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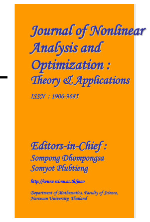

EXPANDING THE APPLICABILITY OF A TWO STEP NEWTON LAVRENTIEV METHOD FOR ILL-POSED PROBLEMS

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ABSTRACT. In [3] we presented a cubically convergent Two Step Directional Newton Method (TSDNM) for approximating a solution of an operator equation in a Hilbert space setting. George and Pareth in [13] use the analogous Two Step Newton Lavrentiev Method (TSNLM) to approximate a solution of an ill-posed equation. In the present paper we show how to expand the applicability of (TSNLM). In particular, we present a semilocal convergence analysis of (TSNLM) under: weaker hypotheses, weaker convergence criteria, tighter error estimates on the distances involved and an at least as precise information on the location of the solution.

KEYWORDS: Newton-Lavrentiev regularization method; ill-posed problem; Hilbert space; semilocal convergence.
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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution of the nonlinear ill-posed operator equation

$$F(x) = f, \tag{1.1}$$

where F is a Fréchet differentiable operator defined on an open and convex subset $D(F)$ of a real Hilbert space X and $f \in X$. Let $\langle ., . \rangle$ and $\|.\|$, stand, respectively for the inner product and the corresponding norm. Let also $U(x, r)$ and $\overline{U(x, r)}$, stand, respectively for the open and closed balls in X with center $x \in X$ and radius $r > 0$. Let $L(X)$ denote the space of bounded linear operators from X into X. We suppose that F is a monotone operator. That is

$$\langle F(x) - F(y), x - y \rangle \geq 0, \qquad \forall x, y \in D(F).$$

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Set $S = \{x : F(x) = f\}$. It is known that S is closed and convex provided that F is monotone and continuous (see, e.g., [28]). It follows that there exists a unique element of minimal norm. Denoted such an element by $\hat{x}$. Then we have $F(\hat{x}) = f$. Suppose that $F'(.)$ is a positive self adjoint operator. Then, $(F'(.) + \alpha I)^{-1} \in L(X)$ for each $\alpha > 0$. Here, we need $\sigma(F'(.)) \subseteq [0, \infty)$ and $\|(F'(.) + \alpha I)^{-1}\| \leq \frac{c}{\alpha}$ for some constant $c > 0$ and for any $\alpha > 0$. Such conditions are weaker than the self adjointness of $F'(.)$. In practice, only noisy data f^δ is available, such that $\|f - f^\delta\| \leq \delta$. Hence, the problem of computing $\hat{x}$ from equation $F(x) = f^\delta$ is ill-posed. Since a small perturbation in the data can cause large deviation in the solution. The Lavrentiev regularization method has been used to solve (1.1) by obtaining an approximation x_α^δ of equation

$$F(x) + \alpha(x - x_0) = f^\delta, \tag{1.2}$$

where $\alpha > 0$ is the regularization parameter and $x_0 \in D(F)$ is an initial point which is an approximation to $\hat{x}$ [11] ,[13], [15], [29, 30]. If $\alpha > 0$ is chosen appropriately then, it is known that $x_\alpha^\delta \to \hat{x}$ as $\alpha \to 0$ and $\delta \to 0$ [27], [30].

Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modelling [1], [4], [8], [20], [25], [26]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

In [3], we introduced a third order Two Step Directional Newton Method (TSDNM) for approximating a zero x^* of a differentiable operator F on a Banach space setting. Motivated by (TSDNM) George and Pareth [13] used the analogous Two Step Newton Lavrentiev Method (TSNLM)

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0)] \tag{1.3}$$

and

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[F(y_{n,\alpha}^\delta) - f^\delta + \alpha(y_{n,\alpha}^\delta - x_0)] \tag{1.4}$$

to generate a cubically convergent iteration $\{x_{n,\alpha}^\delta\}$ approximating x_α^δ, where $x_{0,\alpha}^\delta = x_0, \alpha > \alpha_0 > 0$ and $R_\alpha(x) = F'(x) + \alpha I$ [13].

Note that we have

$$\|R_\alpha(x)^{-1}F'(x)\| \leq 1, \quad \forall x \in D(F). \tag{1.5}$$

The semilocal convergence analysis was based on the following conditions which has been used extensively in the study of iterative procedures for solving ill-posed problems [11], [16], [29].

(C1) There exists a constant $L > 0$ such that for each $x, u \in D(F)$ and $v \in X$, there exists an element $P(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)P(x, u, v), \qquad \|P(x, u, v)\| \leq L\|v\|\|x - u\|.$$

They used the additional restriction that

$$0 < L \leq 1. \tag{1.6}$$

In the present paper, we extend the convergence domain of (TSNLM) and also drop restrictive condition (1.6) under weaker sufficient semilocal convergence criteria.

Moreover, the upper bounds on the distances $\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\|, \|x^\delta_{n,\alpha} - x^\delta_\alpha\|$ are tighter and the information on the location of the solution x^δ_α at least as precise (see Section 3).

There are cases when Lipschitz-type condition $(C1)$ is violated (see Section 4) but the following weaker central-Lipschitz-type condition is satisfied:

(C1)' Let $x_0 \in D(F)$ be fixed. There exists a constant $L_0 > 0, r > 0$ such that for each $x, u \in U(x_0, r) \cup U(\hat{x}, r) \subseteq D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ such that

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq L_0\|v\|(\|x_0 - u\| + \|x - x_0\|).$$

We note that since $\|u - x\| \leq \|u - x_0\| + \|x - x_0\|$ condition $(C1)$ always implies $(C1)'$ with $L_0 = L$ and $\Phi = P$ but not necessarily vice versa. Note that, under $(C1)'$ we also have the following special case:

(C1)" Let $x_0 \in D(F)$ be fixed. There exists a constant $l_0 > 0$ such that for all $w_\theta = x_0 + \theta(\hat{x} - x_0) \in D(F)$ and $v \in X$, there exists an element $\Phi(x_0, w_\theta, v) \in X$ such that

$$[F'(x_0) - F'(w_\theta)]v = F'(x_0)\Phi(x_0, w_\theta, v), \qquad \|\Phi(x_0, w_\theta, v)\| \leq l_0\|v\|\|x_0 - w_\theta\|.$$

Note that $l_0 \leq L_0 \leq L$ hold in general and $\frac{L_0}{l_0}$ and $\frac{L}{L_0}$ can be arbitrarily large [1]-[5].

In section 2 we provide a semilocal convergence analysis for (TSNLM) using $(C1)'$ instead of $(C1)$. We shall refer to [3], [13] for some of the proofs omitted in this study.

2. Semilocal convergence of (TSNLM) under $(C1)'$

We present the semilocal convergence of (TSNLM) using $(C1)'$. We need to introduce some sequences and parameters:

$$e^\delta_{n,\alpha} := \|y^\delta_{n,\alpha} - x^\delta_{n,\alpha}\|, \qquad \forall n = 0, 1, \cdots, \tag{2.1}$$

for $\delta_0 < (17 - 12\sqrt{2})\alpha_0$ for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$,

$$\rho \leq \frac{\sqrt{1 + 2l_0(17 - 12\sqrt{2} - \frac{\delta_0}{\alpha_0})} - 1}{l_0} = \rho_0. \tag{2.2}$$

Let

$$b_\rho = \frac{l_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0}, \tag{2.3}$$

$$r = \frac{1}{L_0}\frac{2b_\rho}{1 - b_\rho + \sqrt{(1 - b_\rho)^2 - 32b_\rho}}, \tag{2.4}$$

$$\gamma_\rho = \frac{L}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0}, \tag{2.5}$$

and

$$p = 2L_0 r, q = 2p^2. \tag{2.6}$$

Note that r is well defined, since $\frac{p}{2} < 1, q \in (0, 1)$ and $b_\rho \in (0, 17 - 12\sqrt{2}]$. We also have that

$$b_\rho \leq \gamma_\rho \tag{2.7}$$

and

$$\frac{1 + L_0 r}{1 - 8L_0^2 r^2} b_\rho = \frac{1 + \frac{p}{2}}{1 - q} b_\rho = L_0 r. \tag{2.8}$$

Estimate (2.7) holds as strict inequality if $l_0 < L$. Parameter γ_ρ was used in [13]. In order for us to simplify the notation, let x_n, y_n and e_n, stand, respectively for $x^\delta_{n,\alpha}, y^\delta_{n,\alpha}$ and $e^\delta_{n,\alpha}$. If we simply use the needed $(C1)''$ instead of $(C1)$ we arrive at:

LEMMA 2.1. Suppose that $(C1)''$ holds and b_ρ is given by (2.3). Then, the following assertion holds

$$e_0 \le b_\rho < 17 - 12\sqrt{2} = 0.029437252\cdots .$$

Proof. Using (2.1), (2.2), (2.3) and $(C1)''$ we obtain in turn that

$$\begin{aligned}
e_0 = \|y_0 - x_0\| &= \|R_\alpha(x_0)^{-1}(F(x_0) - f^\delta)\| \\
&= \|R_\alpha(x_0)^{-1}[F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha(x_0)^{-1}[\int_0^1 (F'(x_0 + t(\hat{x} - x_0)) - F'(x_0))dt(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&\le \|\int_0^1 \Phi(x_0 + t(\hat{x} - x_0), x_0, x_0 - \hat{x})\| + \|x_0 - \hat{x}\| \\
&\quad + \|R_\alpha(x_0)^{-1}(F(\hat{x}) - f^\delta)\| \\
&\le \frac{l_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + \frac{1}{\alpha}\|F(\hat{x}) - f^\delta\| \\
&\le \frac{l_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \\
&\le \frac{l_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0} = b_\rho \le 17 - 12\sqrt{2}.
\end{aligned}$$

The proof of the Lemma is complete.

REMARK 2.2. If $l_0 = L$ Lemma 2.1 reduces to Lemma 11 in [13]. Otherwise, i.e., if $l_0 < L$ it constitutes an improvement according to (2.7).

With the notion introduced so far we can present the semilocal convergence analysis of (TSNLM) using the next three results.

THEOREM 2.3. Suppose that $(C1)'$ holds and $\delta \in (0, \delta_0]$. Then, the following assertions hold

(a) $\|x_n - y_n\| \le p\|y_{n-1} - x_{n-1}\| = pe_{n-1}$,
(b) $\|x_n - x_{n-1}\| \le (1 + \frac{p}{2})e_{n-1}$,
(c) $e_n \le qe_{n-1}$.

Proof. Using (1.3) we get that

$$\begin{aligned}
x_n - y_{n-1} &= y_{n-1} - x_{n-1} - R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - F(x_{n-1}) \\
&\quad + \alpha(y_{n-1} - x_{n-1})) \\
&= R_\alpha(x_{n-1})^{-1}[R_\alpha(x_{n-1})(y_{n-1} - x_{n-1}) \\
&\quad - (F(y_{n-1}) - F(x_{n-1})) - \alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha(x_{n-1})^{-1}\int_0^1 \{F'(x_{n-1}) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\} \\
&\quad \times(y_{n-1} - x_{n-1})dt \\
&= R_\alpha(x_{n-1})^{-1}\int_0^1 \{F'(x_{n-1}) - F'(x_0) + F'(x_0) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\} \\
&\quad \times(y_{n-1} - x_{n-1})dt.
\end{aligned}$$

In view of $(C1)'$ and (1.5) we have that

$$\|x_n - y_{n-1}\| \le \|\int_0^1 \Phi(x_{n-1}, x_0, y_{n-1} - x_{n-1})dt\|$$

$$
\begin{aligned}
&+\|\int_0^1 \Phi(x_0, x_{n-1}+t(y_{n-1}-x_{n-1}), y_{n-1}-x_{n-1})dt\| \\
\leq\ & L_0[\|x_{n-1}-x_0\| + \int_0^1 \|x_{n-1}-x_0+t(y_{n-1}-x_{n-1})\|dt]\|y_{n-1}-x_{n-1}\| \\
\leq\ & 2L_0 r\|y_{n-1}-x_{n-1}\| = p\|y_{n-1}-x_{n-1}\| = pe_{n-1}.
\end{aligned}
$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|x_n - x_{n-1}\| \leq \|x_n - y_{n-1}\| + \|y_{n-1}-x_{n-1}\|.$$

To prove (c) we first use (1.3) to obtain in turn the identity

$$
\begin{aligned}
y_n - x_n &= x_n - y_{n-1} - R_\alpha(x_n)^{-1}(F(x_n) - f^\delta + \alpha(x_n - x_0)) \\
&\quad + R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - f^\delta + \alpha(y_{n-1}-x_0)) \\
&= x_n - y_{n-1} - R_\alpha(x_n)^{-1}(F(x_n) - F(y_{n-1}) + \alpha(x_n - y_{n-1})) \\
&\quad + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}](F(y_{n-1}) - f^\delta + \alpha(y_{n-1}-x_0)) \\
&= R_\alpha(x_n)^{-1}[R_\alpha(x_n)(x_n - y_{n-1}) - (F(x_n) - F(y_{n-1})) \\
&\quad -\alpha(x_n - y_{n-1})] + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}] \\
&\quad \times(F(y_{n-1}) - f^\delta + \alpha(y_{n-1}-x_0)). \qquad (2.9)
\end{aligned}
$$

Then, again by $(C1)'$ and (2.9) we obtain that

$$
\begin{aligned}
e_n &\leq \|R_\alpha(x_n)^{-1}\int_0^1 [F'(x_n) - F'(y_{n-1}+t(x_n-y_{n-1}))]dt(x_n-y_{n-1})\| \\
&\quad +\|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - f^\delta \\
&\quad +\alpha(y_{n-1}-x_0))\| \\
&\leq \|R_\alpha(x_n)^{-1}\int_0^1 [F'(x_n) - F'(y_{n-1}+t(x_n-y_{n-1}))]dt(x_n-y_{n-1})\| \\
&\quad +\|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))(y_{n-1}-x_n)\| \\
&\leq L_0[\|x_n - x_0\| + \int_0^1 \|y_{n-1}-x_0+t(x_n-y_{n-1})\|dt]\|x_n-y_{n-1}\| \\
&\quad +L_0[\|x_n-x_0\| + \|x_{n-1}-x_0\|]\|x_n-y_{n-1}\| \\
&\leq 4L_0 r\|y_{n-1}-x_n\| = 4L_0 r(2L_0 r)e_{n-1} \\
&= qe_{n-1}.
\end{aligned}
$$

This completes the proof of the Theorem.

THEOREM 2.4. Under the hypotheses of Theorem 2.3 further suppose that

$$\rho < \rho_0 \ \text{ and } \ L_0 \leq 1. \qquad (2.10)$$

Moreover, suppose that

$$\overline{U(x_0, r)} \subseteq D(F). \qquad (2.11)$$

Then, $x_n, y_n \in U(x_0, r)$ for each $n = 0, 1, 2, \cdots$.

Proof. We note by (2.10) that we have

$$q \in (0, 1). \qquad (2.12)$$

Using Lemma 2.1, Theorem 2.3 and (2.11) we get that

$$\|x_1 - x_0\| \leq (1 + L_0 r)e_0 \leq (1 + L_0 r)b_\rho < r.$$

Hence, $x_1 \in U(x_0, r)$. Similarly, we obtain that

$$\|y_1 - x_0\| \quad \leq \quad \|y_1 - x_1\| + \|x_1 - x_0\| \qquad (2.13)$$

$$\begin{aligned} &\leq qe_0 + (1+\frac{p}{2})b_\rho &(2.14)\\ &\leq [q+1+\frac{p}{2}]b_\rho < L_0 r \leq r, &(2.15)\end{aligned}$$

which implies $y_1 \in U(x_0, r)$. Moreover, we have that

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq (1+\frac{p}{2})\|y_1 - x_1\| + (1+\frac{p}{2})b_\rho \\ &\leq (1+\frac{p}{2})qb_\rho + (1+\frac{p}{2})b_\rho \\ &= (1+q)(1+\frac{p}{2})b_\rho < L_0 r \leq r, \end{aligned}$$

which also implies $x_2 \in U(x_0, r)$. Furthermore, we obtain that

$$\begin{aligned} \|y_2 - x_0\| &\leq \|y_2 - x_2\| + \|x_2 - x_0\| \\ &\leq q\|y_1 - x_1\| + (1+q)(1+\frac{p}{2})b_\rho \\ &\leq q^2(1+\frac{p}{2})b_\rho + (1+q)(1+\frac{p}{2})b_\rho \\ &\leq (1+q+q^2)(1+\frac{p}{2})b_\rho < L_0 r \leq r. \end{aligned}$$

Hence, we proved that $y_2 \in U(x_0, r)$. Proceeding in an analogous way we prove that $x_n, y_n \in U(x_0, r)$. That completes the proof of the Theorem.

THEOREM 2.5. Suppose that the hypotheses of Theorem 2.4 hold. Then, sequence $\{x^\delta_{n,\alpha}\}$ remains in $U(x_0, r)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^\delta_\alpha \in \overline{U(x_0, r)}$ of equation (1.2). Moreover, the following estimates hold

$$\|x_n - x^\delta_\alpha\| \leq b_0 e^{-\gamma_0 n}, \tag{2.16}$$

where $b_0 = (1+\frac{p}{2})b_\rho$ and $\gamma_0 = -\ln q > 0$.

Proof. Using (b) of Theorem 2.3 and (2.10) we get that

$$\|x_{n+m} - x_n\| \leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\|. \tag{2.17}$$

But, we have

$$\|x_{n+i+1} - x_{n+i}\| \leq (1+\frac{p}{2})q^{n+i}e_0. \tag{2.18}$$

In view of (2.18), inequality (2.17) gives that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq [1+q+q^2+\cdots+q^{m-1}]q^n(1+\frac{p}{2})e_0 \\ &\leq \frac{1-q^m}{1-q}(1+\frac{p}{2})q^n e_0. &(2.19)\end{aligned}$$

It follows from (2.19) that sequence $\{x_n\}$ is complete in a Hilbert space X and as such it converges to some $x^\delta_\alpha \in \overline{U(x_0, r)}$ (since $\overline{U(x_0, r)}$ is closed set). By letting $m \to \infty$ we obtain (2.16). Finally, to prove x^δ_α is a solution of (1.2), note that

$$\begin{aligned} \|F(x_n) - f^\delta + \alpha(x_n - x_0)\| &= \|R_\alpha(x_n)(x_n - y_n)\| \\ &\leq (\|F'(x_n)\| + \alpha)e_n \\ &\leq (\|F'(x_n)\| + \alpha)q^n b_\rho \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

That completes the proof of the Theorem.

REMARK 2.6. (a) The convergence order of (TSNLM) is three [13] under $(C1)$. In Theorem 2.5 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [4]) defined by

$$\varrho \approx \ln\left(\frac{\|x_{n+1}-x_\alpha^\delta\|}{\|x_n-x_\alpha^\delta\|}\right) / \ln\left(\frac{\|x_n-x_\alpha^\delta\|}{\|x_{n-1}-x_\alpha^\delta\|}\right).$$

The (COC) ϱ will then be close to 3 which is the order of convergence of (TSNLM).

(b) In the rest of this section we suppose that

$$\rho_0^* \leq r \tag{2.20}$$

which is possible for x_0 sufficiently close to $\hat{x}$.

Next, we present the results concerning error bounds under source conditions. We need a condition on the source function.

(C2) (George and Pareth [13]) There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying $lim_{\lambda\to 0}\varphi(\lambda) = 0$ and $v \in X$ with $\|v\| \leq 1$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

and

$$sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_\varphi\varphi(\alpha), \qquad \forall\lambda \in (0, a].$$

REMARK 2.7. It can easily be seen that functions

$$\varphi(\lambda) = \lambda^\nu, \quad \lambda > 0$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} (ln\frac{1}{\lambda})^{-\beta} & , \quad 0 < \lambda \leq e^{-(\beta+1)} \\ 0 & , \quad otherwise \end{cases}$$

for $\beta \geq 0$ satisfy $(C2)$ (cf. [23]).

PROPOSITION 2.8. (cf. [30], Proposition 3.1) Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let x_α^δ be the unique solution of (1.2) and $x_\alpha := x_\alpha^0$. Then

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha}.$$

THEOREM 2.9. (cf. [29], Proposition 4.1 or [30], Theorem 3.3) Suppose that $(C1)', (C2)$ and hypotheses of Proposition 2.8 hold. Let $\hat{x} \in D(F)$ be a solution of (1.1). Then, the following assertion holds

$$\|x_\alpha - \hat{x}\| \leq (L_0r + 1)\varphi(\alpha).$$

THEOREM 2.10. Suppose hypotheses of Theorem 2.5 and Theorem 2.9 hold. Then, the following assertion holds

$$\|x_n - \hat{x}\| \leq b_0e^{-\gamma_0 n} + c_1(\varphi(\alpha) + \frac{\delta}{\alpha})$$

where $c_1 = \max\{1, (L_0r + 1)\}$.

Let

$$\bar{c} := max\{b_0 + 1, (L_0r + 1)\}, \tag{2.21}$$

and let

$$n_\delta := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha}\}. \tag{2.22}$$

THEOREM 2.11. Let $\bar{c}$ and n_δ be as in (2.21) and (2.22) respectively. Suppose that hypothese of Theorem 2.10 hold. Then, the following assertions hold

$$\|x_{n_\delta} - \hat{x}\| \leq \bar{c}(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{2.23}$$

Note that the error estimate $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (2.23) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (2.23) we have the following.

THEOREM 2.12. Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 2.11 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (2.22). Then

$$\|x_{n_\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

In this section, we present a parameter choice rule based on the balancing principle studied in [22], [27]. In this method, the regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, M\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha_i}\}.$$

Then for $i = 0, 1, \cdots, M$, we have

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq c\frac{\delta}{\alpha_i}, \qquad \forall i = 0, 1, \cdots M.$$

Let $x_i := x^\delta_{n_i,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operate only with corresponding $x_i, \quad i = 0, 1, \cdots, M$. Proof of the following theorem is analogous to the proof of Theorem 3.1 in [29].

THEOREM 2.13. (cf. [29], Theorem 3.1) Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Suppose the hypotheses of Theorem 2.11 and Theorem 2.12 hold and let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\} < M,$$

$$k := max\{i : \|x_i - x_j\| \leq 4\bar{c}\frac{\delta}{\alpha_j}, \qquad j = 0, 1, 2, \cdots, i\}.$$

Then $l \leq k$ and

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{c}\mu$.

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 2.13 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose M big enough but not too large and $\alpha_i := \mu^i\alpha_0, i = 0, 1, 2, \cdots, M$.
- Choose $\rho \leq \rho_0^*$.

2.1. **Algorithm.**

1. Set $i = 0$.
2. Choose $n_i = min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha_i}\}$.
3. Solve $x_i = x^\delta_{n_i,\alpha_i}$ by using the iteration (1.3).
4. If $\|x_i - x_j\| > 4\bar{c}\frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return x_k.
5. Else set $i = i + 1$ and return to Step 2.

3. Semilocal convergence of (TSNLM) under (C1)

We present the semilocal convergence of (TSNLM) under $(C1)$. As in [3], [13] let us define function $g : (0, \infty) \to (0, \infty)$ by

$$g(t) = \frac{L^2}{8}(4 + 3Lt)t^2. \tag{3.1}$$

Note that if

$$t < \rho_1 = \frac{0.706442399}{L} \tag{3.2}$$

then

$$g(t) < 1. \tag{3.3}$$

Set

$$\rho^* = \min\{\rho_1, \rho_0\}. \tag{3.4}$$

Then, as in Section 2 (see also [3] and [13]) we were at the main semilocal convergence result for (TSNLM):

THEOREM 3.1. Suppose that $(C1), (C1)''$ hold and

$$\rho < \rho^* \tag{3.5}$$

where ρ^* is defined by (3.4) and $\overline{U(x_0, R)} \in D(F)$ with

$$R = (\frac{1}{1 - g(b_\rho)} + \frac{L}{2}\frac{b_\rho}{1 - g(b_\rho)^2})b_\rho. \tag{3.6}$$

Then, the following assertions hold

$$\|x_n - y_{n-1}\| \leq \frac{Le_{n-1}}{2}\|y_{n-1} - x_{n-1}\|,$$

$$\|x_n - x_{n-1}\| \leq (1 + \frac{Le_{n-1}}{2})\|y_{n-1} - x_{n-1}\|,$$

$$\|y_n - x_n\| \leq g(e_{n-1})\|y_{n-1} - x_{n-1}\|,$$

$$g(e_n) \leq g(b_\rho)^{3^n},$$

$$e_n \leq g(b_\rho)^{\frac{3^n-1}{2}}b_\rho.$$

Sequence $\{x_n\}$ generated by (1.3) remains in $U(x_0, R)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^\delta_\alpha \in \overline{U(x_0, R)}$ of equation (1.2). Moreover, the following assertion hold

$$\|x_n - x^\delta_\alpha\| \leq de^{-\gamma^{3^n}},$$

where $d = (\frac{1}{1-g(b_\rho)^3} + \frac{Lb_\rho}{2}\frac{1}{1-g(b_\rho)^3}g(b_\rho))b_\rho$ and $\gamma = -\ln g(b_\rho)$.

REMARK 3.2. Even if $l_0 = L$ Theorem 3.1 improves Theorem 3 in [13], since we do not assume $L \leq 1$. Otherwise, i.e., if $l_0 < L$, it constitutes a further improvement with advantages as stated in the introduction of this study since $b_\rho < \gamma_\rho$. Note that the results in [13] use γ_ρ instead of b_ρ. Therefore our constants c, γ are tighter if $l_0 < L$ or $L_0 < L$.

REMARK 3.3. In the rest of this section we suppose that

$$\rho^* \leq R \tag{3.7}$$

which is possible for x_0 sufficiently close to $\hat{x}$. The rest of the results of Section 2 hold in this setting if we replace $L_0, b's, \gamma_0, e^{-\gamma_0 n}, \rho_0^*$ respectively by $L, c's, \gamma, e^{-\gamma 3^n}, \rho^*$.

4. Examples

In this section we first consider the example considered in [29] for illustrating the algorithm considered in section 2. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dim V_n = n+1$. Precisely we choose V_n as the linear span of $\{v_1, v_2, \cdots, v_{n+1}\}$ where $v_i, i = 1, 2, \cdots, n+1$ are the linear splines in a uniform grid of $n+1$ points in $[0,1]$.

EXAMPLE 4.1. (see [29], section 4.3) Let $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t, & 0 \leq t \leq s \leq 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds\right](x-y)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds. \tag{4.1}$$

Note that for $u, v > 0$,

$$\begin{aligned}(F'(v) - F'(u))w &= 3\int_0^1 k(t,s)[(v(s))^2 - (u(s))^2]w(s)ds \\ &:= F'(u)\Phi(v,u,w),\end{aligned}$$

where $\Phi(v,u,w) = \frac{(v^2-u^2)w}{u^2}$.

Observe that

$$\begin{aligned}\Phi(v,u,w) &= \frac{(v^2-u^2)w}{u^2} \\ &= \frac{(u+v)(v-u)w}{u^2}.\end{aligned}$$

So condition $(C1)'$ satisfies with $L_0 \geq \|\frac{u+v}{u^2}\|$.

In our computation, we take $f(t) = \frac{6\cos(\pi t)+\cos^3(\pi t)+14t-7}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = \cos(\pi t).$$

We use

$$x_0(t) = \cos(\pi t) + \frac{3(t\pi^2 - t^2\pi^2 + sin^2(\pi t))}{4\pi^2}$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))\frac{1}{4}$$

where $\varphi(\lambda) = \lambda$.

Observe that while performing numerical computation on finite dimensional subspace (V_n) of X, one has to consider the operator $P_n F'(.) P_n$ instead of $F'(.)$, where P_n is the orthogonal projection on to V_n. Thus incurs an additional error $\|P_n F'(.) P_n - F'(.)\| = O(\|F'(.)(I - P_n)\|)$.

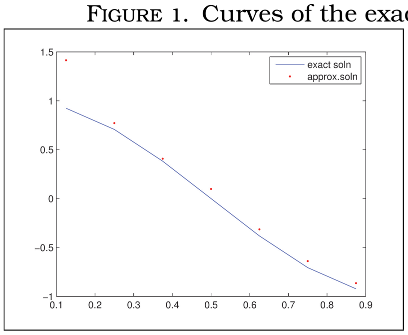
Let $\|F'(.)(I - P_n)\| \leq \varepsilon_n$. For the operator $F'(.)$ defined in (4.1), $\varepsilon_n = O(n^{-2})$ (cf. [14]). Thus we expect to obtain the rate of convergence $O((\delta + \varepsilon_n)^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.1)(\delta + \varepsilon_n)$, $\mu = 1.1$. The results of the computation are presented in Table 1. The plots of the exact solution for ($n = 128$ to $n = 1024$) and the approximate solution obtained are given in Figures 1 and 2.

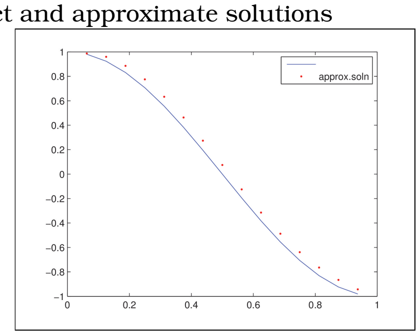
TABLE 1. Iterations and corresponding error estimates

n	k	n_k	$\delta + \varepsilon_n$	α_k	$\|x_k - \hat{x}\|$	$\frac{\|x_k - \hat{x}\|}{(\delta+\varepsilon_n)^{1/2}}$
8	2	2	0.0135	0.0180	0.3575	3.0782
16	2	2	0.0134	0.0178	0.2573	2.2247
32	2	2	0.0133	0.0178	0.1871	1.6196
64	2	2	0.0133	0.0177	0.1394	1.2073
128	2	2	0.0133	0.0177	0.1079	0.9344
256	2	2	0.0133	0.0177	0.0880	0.7619
512	2	2	0.0133	0.0177	0.0761	0.6588
1024	2	2	0.0133	0.0177	0.0694	0.6007

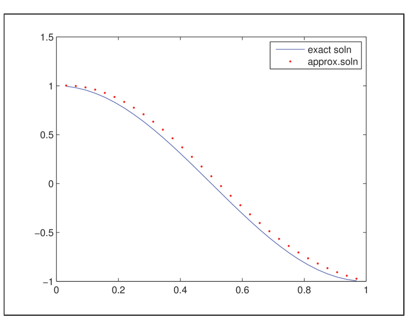
FIGURE 1. Curves of the exact and approximate solutions



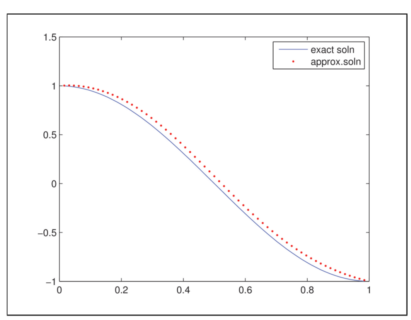
n=8



n=16

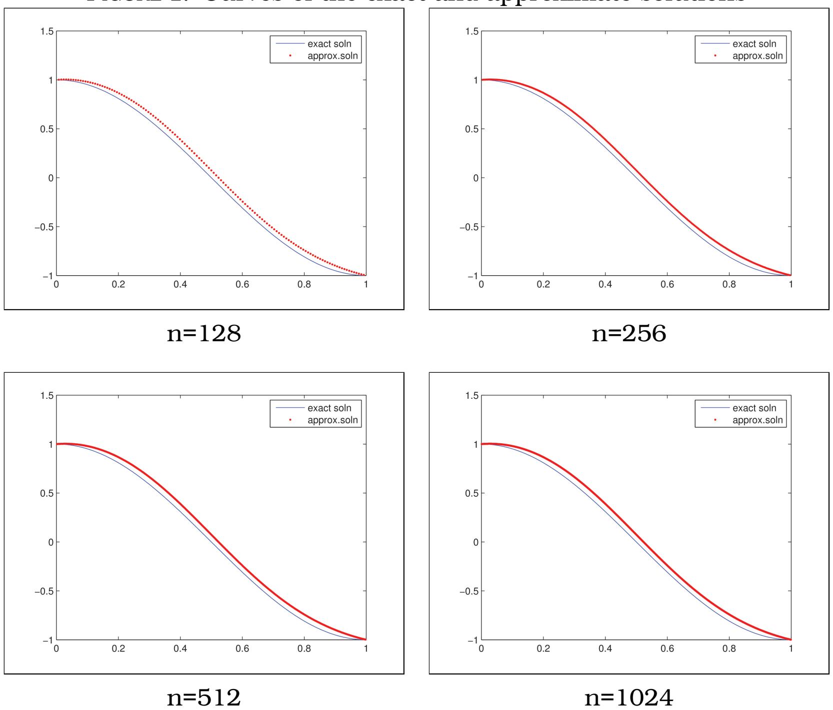


n=32



n=64

FIGURE 2. Curves of the exact and approximate solutions



Next we present two examples where $(C1)$ is not satisfied but $(C1)'$ is satisfied.

EXAMPLE 4.2. Let $X = Y = \mathbb{R}$, $D = [0, \infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{4.2}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. However central Lipschitz condition $(C1)'$ holds for $L_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \\ &\leq L_0 |x - x_0|. \end{aligned}$$

EXAMPLE 4.3. We consider the integral equations

$$u(s) = f(s) + \tau \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{4.3}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a, b], \tau$ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n} \tag{4.4}$$

$$u(a) = f(a), u(b) = f(b). \tag{4.5}$$

These type of problems have been considered in [1], [2], [19].

Equation of the form (4.3) generalize equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt \tag{4.6}$$

studied in [1], [2], [19]. Instead of (4.3) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \tau \int_a^b G(s,t)u(t)^{1+1/n}dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \tau(1+\frac{1}{n})\int_a^b G(s,t)u(t)^{1/n}v(t)dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\tau|(1+\frac{1}{n})\int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_1\|x-y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}dt \leq L_2 \max_{x\in[0,1]} x(s), \tag{4.7}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (4.7)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (4.7) is not satisfied in this case. However, condition $(C1)'$ holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &= |\tau|(1+\frac{1}{n})\max_{s\in[a,b]}|\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt| \\ &\leq |\tau|(1+\frac{1}{n})\max_{s\in[a,b]} G_n(s,t)\end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n}f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}}\|v\|$.

Hence,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}}\max_{s\in[a,b]}\int_a^b G(s,t)dt\|x-x_0\| \\ &\leq L_0\|x-x_0\|,\end{aligned}$$

where $L_0 = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} N$ and $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$. Then condition $(C1)'$ holds for sufficiently small τ.

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