$$

and (2.26) implies that

$$D(FTz, FTy) < \frac{1}{K}D(Fz, Fy) = \frac{1}{K} \cdot D(FTz, FTy).$$

This is impossible since $K \geq 1$. Thus T has a unique fixed point in X.

(2). Let v be the unique fixed point of T. Suppose to the contrary that $Fv \neq v$. Since F is one-to-one, $F^2v \neq Fv$. Then

$$\frac{1}{1+K}D(Fv, FTv) = 0 < D(Fv, F^2v).$$

It follows from (2.26) that

$$D(FTv, FTFv) = D(FTv, F^2Tv) = D(Fv, F^2v) < \frac{1}{K}D(Fv, F^2v) \leq D(Fv, F^2v).$$

It is a contradiction. This proves that v is a unique common fixed point of T and F. □

The following example shows that Theorem 2.2 is a proper generalization of Theorem 1.1.

Example 2.9. Let $X = [0, +\infty)$, let D be the usual metric on $\mathbb{R}$, that is $K = 1$, and let T, F be defined by

$$Tx = \frac{x^2}{x+1}, Fx = \mathrm{e}^x - 1$$

for all $x \in X$. We have

$$\begin{aligned} D(Tx, T2x) &= \frac{x^2(2x+3)}{(2x+1)(x+1)} \\ D(x, 2x) &= x \\ D(x, Tx) &= \frac{x}{x+1} \\ D(2x, T2x) &= \frac{2x}{2x+1} \\ D(x, T2x) &= \left|\frac{2x^2 - x}{2x+1}\right| \end{aligned}$$

$$D(2x, Tx) = \frac{x^2+2x}{x+1}.$$

Let the condition (1.2) hold. Since

$$\theta(r).D(x,Tx) = \theta(r)\frac{x}{x+1} \le \frac{x}{x+1} \le x = D(x,2x)$$

for all $x \in X$, then

$$D(Tx, T2x) \le rM(x, 2x)$$

where

$$M(x,2x) = \max\Big\{x, \frac{x}{x+1}, \frac{2x}{2x+1}, \frac{1}{2}\Big(\Big|\frac{2x^2-x}{2x+1}\Big| + \frac{x^2+2x}{x+1}\Big)\Big\} \le \frac{x^2+2x}{x+1}.$$

Then we have

$$\frac{x^2(2x+3)}{(2x+1)(x+1)} \le r\frac{x^2+2x}{x+1}$$

that is

$$\frac{x(2x+3)}{(2x+1)(x+1)} \le r\frac{x+2}{x+1}$$

for all $x \in X$. Taking the limit as $x \to +\infty$, we get $r \ge 1$. It is a contradiction. This proves that Theorem 1.1 is not applicable to T.

On the other hand, we have

$$D(FTx, FTy) = \Big|e^{\frac{x^2}{x+1}} - e^{\frac{y^2}{y+1}}\Big|$$
$$D(Fx, Fy) = |e^x - e^y|.$$

We consider two following cases.

Case 1. $x \ge y$. Then $D(FTx, FTy) \le \frac{1}{2}D(Fx, Fy)$ is equivalent to

$$2e^{\frac{x^2}{x+1}} - e^x \le 2e^{\frac{y^2}{y+1}} - e^y.$$

Now we shall prove that $\varphi(x) = 2e^{\frac{x^2}{x+1}} - e^x$ is decreasing on $[0, +\infty)$. Indeed, we have

$$\varphi'(x) = e^x\Big(2\frac{x^2+2x}{(x+1)^2}e^{\frac{-x}{x+1}} - 1\Big).$$

Note that $\psi(x) = 2\frac{x^2+2x}{(x+1)^2}e^{\frac{-x}{x+1}} - 1$ satisfies $\psi'(x) = e^{\frac{-x}{x+1}}\frac{4-2x^2}{(x+1)^4}$. It implies that

$$\max_{[0,+\infty)} \psi(x) = \psi(\sqrt{2}) < 0.$$

Therefore, $\varphi'(x) < 0$ on $[0, +\infty)$. This proves that $\varphi(x)$ is decreasing. Then we have

$$D(FTx, FTy) < \frac{1}{2}D(Fx, Fy) \tag{2.33}$$

for all $x, y \in X$. This proves that (2.23) holds with $r = \frac{1}{2}$.

Case 2. $x < y$. Then $D(FTx, FTy) \le \frac{1}{2}D(Fx, Fy)$ is equivalent to

$$2e^{\frac{y^2}{y+1}} - e^y \le 2e^{\frac{x^2}{x+1}} - e^x.$$

As the same as Case 1, we also get that (2.23) holds with $r = \frac{1}{2}$.

By two above cases, we see that (2.23) holds with $r = \frac{1}{2}$. Note that other conditions in Corollary 2.4 are also satisfied, then Corollary 2.4 is applicable to T and F. We see that $x = 0$ is the unique fixed point of T.

The following example shows that Corollary 2.4 is a proper generalization of [2, Corollary 1].

Example 2.10. For X and F, T as in Example 2.9, we have

$$D(Tx, T2x) = \frac{x^2(2x+3)}{(2x+1)(x+1)}, D(x, 2x) = x.$$

If the condition in [2, Corollary 1] holds, then $\frac{x^2(2x+3)}{(2x+1)(x+1)} \leq r.x$, that is

$$\frac{x(2x+3)}{(2x+1)(x+1)} \leq r \tag{2.34}$$

for all $x \in X$. Taking the limit as $x \to +\infty$ in (2.34), we get $r \geq 1$. It is a contradiction. This proves that [2, Corollary 1] is not applicable to T. As in Example 2.9, Corollary 2.4 is applicable to F and T. Note that $x = 0$ is the unique fixed point of T.

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