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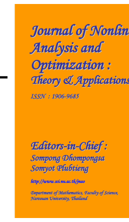

FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN ORDERED PARTIAL METRIC SPACES

BINAYAK S. CHOUDHURY[1] AND AMARESH KUNDU[2,*]

[1]Department of Mathematics, Bengal Engineering and Science University, Shibpur, P.O - B. Garden, Howrah - 711103, India
[2]Department of Mathematics, Siliguri Institute of Technology, P.O - Sukna, Darjeeling - 734009, West Bengal, India

ABSTRACT. In this paper we develope the weak contraction mapping principle in the context of partial metric spaces which are generalizations of metric spaces meant for the study of denotational semantics of programming languages. We consider certain control conditions for this purpose and accomplish the task in partial metric spaces. Additionally, a partial order is defined on this space. An illustrative example is given. The method we use in this paper is a combination of analytic and order theoretic methodologies.

KEYWORDS : Partially ordered set; Partial metric; Non-decreasing mapping; Weak contraction; Control functions; Fixed point.
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1. INTRODUCTION

Fixed points play an important role in computer science especially for justifications of induction and recursive definitions. In 1994 Matthews [23, 24] introduced the conception of partial metric spaces as generalizations of metric spaces where self distance may be non-zero. The motivation for such a generalization comes from the study of denotational semantics of programming languages in computer science [38] where it was felt that a metric approach to this study is not possible unless the definition of the metric is suitably modified. Our interest is to prove fixed point results in this space. Fixed points have important roles to play in computer science, especially in semantics [37]. The study of fixed points in partial metric spaces was initiated in [23] where Matthews established a contraction mapping theorem in partial metric spaces. Other fixed point results followed this work. Some instances of these works are in [1, 4-6, 21, 22, 31, 33, 35, 40].

* *Corresponding author.*
Email address : kunduamaresh@yahoo.com (Amaresh Kundu).
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In metric spaces we find a lot of efforts to generalize the Banachs contraction mapping principle as, for instances, in [7, 9, 25, 39]. Particularly, Alber and Guerre-Delabriere in [3] introduced the concept of weak contraction in Hilbert spaces. Rhoades in [34] has shown that the result which Alber et al. proved in [3] is also valid in complete metric spaces. A weak contraction is intermediate to contraction mapping and a nonexpansive mapping. The weak contraction principle established by Rhoades in metric spaces as mentioned above has been generalized in a number of ways. Dutta and Choudhury [15] has proved a generalization employing a method different from that used by Rhoades. Another approach of generalisation was initiated by Eslamian and Abkar [17] and was further adopted by Choudhury and Kundu [14].
A separate methodology was applied to this problem by Popescu [32] and proved that some of the control conditions used by Doric [16] are not required. There are several other fixed point results of weakly contractive mappings and their generalizations. Some instances of these works are noted in [10, 12-13, 16, 28, 29, 41].

In recent years fixed point theory has experienced a rapid development in partially ordered metric spaces. References [2, 8, 11, 19, 27, 30] are some instances of these works. Particularly, Harjani et. al have established a generalized weak contraction principle in partially ordered metric spaces [20].

The purpose of this paper is to weaken the contractive conditions in partial metric spaces having a partial ordering defined on them. We have shown that the weak contractions necessarily have fixed points in partially ordered partial metric spaces. using the notion of weak control conditions two fixed point theorems in ordered partial metric spaces in view of Popescu [32] conditions has been proved. Here our effort is to show that a parallel development is also possible in partial metric spaces with a partial order. Our approach is a blending of analytic and order theoretic methods. We have given an illustrative examples.

The following are some essential concepts for our discussion in this paper.

Definition 1.1. [23] Let X be a nonempty set and let $p : X \times X \to \mathrm{R}^+$ be such that the following are satisfied, for all $x, y, z \in X$

$(P1)$ $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$
$(P2)$ $p(x, x) \leq p(x, y)$
$(P3)$ $p(x, y) = p(y, x)$
$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.
It is clear that, if $p(x, y) = 0$, then from $(p1)$ and $(p2)$ $x = y$. But if $x = y$, $p(x, y)$ may not be 0. If p be a partial metric on X, then the function $d_p : X \times X \to R^+$ defined as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

satisfies the conditions of an usual metric on X [23]. Each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$ [23].

The concepts of convergence, Cauchy sequence, completeness and continuity in partial metric space is given in the following definition.

Definition 1.2. [23] Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in the partial metric space (X, p) converges to the limit x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(2) A sequence $\{x_n\}$ in the partial metric space (X, p) is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.

(3) A partial metric space (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that
$p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$.

(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(fx_0, \varepsilon)$.

The following implication follows from the above definition.

If a function $f : X \to X$ where (X, p) is a partial metric space is continuous then $fx_n \to fx$ whenever $x_n \to x$ as $n \to \infty$.

Lemma 1.3. *[23] Let (X, p) be a partial metric space.*

(1) A sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).

(2) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x, x_n) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_m, x_n)$.

Definition 1.4. A function $f : R \to R$ is said to be monotone non-decreasing (or monotone increasing) if $x \geq y$ implies $f(x) \geq f(y)$.

The following are examples of a partial metric spaces.

Example 1.5. [22] Let X =[0,1] and $p : X \times X \to \mathrm{R}^+$ be defined as $p(x, y) = \max\{x, y\}$. Then, (X, p) is a partial metric space and it is also complete.

We construct the following example of a partial metric spaces.

Example 1.6. Let $X = \{0, 1, 2, 3, 4, ...\}$. We define $p : X \times X \to \mathrm{R}^+$ as

$$p(x, y) = \begin{cases} x + y + 2, \text{if } x \neq y, \\ 1, \text{if } x = y. \end{cases}$$

Then p is a partial metric on X.

The properties $(P1)$, $(P2)$ and $(P3)$ are directly verified by inspection. We prove $(P4)$ in the following. Let $a, b, c \in X$. If $a \neq c$ then

i) $p(a, c) = a + c + 2 < a + b + 2 + b + c + 2 - 1 = p(a, b) + p(b, c) - p(b, b)$ (if $b \neq a$ and $b \neq c$).

ii) $p(a, c) = a + c + 2 < 1 + a + c + 2 = p(a, b) + p(b, c) - p(b, b)$ (if $b = a$ and $b \neq c$).

If $a = c$ then $p(a,c) = 1 \leq p(a,b) + p(b,c) - 1 = p(a,b) + p(b,c) - p(b,b)$. Thus (P4) is satisfied.

In view of (1.1) the function $d_p : X \times X \to R^+$ defined as
$$d_p(x,y) = \begin{cases} 2x + 2y + 2, \text{if } x \neq y, \\ 0, \text{if } x = y. \end{cases}$$
It is a metric on X.

2. MAIN RESULTS

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and let there be a partial metric p on X such that (X,p) is a complete partial metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that*

$$\psi(p(fx, fy)) \leq \psi(M(x,y)) - \beta(M(x,y)) \text{ whenever } x, y \in X \text{ and } x \preceq y, \quad (2.1)$$

with

$$M(x,y) = \max\{p(x,y), p(Tx,x), p(y,Ty), \frac{1}{2}[p(y,Tx) + p(x,Ty)]\} \quad (2.2)$$

where

i) $\psi : [0,\infty) \to [0,\infty)$ is a monotone non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$,

ii) $\beta : [0,\infty) \to [0,\infty)$ is a function satisfying $\beta(0) = 0$, $\liminf_{n\to\infty} \beta(a_n) > 0$ whenever $\lim_{n\to\infty} a_n = a > 0$,

iii) $\beta(t) > \psi(t) - \psi(t^-)$ for all $t > 0$, where $\psi(t^-)$ is the left limit of ψ at t.

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Starting with $x_0 \in X$, and following the same steps as in theorem 2.1, we obtain a sequence $\{x_n\}$ in X defined as

$$fx_n = x_{n+1} \text{ for all } n \geq 0, \quad (2.3)$$

for which

$$x_0 \preceq fx_0 = x_1 \preceq fx_1 = x_2 \preceq fx_2 \preceq .... \preceq fx_{n-1} = x_n \preceq fx_n = x_{n+1} \preceq ..... \quad (2.4)$$

If $x_n = x_{n+1}$, then f has a fixed point. Therefore we assume that

$$x_n \neq x_{n+1}, \text{ for all } n \geq 0.$$

Then it follows from the definition of p that

$$p(x_n, x_{n+1}) \neq 0 \text{ for all } n \geq 0. \quad (2.5)$$

Let, if possible, for some n

$$p(x_{n-1}, x_n) < p(x_n, x_{n+1}). \quad (2.6)$$

By triangular inequality of partial metric space,

$$\begin{aligned} \frac{1}{2}(p(x_{n-1}, x_{n+1}) + p(x_n, x_n)) &\leq \frac{1}{2}(p(x_{n-1}, x_n) + p(x_n, x_{n+1})) \\ &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned}$$

Now,

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_n, x_n) + p(x_{n-1}, x_{n+1})]\} \\ &= p(x_n, x_{n+1}) \quad \text{[by (2.6)]} \end{aligned}$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), using (2.2), (2.3), (2.4) and the monotone property of ψ, for all $n \geq 0$, we have

$$\begin{aligned}\psi(p(x_n, x_{n+1})) &= \psi(p(fx_{n-1}, fx_n))\\ &\leq \psi(M(x_{n-1}, x_n)) - \beta(M(x_{n-1}, x_n))\\ .&\leq \psi(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1}))\end{aligned} \tag{2.7}$$

A consequence of the properties of β given in condition (ii) of the theorem is that $\beta(a) > 0$ for $a > 0$. Then from (2.5), $\beta(p(x_n, x_{n+1})) > 0$. With this, (2.7) leads to a contradiction. Therefore, for all $n \geq 1$,

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

Thus the sequence $\{p(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{2.8}$$

Suppose that $r > 0$. If there exists n such that $p(x_n, x_{n+1}) = r$, then, by (2.7) we have $\psi(r) \leq \psi(r) - \beta(r)$. Since $\beta(r) > 0$, this is a contradiction. So $p(x_n, x_{n+1}) > r$, for all $n \geq 0$. Then taking limit infimum as $n \to \infty$ in (2.7), using (2.8) and the fact that $\{p(x_n, x_{n+1})\}$ is monotone decreasing, we have

$$\psi(r^+) \leq \psi(r^+) - \lim_{n\to\infty} \inf \beta(p(x_n, x_{n+1})).$$

By virtue of condition (ii), $\lim_{n\to\infty} \inf \beta(p(x_n, x_{n+1})) > 0$. So the above inequality leads to a contradiction. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.9}$$

It follows by (P1) and (P2) of definition 1.1 that

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.10}$$

Since from (1.1), $d_p(x, y) \leq 2p(x, y)$ for all $x, y \in X$, for all $n \geq 0$, from (2.9) it follows that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.11}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in (X, d_p). If not, then there exists some $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$n(k) > m(k) > k,$$

$$d_p(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{2.12}$$

and

$$d_p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.13}$$

Now, for all $k \geq 0$, we have

$$\begin{aligned}\varepsilon \leq d_p(x_{m(k)}, x_{n(k)}) &\leq d_p(x_{m(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})\\ &< \varepsilon + d_p(x_{n(k)-1}, x_{n(k)}) \qquad \text{(by(2.13))}.\end{aligned}$$

Taking $k \to \infty$ in the above inequality, using (2.11), we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.14}$$

Also, for all $k \geq 0$, we have

$d_p(x_{m(k)-1}, x_{n(k)-1}) \leq d_p(x_{m(k)-1}, x_{m(k)}) + d_p(x_{m(k)}, x_{n(k)}) + d_p(x_{n(k)}, x_{n(k)-1})$
and $d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})$.

Taking limit as $k \to \infty$ in the above two inequalities, using (2.11) and (2.14) we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.15}$$

For k=1,2,3...

$$d_p(x_{n(k)-1}, x_{m(k)}) \leq d_p(x_{n(k)-1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)}) \tag{2.16}$$

and

$$d_p(x_{n(k)}, x_{m(k)}) \leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)}). \tag{2.17}$$

Making $k \to \infty$ in (2.16) and (2.17) respectively, and using (2.14) and (2.11) we have

$$\lim_{k\to\infty} d_p(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.18}$$

Again for k=1,2,3...

$$d_p(x_{n(k)}, x_{m(k)-1}) \leq d_p(x_{n(k)}, x_{m(k)}) + d_p(x_{m(k)}, x_{m(k)-1})$$

$$and \ \ d_p(x_{n(k)}, x_{m(k)}) \leq d_p(x_{n(k)}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{m(k)}).$$

Making $k \to \infty$ in the above two inequalities and using (2.14) and (2.11) we obtain

$$\lim_{k\to\infty} d_p(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.19}$$

Since for all $x, y \in X$, $d_p(x, y) \leq 2p(x, y) - p(x, x) - p(y, y)$ by using, (2.14), (2.15), (2.18) and (2.19) we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}, \tag{2.20}$$

$$\lim_{k\to\infty} p(x_{n(k)+1}, x_{m(k)}) = \frac{\varepsilon}{2}, \tag{2.21}$$

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)+1}) = \frac{\varepsilon}{2} \tag{2.22}$$

and

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.23}$$

Next we show that for sufficiently large k, $p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}$.

If not, then there exists a subsequence $\{k(i)\}$ of $\mathbb{N}$ such that for all $i > 0$,

$$\frac{\varepsilon}{2} < p(x_{m(k(i))}, x_{n(k(i))})). \tag{2.24}$$

In view of (2.4), substituting $x = x_{m(k(i))-1}$ and $y = x_{n(k(i))-1}$ in (2.1), for all $i > 0$, we have

$$\begin{aligned} & \psi(p(x_{m(k(i))}, x_{n(k(i))})) \\ = \ & \psi(p(fx_{m(k(i))-1}, fx_{n(k(i))-1})) \\ \leq \ & \psi(M(x_{m(k(i))-1}, x_{n(k(i))-1})) - \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})). \end{aligned} \tag{2.25}$$

$$\begin{aligned} & M(x_{m(k(i))-1}, x_{n(k(i))-1}) \\ & = \max\{p(x_{m(k(i))-1}, x_{n(k(i))-1}), p(x_{m(k(i))-1}, x_{m(k(i))}), p(x_{n(k(i))-1}, x_{n(k(i))}), \\ & \qquad \tfrac{1}{2}(p(x_{m(k(i))-1}, x_{n(k(i))}) + p(x_{n(k(i))-1}, x_{m(k(i))}))\} \end{aligned}$$

Taking limit as $i \to \infty$ in (2.25), using (2.20)- (2.23) in the above inequality, taking into account the inequality (2.24) and the monotone property of ψ, we obtain

$$\psi(\frac{\varepsilon}{2}^{+}) \leq \psi(\frac{\varepsilon}{2}^{+}) - \liminf_{i\to\infty} \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})).$$

But by a property of β, the last inequality implies that

$$\liminf_{i\to\infty} \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})) > 0.$$

Then the above inequality gives a contradiction. Thus for sufficiently large k,

$$p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}. \tag{2.26}$$

Again from (1.1) we have

$$d_p(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}).$$

Taking $k \to \infty$ and using (2.14) and (2.10), we have $p(x_{m(k)}, x_{n(k)}) \geq \frac{\varepsilon}{2}$. Then the above observation along with (2.26) implies that, there exists a positive integer k_1 such that for all $k \geq k_1$,

$$p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{2.27}$$

Substituting $x = x_{m(k)}$ and $y = y_{n(k)}$ in (2.1), (2.2), using (2.3), (2.4), we obtain

$$\begin{aligned} \psi(p(x_{m(k)+1}, x_{n(k)+1})) &= \psi(p(fx_{m(k)}, fx_{n(k)})) \\ &\leq \psi(M(x_{m(k)}, x_{n(k)})) - \beta(M(x_{m(k)}, x_{n(k)})) \end{aligned} \tag{2.28}$$

where,

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}) &= \max\{p(x_{m(k)}, x_{n(k)}), p(x_{m(k)}, x_{m(k)+1}), p(x_{n(k)}, x_{n(k)+1}), \\ &\qquad \frac{1}{2}(p(x_{m(k)}, x_{n(k)+1}) + p(x_{m(k)+1}, x_{n(k)}))\} \end{aligned} \tag{2.29}$$

Then, for all $k \geq k_1$, $M(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}$ and (2.28) becomes

$$\psi(p(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2}) < \psi(\frac{\varepsilon}{2}). \tag{2.30}$$

Thus, by (2.30), using the monotone property of ψ, for all $k \geq k_1$, we have

$$p(x_{m(k)+1}, x_{n(k)+1}) < \frac{\varepsilon}{2}. \tag{2.31}$$

Taking the limit as $k \to \infty$ in (2.28), using (2.29) and (2.31), we obtain $\psi(\frac{\varepsilon}{2}^{-}) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2})$, which contradicts condition (iii).

Therefore the sequence $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, by lemma 1.3, (X, d_p) is complete and consequently the sequence $\{x_n\}$ is convergent to z in X, that is,

$$\lim_{n\to\infty} x_n = z. \tag{2.32}$$

Thus, by lemma 1.3,

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.33}$$

Again by (1.1), for all $m, n \geq 0$

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Taking limit $m, n \to \infty$, using (2.10) and the fact that $\{x_n\}$ is a Cauchy sequence in (X, d_p), we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Then, from (2.33), it follows that

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.34}$$

Next we prove that $fz = z$. Suppose that

$$p(z, fz) > 0. \tag{2.35}$$

By virtue of (2.32), the continuity of f implies that $fx_n \to fz$ as $n \to \infty$. Then, by lemma 1.3, we have

$$p(fz, fz) = \lim_{n\to\infty} p(fx_n, fz) = \lim_{n\to\infty} p(x_{n+1}, fz). \tag{2.36}$$

Now,

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(x_{n+1}, fz). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (2.32), (2.34) and (2.36), we obtain

$$\begin{aligned} p(z, fz) &\leq \lim_{n\to\infty} p(z, x_{n+1}) + \lim_{n\to\infty} p(x_{n+1}, fz) \\ &= p(fz, fz). \end{aligned}$$

$M(z, z) = \max\{p(z, z), p(z, fz), p(z, fz), \frac{1}{2}(p(z, fz) + p(z, fz))\} = p(z, fz)$ [by (2.34)]

Using the last inequality and the monotone property of ψ, from (2.36) we obtain,

$$\begin{aligned} \psi(p(z, fz)) &\leq \psi(p(fz, fz)) \leq \psi(M(z, z)) - \beta(M(z, z)) \text{ (by (2.1) and (2.4)).} \\ &\leq \psi(p(z, fz)) - \beta(p(z, fz)) \end{aligned} \tag{2.37}$$

In view of (i), (ii) and (2.35) we obtain $p(z, fz) = 0$. Then from (P1) and (P2) of the definition 1.1, it follows that $z = fz$. □

Our next theorem is obtained by replacing the continuity of f by an ordered theoretic condition.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. We assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then*

$$x_n \preceq z \text{ for all } n \geq 0. \tag{2.38}$$

Let $f : X \to X$ be a non-decreasing mapping such that

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \beta(M(x, y)) \text{ for all } x, y \in X \text{ and } x \prec y\ (x \neq y), \tag{2.39}$$

where ψ and β satisfies all the condition of theorem 2.3. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the steps identically as in the proof of the theorem 2.1 we obtain (2.32) and (2.34). In view of (2.4) we claim that $\{x_n\}$ is a non-decreasing sequence converges to z in (X, p) such that for all $n \geq 1$, $x_n \preceq z$. If $x_n = z$, for some n, then, from (2.29) and (2.54), it follows that $x_n = x_{n+1}$, in which case we have a fixed point. So we assume that $x_n \neq z$ for all $n \geq 0$. From (2.34) it is observed that $p(z, z) = 0$. Suppose $\varepsilon = p(z, fz) > 0$.

Thus for each k_0 there exists $k_0 \in \mathbb{N}$ such that for $n \geq k_0$,

$$p(x_{n+1}, z) < \frac{\varepsilon}{2}, \quad p(x_n, z) < \frac{\varepsilon}{2} \text{ and in view of (2.9) } p(x_{n+1}, x_n) < \frac{\varepsilon}{2}.$$

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(fx_n, fz) \end{aligned}$$

Taking $n \to \infty$, and using (2.34), we have

$$p(z, fz) \leq \lim_{n\to\infty} p(fx_n, fz) \tag{2.40}$$

Since $x_n \preceq z$, putting $x = x_n$ and $y = z$ in (2.39), using (2.40), and the property of ψ, we get

$$\begin{aligned}\psi(p(z, fz)) &\leq \lim_{n\to\infty} \psi(p(fx_n, fz)) \\ &\leq \lim_{n\to\infty} \psi(M(x_n, z)) - \lim_{n\to\infty} \beta(M(x_n, z)) \qquad (2.41)\end{aligned}$$

where

$$\begin{aligned}\varepsilon &= p(z, fz) \leq M(x_n, z) \\ &= \max\{p(x_n, z), p(z, fz), p(fx_n, x_n), \frac{1}{2}(p(x_n, fz) + p(z, fx_n))\} \\ &= \max\{p(x_n, z), p(x_{n+1}, x_n), p(fz, z), \frac{1}{2}(p(x_n, z) + p(z, fz) + p(z, x_{n+1}))\} \\ &\leq \max\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \varepsilon, \varepsilon\} = \varepsilon.\end{aligned}$$

Then by (2.41),

$$\psi(\varepsilon^-) \leq \psi(\varepsilon) - \beta(\varepsilon) \qquad (2.42)$$

In view of the properties of (i)-(iii) we arrive at a contradiction, unless $p(fz, z) = 0$. Since $p(z, z) = 0$ and $p(z, fz) = 0$, from (P1) and (P2) of definition 1.1, it follows that $z = fz$. □

Remark 2.3. Under the assumption when partial metric is a metric we have the result of Popescu [32].

Example 2.4. Let X = [0, 1]∪{2, 3, 4, ...}. $p(x, y) = max\{x, y\}$ for $x, y \in X$. We define a partial order as follows

1) $0 \preceq x$ for all $x \in [0, 1]$ and $0 \preceq 2$, $1 \preceq 3$.

2) for all $x, y \in \{2, 3, 4, ...\}$ $x \preceq y$ iff $x \leq y$ and $(y - x)$ is divisible by 2.

That is we have the following two chains $0 \leq 2 \leq 4...$ and $0 \leq 1 \leq 3...$ Then "$\preceq$" satisfies all the conditions of a partially ordered set.

Also, (X, p) is a complete partial metric space.
Let $\psi : [0, \infty) \to [0, \infty)$ be defined as:

$$\psi(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ t^2, & \text{if } t > 1. \end{cases}$$

and

$\beta : [0, \infty) \to [0, \infty)$ be defined as:

$$\beta(t) = \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\ 2t - 1, & \text{if } t > 1. \end{cases}$$

Let $f : X \to X$ be defined as:

$$\psi(t) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ x - 1, & \text{if } x \in \{2, 3, 4, ...\}. \end{cases}$$

without loss of generality, we assume that $x > y$.

$p(x,y) = \max\{x,\, y\} = x$, $p(x, fx) = \max\{x,\, fx\} = x$,
$p(y, fy) = \max\{y,\, fy\} = y$.

$$p(y, fx) = \begin{cases} \max\{y, x - \frac{x^2}{2}\}, & \text{if } \ 0 \le x \le 1 \\ \max\{y, x-1\}, & \text{if } \ x \in \{2,3,4,...\}. \end{cases}$$

or

$$p(x, fy) = \begin{cases} \max\{x, y - \frac{y^2}{2}\} = x, & \text{if } \ 0 \le y \le 1 \\ \max\{x, y-1\} = x, & \text{if } \ y \in \{2,3,4,...\}. \end{cases}$$

Then

$$M(x,y) = \max\{p(x,y), p(fx,x), p(y,fy), \tfrac{1}{2}(p(y,fx)+p(x,fy))\} = x$$

Therefore, we discuss the following cases.

Case-1: $x, y \in [0,1]$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(x - \frac{x^2}{2}, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(x - \frac{x^2}{2}\right)$ [since $x + y < 2$]

$= x - \frac{x^2}{2}$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Case-2: $x \in \{3,4,...\}$ and $y \in [0,1]$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(x-1, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(x-1\right)$

$= \left(x-1\right)^2 = x^2 - 2x + 1$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Case-3: $x = 2$ and $y \in \left[0,1\right], fx = 1$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(1, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(1\right)$ =1

$= \psi\left(2\right) - \beta\left(2\right)$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Hence the required conditions of theorem 2.1 are satisfied and it is seen that "0" is the fixed point of f in X.

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