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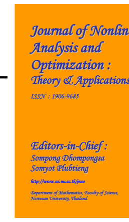

COUPLED COMMON FIXED POINT THEOREMS OF CIRIC TYPE $g-$WEAK CONTRACTIONS WITH CLR_g PROPERTY

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In this paper we define Ciric type $g-$weak contractions in the context of coupled fixed points and prove the existence of coupled common fixed points for a pair of w-compatible maps using CLR_g property. Further, we consider a pair of maps satisfying a new class of implicit relation with CLR_g property and prove the existence of coupled common fixed points. The results of Long, Rhoades and Rajovic [15] and our results are independent. Examples are provided to illustrate this phenomenon.. ...

KEYWORDS : Coupled fixed point, coupled coincidence point, coupled common fixed point, $w-$compatible maps, property (E. A), CLR_g property, implicit relation..
AMS Subject Classification: 47H10, 54H25

1. INTRODUCTION

In 2006, Bhaskar and Lakshmikantham [9] established a coupled contraction principle and proved the existence of coupled fixed points in partially ordered compete metric spaces. In 2009, Lakshmikantham and Ciric [14] introduced the concept of commuting maps, coupled coincidence points and coupled common fixed points and established coupled coincidence, coupled common fixed point theorems in partially ordered complete metric spaces. In 2010, Abbas, Khan, Radenovic [3] introduced the concept of $w-$compatible maps in the context of coupled fixed points in cone metric spaces. Recently Long, Rhoades and Rajovic [15] established coupled coincidence point theorems in complete metric spaces and cone metric spaces too. Some works in this line of research in different spaces are [3, 8, 10, 13, 22, 23, 24, 25].

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers, R is the set of all real numbers and $R_+ = [0, \infty)$.

In the following definitions, we suppose that X is a non-empty set.

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Definition 1.1. [9] An element (x, y) in $X \times X$ is called a $coupled\ fixed\ point$ of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.2. [14] An element (x, y) in $X \times X$ is called a $coupled$ $coincidence\ point$ of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 1.3. [14] An element (x, y) in $X \times X$ is called a $coupled\ common$ $fixed\ point$ of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.4. [14] The mappings $F : X \times X \to X$ and $g : X \to X$ are called $commutative$ if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Definition 1.5. [3] The mappings $F : X \times X \to X$ and $g : X \to X$ are called $w-compatible$ if $gF(x, y) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

We denote $\Phi_1 = \big\{\varphi/\varphi : R_+ \to R_+$ satisfying φ is non-decreasing and $\lim\limits_{n\longrightarrow\infty} \varphi^n(t) = 0$ for $t > 0\big\}$.

Long, Rhoades and Rajovic [15] proved the following theorem in complete metric spaces.

Theorem 1.1. [15] *Let* (X, d) *be a complete metric space. Assume that* $F : X \times X \to X,\ g : X \to X$ *are two mappings satisfying*

(H_1) : *there exists* $\varphi \in \Phi_1$ *such that*

$$d\big(F(x, y), F(u, v)\big) \le \varphi\big(M_F^g(x, y, u, v)\big) \text{ for all } x, y, u, v \in X; \tag{1.1}$$

where

$$M_F^g(x, y, u, v) = max\{d(gx, gu),\ d(gy, gv),\ d(gx, F(x, y)),\ d(gu, F(u, v)),$$
$$d(gy, F(y, x)),\ d(gv, F(v, u)),\ \tfrac{d(gx,F(u,v))+d(gu,F(x,y))}{2},\ \tfrac{d(gy,F(v,u))+d(gv,F(y,x))}{2}\},$$

(H_2) : $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a closed subset of* X.

Then (i) F *and* g *have a coupled coincidence point in* X *and*

(ii) F *and* g *have a unique common fixed point whenever* F *and* g *are* $w-$*compatible.*

Popa [16] introduced *implicit relations* and established the existence of fixed points and common fixed points in metric spaces. The importance of using an implicit relation in proving fixed point theorems is that it includes many known contractive conditions so that the known results follow as corollaries. Some works on this line of research are [4, 5, 6, 7, 17].

In 2002, Amari and Moutawakil [1] introduced the notion of property ($E.\ A$) and proved the existence of common fixed points for a pair of self maps. Many researchers [2, 11, 18] worked in this direction.

In 2011, Sintunavarat and Kumam [20] introduced a new property called *common limit in the range of* g (CLR_g) in both metric and fuzzy metric spaces and proved common fixed point theorems in fuzzy metric specs. CLR_g property never requires the closedness of the range space of g for the existence of fixed points. For more details and works on CLR_g property we refer [19, 20, 21].

Recently Jain, Tas, Sanjay Kumar and Gupta [13] extended the notation of property ($E.\ A$) and CLR_g property to the context of coupled fixed points in metric spaces and fuzzy metric spaces and proved the coupled fixed point results in fuzzy metric spaces.

Definition 1.6. [13] Let (X, d) be a metric space. Two mappings $F : X \times X \to X$ and $g : X \to X$ are said to satisfy $property$ ($E.\ A$) if there exist

two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = t_1$ and $\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = t_2$
for some $t_1, t_2 \in X$.

Definition 1.7. [13] Let (X, d) be a metric space. Two mappings $F : X \times X \to X$ and $g : X \to X$ are said to satisfy *common limit in the range of* g (CLR_g) *property* if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = gt_1$ and $\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = gt_2$
for some $t_1, t_2 \in X$.

Remark 1.8. If F and g satisfy 'property $(E.A)$ with range of g is closed' then F and g satisfy 'CLR_g property'. But its converse is not true due to the following example.

Example 1.9. Let $X = (-4, 4)$. We define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x, y) = \frac{x - y}{4},\ x, y \in X$$

and

$$gx = \frac{x}{2},\ x \in X.$$

Here $g(X) = (-2, 2)$ is not a closed set. Now we choose two sequences $\{x_n\}$ and $\{y_n\}$ in X by
$x_n = -2 - \frac{1}{n}$ and $y_n = 2 + \frac{1}{n},\ n = 1, 2, 3...$. Hence

$$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = -1 = g(-2)$$

and

$$\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = 1 = g(2).$$

Thus the pair (F, g) satisfy CLR_g property.

Hence CLR_g property is more general than property $(E.A)$ with $g(X)$ is closed.

In this paper, we prove a coupled common fixed point theorem for Ciric type $g-$weak contractions by using CLR_g property. Further, we consider a pair of maps satisfying a new class of implicit relation with CLR_g property and prove the existence of coupled common fixed points.

In the following, we define
$\Phi = \{\varphi/\varphi : R_+ \to R_+$ satisfying φ is continuous and $\varphi(t) = 0$ if and only if $t = 0\}$. Here we note that the classes of functions Φ_1 and Φ are independent, in the sense that neither Φ_1 is contained in Φ nor Φ is contained in Φ_1. We illustrate it in the following examples.

Example 1.10. $\varphi = [0, +\infty) \to [0, +\infty)$ defined by $\varphi(t) = \begin{cases} t^2 & if\ t \in [0, 1] \\ \frac{1}{t} & if\ t \in (1, \infty). \end{cases}$

Clearly $\varphi \in \Phi$, but φ is not an increasing function. Hence φ does not belong to Φ_1.

Example 1.11. $\varphi = [0, +\infty) \to [0, +\infty)$ defined by $\varphi(t) = \begin{cases} \frac{t^2}{12} & if\ t \in [0, 1] \\ \frac{t}{10} & if\ t \in (1, \infty). \end{cases}$

Clearly $\varphi \in \Phi_1$, but φ is not a continuous function. Hence φ does not belong to Φ.

2. PRELIMINARIES

We define Ciric type $g-$weak contractions and a class of implicit relation in the context of coupled fixed points.

Definition 2.1. Let (X,d) be a metric space. Let $F : X \times X \to X,\ g : X \to X$ be two maps of a metric space X. We say that F is a *Ciric type g-weak contraction map* if there exists $\varphi \in \Phi$ such that

$$d(F(x,y),F(u,v)) \leq M(x,y,u,v) - \varphi(M(x,y,u,v)) \text{ for all } x,y,u,v \in X; \quad (2.1)$$

where

$$M(x,y,u,v) = max\{d(gx,gu),\ d(gy,gv),\ d(gx,F(x,y)),\ d(gy,F(y,x)), d(gu,F(u,v)),$$
$$d(gv,F(v,u)),\ d(gx,F(u,v)), d(gy,F(v,u)),\ d(gu,F(x,y)),\ d(gv,F(y,x))\}.$$

Remark 2.2. Suppose that F and g satisfy the inequality (1.1) with $\varphi \in \Phi_1$. If φ is continuous then F is a Ciric type $g-$ weak contraction. But its converse need not be true (Example 2.3).

For, we assume that (1.1) holds.

$$i.e.,\ d(F(x,y),F(u,v)) \leq \varphi(M_F^g(x,y,u,v))$$
$$\leq M(x,y,u,v) - (I-\varphi)M(x,y,u,v)$$
$$= M(x,y,u,v) - \phi_\varphi(M(x,y,u,v)),$$

where $\phi_\varphi = I - \varphi$ and it is clear that $\phi_\varphi(t) = 0$ if and only if $t = 0$.

Example 2.3. Let $X = [-1,1)$ with the usual metric. We define $F : X \times X \to X$ and $g : X \to X$ by $F(x,y) = \begin{cases} \frac{1}{4} & if\ x \geq y \\ -\frac{1}{4} & if\ x < y; \end{cases}$ and $gx = \begin{cases} \frac{1}{2} & if\ x \neq 0 \\ 0 & if\ x = 0. \end{cases}$

We define $\varphi(t) = \frac{1}{8}t,\ t \geq 0$. Clearly $\varphi \in \Phi$ and F is a Ciric type $g-$weak contraction.

But for $x = 1,\ y = u = 0$ and $v = 1$, we have

$d(F(x,y),F(u,v) = \frac{1}{2} \not\leq \varphi(max\{\frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{2}\}) = \varphi(\frac{1}{2})$ for any $\varphi \in \Phi$,

since $\varphi(t) < t$ for $t > 0$.

Hence the inequality (1.1) fails to hold.

Definition 2.4. Let Λ be the set of all continuous functions $T : R_+^{11} \to R$ satisfying the following conditions:

(T_1) : there exists a mapping $f : R_+ \to R_+,\ f(t) < t$ for $t > 0$ such that

$T(u,0,0,0,0,v_1,v_2,v_1,v_2,0,0) \leq 0$ for $u > 0$ or

$T(u,v_1,v_2,0,0,0,0,v_1,v_2) \leq 0$ for $u > 0$

implies that $u \leq f(max\{v_1,v_2\})$.

$(T_2) : T(u,0,0,u,u,0,0,0,0,0,u) > 0$ for $u > 0$.

Example 2.5. $T(t_1,t_2,...,t_{11}) = t_1 - k\ max\{t_2,t_3\}$, where $k \in [0,1)$.

Let $T(u,v_1,v_2,0,0,0,0,v_1,v_2,v_1,v_2) = u - k\ max\{v_1,v_2\} \leq 0$

$i.e.,\ u \leq k\ max\{v_1,v_2\}$.

Thus $u \leq f(max\{v_1,v_2\})$ with $f(t) = kt$. Hence T_1 satisfied.

Also $T(u,0,0,u,u,0,0,0,0,0,u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.6. $T(t_1, t_2, ..., t_{11}) = t_1 - \varphi(max\{t_2, t_3, t_4, t_5, t_6, t_7, \frac{t_8+t_9}{2}, \frac{t_{10}+t_{11}}{2}\})$
with $\varphi(t) < t$ for $t > 0,\ \varphi(t) = 0$ if and only if $t = 0$ and φ is continuous.
Let $u > 0$ and $T(u, v_1, v_2, 0, 0, 0, 0, v_1, v_2, v_1, v_2) = u - \varphi\big(max\{v_1, v_2\}\big) \leq 0.$
Hence $u \leq f(max\{v_1, v_2\})$ with $f = \varphi$.
Also $T(u, 0, 0, u, u, 0, 0, 0, 0, 0, u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.7. $T(t_1, t_2, ..., t_{11}) = t_1 - \alpha\frac{t_8t_9+t_{10}t_{11}}{1+t_2+t_3+t_4+t_5+t_6+t_7}$ where $0 \leq \alpha < 1$.
Let $u > 0,\ T(u, 0, 0, 0, 0, v_1, v_2, v_1, v_2, 0, 0) = u - \alpha\frac{v_1v_2}{1+v_1+v_2} \leq 0.$
$i.e.,\ u \leq \alpha\frac{v_1v_2}{1+v_1+v_2} \leq \alpha\ max\{v_1, v_2\}$. Hence $u \leq f(max\{v_1, v_2\})$ with $f(t) = \alpha$ for all $t \geq 0$. Also $T(u, 0, 0, u, u, 0, 0, 0, 0, 0, u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.8. $T(t_1, t_2, ..., t_{11}) = t_1 - (a_1t_2 + a_2t_3 + \cdots + a_{10}t_{11})$
where $\sum_{i=0}^{10} a_i < 1$. Let $u > 0$,
$T(u, 0, 0, 0, 0, v_1, v_2, v_1, v_2, 0, 0) = u - [(a_6 + a_8)v_1 + (a_7 + a_9)v_2] \leq 0.$
$i.e.,\ u \leq (a_6 + a_8)max\{v_1, v_2\} + (a_7 + a_9)max\{v_1, v_2\}$
$= (a_6 + a_7 + a_8 + a_9)max\{v_1, v_2\}.$
Thus $u \leq f(max\{v_1, v_2\})$ with $f(t) = (a_6 + a_7 + a_8 + a_9)t$.
Also $T(u, 0, 0, u, u, 0, 0, 0, 0, 0, u) = u - (a_3 + a_4 + a_{11})u > 0$ for $u > 0$.
Hence $T \in \Lambda$.

3. MAIN RESULTS

The following is the main result of this section.

Theorem 3.1. *Let $(X,\ d)$ be a metric space and $F : X \times X \to X,\ g : X \to X$ be two maps, the pair (F, g) satisfy CLR_g property and F is a Ciric type $g-$weak contraction map then F and g have a coupled coincidence point. Further, F and g have a unique coupled common fixed point provided F and g are w-compatible.*

Proof. Since F and g satisfy CLR_g property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = gx$ and $\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = gy$ for some $x, y \in X$.
Now, we prove that $gx = F(x, y)$ and $gy = F(y, x)$.
Assume that $d(gx, F(x, y)) > 0$ or $d(gy, F(y, x)) > 0$.
Now, we consider

$$\begin{aligned} d(gx, F(x, y)) &\leq d(gx, F(x_n, y_n)) + d\big(F(x_n, y_n), F(x, y)\big) \\ &\leq d(gx, F(x_n, y_n)) + M(x_n, y_n, x, y) - \varphi(M(x_n, y_n, x, y)) \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(x_n, y_n, x, y) = max\{&d(gx_n, gx),\ d(gy_n, gy),\ d(gx_n, F(x_n, y_n)), d(gy_n, F(y_n, x_n)), \\ &d(gx, F(x, y)), d(gy, F(y, x)), d(gx_n, F(x, y)),\ d(gy_n, F(y, x)), \\ &d(gx, F(x_n, y_n)), d(gy, F(y_n, x_n))\}. \end{aligned}$$

On taking limits as $n \to \infty$, in $M(x_n, y_n, x, y)$, we get
$\lim_{n\longrightarrow\infty} M(x_n, y_n, x, y) = max\{d(gx, F(x, y)),\ d(gy, F(y, x))\}.$
Now, on taking limits as $n \to \infty$ in (3.2), we get
$d(gx, F(x, y)) \leq max\{d(gx, F(x, y)),\ d(gy, F(y, x))\}$

$$-\varphi(max\{d(gx, F(x,y)),\ d(gy, F(y,x))\})$$
$$< max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}. \quad (3.3)$$

Similarly we get,

$$d(gy, F(y,x)) < max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}. \quad (3.4)$$

Hence from (3.3) and (3.4) we get

$max\{d(gx, F(x,y)),\ d(gy, F(y,x))\} < max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}$,
a contradiction.

Hence $gx = F(x,y)$ and $gy = F(y,x)$.

Thus (x,y) is a coupled coincidence point of F and g.

Let (x,y) and (x^*, y^*) be two coupled coincidence points of F and g.

Now, we prove that $gx = gx^*$ and $gy = gy^*$.

We assume that $d(gx, gx^*) > 0$ or $d(gy, gy^*) > 0$.
Now, we consider

$$\begin{aligned} d(gx, gx^*) &= d(F(x,y), F(x^*, y^*)) \\ &\leq M(x,y,x^*,y^*) - \varphi(M(x,y,x^*,y^*)) \\ &< M(x,y,x^*,y^*) \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} M(x,y,x^*,y^*) = max\{&d(gx, gx^*),\ d(gy, gy^*),\ d(gx, F(x,y)),\ d(gy, F(y,x)), \\ &d(gx^*, F(x^*,y^*)),\ d(gy^*, F(y^*,x^*)),\ d(gx, F(x^*,y^*)), \\ &d(gy, F(y^*,x^*)),\ d(gx^*, F(x,y)),\ d(gy^*, F(y,x))\} \\ = max\{&d(gx, gx^*),\ d(gy, gy^*)\}. \end{aligned}$$

Similarly we get

$$d(gy, gy^*) < max\{d(gx, gx^*),\ d(gy, gy^*)\}. \quad (3.6)$$

Henc, from (3.5) and (3.6), we get

$max\{d(gx, gx^*),\ d(gy, gy^*)\} < max\{d(gx, gx^*),\ d(gy, gy^*)\}$,
a contradiction.

Hence $gx = gx^*$ and $gy = gy^*$. (3.7)
Now, we prove that $gx = gy^*$ and $gy = gx^*$.

We assume that $d(gx, gy^*) > 0$ or $d(gy, gx^*) > 0$.
Consider

$$\begin{aligned} d(gx, gy^*) &= d(F(x,y), F(y^*, x^*)) \\ &\leq M(x,y,y^*,x^*) - \varphi(M(x,y,y^*,x^*)) \\ &< M(x,y,y^*,x^*) \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} M(x,y,y^*,x^*) = max\{&d(gx, gy^*),\ d(gy, gx^*),\ d(gx, F(x,y)),\ d(gy, F(y,x)), \\ &d(gy^*, F(y^*,x^*)),\ d(gx^*, F(x^*,y^*)),\ d(gx, F(y^*,x^*)), \\ &d(gy, F(x^*,y^*)),\ d(gy^*, F(x,y)),\ d(gx^*, F(y,x))\} \\ = max\{&d(gx, gy^*),\ d(gy, gx^*)\}. \end{aligned}$$

Similarly we get

$$d(gy, gx^*) < max\{d(gx, gy^*),\ d(gy, gx^*)\}. \quad (3.9)$$

Hence, from (3.8) and (3.9), we get

$max\{d(gx, gy^*),\ d(gy, gx^*)\} < max\{d(gx, gy^*),\ d(gy, gx^*)\}$,
a contradiction. Hence

$gx = gy^*$ and $gy = gx^*$. (3.10)

Thus, from (3.7) and (3.10), we get

$$gx = gx^* = gy = gy^*. \tag{3.11}$$

Let (x, y) be a coupled coincidence point of F and g, hence $gx = F(x, y)$ and $gy = F(y, x)$. Let us take $u = gx$ and $v = gy$. Since F and g are $w-$compatible, we have

$$gu = ggx = gF(x, y) = F(gx, gy) = F(u, v)$$

and

$$gv = ggy = gF(y, x) = F(gy, gx) = F(v, u).$$

Hence (u, v) is a coupled coincidence point, hence from (3.7) we have

$gu = gx$ and $gv = gy$. Thus

$u = gx = gu = F(u, v)$ and $v = gy = gv = F(v, u)$. (3.12)

Hence (u, v) is a coupled common fixed point.

And from (3.11) we have $u = v$.

Let (u_1, v_1) be another coupled common fixed point of F and g.

i.e., $u_1 = gu_1 = F(u_1, v_1)$ and $v_1 = gv_1 = F(v_1, u_1)$ (3.13)

From (3.11), (3.12) and (3.13), we get

$u_1 = gu_1 = gu = u$ and $v_1 = gv_1 = gv = v$.

Hence coupled common fixed point is unique. □

Corollary 3.1. *Let $(X,\ d)$ be a metric space and $F : X \times X \to X,\ g : X \to X$ be two maps, the pair (F, g) satisfy property (E.A), $g(X)$ is closed and F is a Ciric type $g-$weak contraction map then F and g have a coupled coincidence point. Further, F and g have a unique coupled common fixed point provided F and g are w-compatible.*

Proof. Since the pair (F, g) satisfies property ($E.A$) and $g(X)$ is closed, by Remark 1.8 we have F and g satisfy CLR_g property and hence by Theorem 3.1 the conclusion of this corollary follows. □

Example 3.2. Let $X = [0, 1)$ with the usual metric.

We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{3} & if\ x, y \in [0, \frac{1}{3})\ with\ x \geq y \\ \frac{1}{2} & if\ x, y \in [\frac{1}{3}, 1)\ with\ x \geq y \\ 0 & otherwise; \end{cases}$$

and $g : X \to X$ defined by $gx = \begin{cases} x & if\ x \in [0, \frac{1}{3}) \\ \frac{9}{10} & if\ x \in [\frac{1}{3}, 1). \end{cases}$

Now, we choose the sequences $\{x_n\}$ and $\{y_n\}$ in X by $x_n = \frac{1}{n+3}$ and $y_n = \frac{1}{3n+1}$, $n = 1, 2, 3, ...,$ then

$\lim\limits_{n \longrightarrow \infty} F(x_n, y_n) = \lim\limits_{n \longrightarrow \infty} g(x_n) = 0 = g0$ and

$\lim\limits_{n \longrightarrow \infty} F(y_n, x_n) = \lim\limits_{n \longrightarrow \infty} g(y_n) = 0 = g0.$

Hence the pair (F, g) satisfy CLR_g property.

We define $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{1}{8}t,\ t \geq 0$; here we observe that $t - \varphi(t)$ is an increasing function.

Now, we consider the following cases to check the inequality (2.1).

First we consider the case $x, y, u, v \in [0, \frac{1}{3})$.
Now, we have the following four subcases.

Subcase (i) : $x \geq y$ and $u \geq v$.
Now

$$d(F(x,y),\ F(u,v)) \leq \begin{cases} \frac{1}{3}[(x-u)+(v-y)] & if\ x \geq u,\ v \geq y \\ \frac{1}{3}[(x-u)+(y-v)] & if\ x \geq u,\ v < y \\ \frac{1}{3}[(u-x)+(v-y)] & if\ x < u,\ v \geq y \\ \frac{1}{3}[(u-x)+(y-v)] & if\ x < u,\ v < y. \end{cases} \tag{3.14}$$

and
$M(x,y,u,v) = max\{|x-u|,\ |y-v|,\ \frac{2x+y}{3},\ y,\ \frac{2u+v}{3},\ v,\ |\frac{3x-u+v}{3}|,\ y,\ |\frac{3u-x+y}{3}|,\ v\}$.

$$Hence\ (3.14) \leq \begin{cases} \frac{7}{8}[\frac{3x-u+v}{3}] & whenever\ (x \geq u, v \geq y)\ or\ (x \geq u, v < y) \\ \frac{7}{8}[\frac{3u-x+y}{3}] & whenever\ (x < u, v \geq y)\ or\ (x < u, v < y) \end{cases}$$
$$= M(x,y,u,v) - \varphi(M(x,y,u,v)).$$

Subcase (ii) : $x \geq y,\ u < v$.
In this subcase, we have

$$d(F(x,y),\ F(u,v)) = \frac{1}{3}(x-y) \leq \begin{cases} \frac{7}{8}x & if\ max\{x,y,u,v\} = x \\ \frac{7}{8}v & if\ max\{x,y,u,v\} = v \end{cases}$$
$$= M(x,y,u,v) - \varphi(M(x,y,u,v))$$

where
$M(x,y,u,v) = max\{|x-u|,\ |y-v|,\ \frac{2x+y}{3}, y, u, \frac{2v+u}{3}, x,\ |y - \frac{v-u}{3}|,\ |u - \frac{x-y}{3}|,\ v\}$.

Subcase (iii) : $x < y,\ u \geq v$.
By symmetry in the inequality (2.1), it is clear that the inequality (2.1) holds as in Sub case (ii).

Subcase (iv) : $x < y,\ u < v$. Inequality (2.1) holds trivially.

In the following cases, $i.e.,$
(i) $x, y, u, v \in [\frac{1}{3}, 1)$ or $u, v \in [0, \frac{1}{3})$ and $x, y \in [\frac{1}{3}, 1)$ with $x \geq y,\ u < v$;
(ii) $u \in [0, \frac{1}{3})$ and $x, y, v \in [\frac{1}{3}, 1)$ with $x \geq y$;
(iii) $v \in [0, \frac{1}{3})$ and $x, y, u \in [\frac{1}{3}, 1)$ with $x \geq y$.
In these cases, we have
$d\big(F(x,y),\ F(u,v)\big) = \frac{1}{2} \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v))$,
where $M(x,y,u,v) = \frac{9}{10}$.

Now we consider the following cases:
(i) $x, y \in [0, \frac{1}{3})$ and $u, v \in [\frac{1}{3}, 1)$ with $x \geq y,\ u < v$;
(ii) $x, y, u \in [0, \frac{1}{3})$ and $v \in [\frac{1}{3}, 1)$ with $x \geq y$;
(iii) $x, y, v \in [0, \frac{1}{3})$ and $u \in [\frac{1}{3}, 1)$ with $x \geq y$.
In these cases, we have
$d\big(F(x,y),\ F(u,v)\big) = \frac{1}{3}(x-y) \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v))$,
where $M(x,y,u,v) = \frac{9}{10}$.

Also, we consider the following case :
$x, y \in [0, \frac{1}{3})$ and $u, v \in [\frac{1}{3}, 1)$ with $x \geq y,\ u \geq v$ then
$d\big(F(x,y),\ F(u,v)\big) = \frac{3-2(x-y)}{6} \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v)),$
where $M(x,y,u,v) = \frac{9}{10}$.

Further, we have the following cases :
(i) $x < y,\ u < v;\ x, y, u, v \in X;$
(ii) $u,\ v$ are in different intervals and $x,\ y$ are in different intervals;
(iii) $x,\ y$ are in different intervals with $u < v$;
(iv) $u,\ v$ are in different intervals with $x < y$.
In these cases, we have $d\big(F(x,y),\ F(u,v)\big) = 0$.

Since the inequality (2.1) is symmetric, the other cases $i.e.,\ x$ is replaced by u and y is replaced by v also hold.
Now at $(x,y) = (0,0)$ we have $gx = F(x,y),\ gy = F(y,x)$ and $gF(x,y) = F(gx, gy)$. Hence F and g satisfy all the hypotheses of Theorem 3.1 and $(0,0)$ is a coupled common fixed point. In fact $(0,0)$ is unique.

Remark 3.3. In Theorem 3.1, we considered Ciric type $g-$ weak contraction which is more general than the inequality (1.1) and relaxed the condition $F(X \times X) \subseteq g(X)$ but imposed a condition namely φ *is continuous on* R_+. Thus Theorem 3.1 is a partial generalization of Theorem 1.1.

Theorem 3.2. *Let* (X,d) *be a metric space,* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that*

(i) F *and* g *satisfy* CLR_g *property,*
(ii) *there exists* $T \in \Lambda$ *such that*

$$\begin{aligned} &Td(F(x,y),F(u,v)),\ d(gx,gu), d(gy,gv), d(gx,F(x,y)), d(gy,F(y,x)),\\ &\quad d(gu,F(u,v)), d(gv,F(v,u)), d(gx,F(u,v)),\ d(gy,F(v,u)),\\ &\quad d(gu,F(x,y)),\ d(gv,F(y,x))) \leq 0 \ \textit{for all}\ x,y,u,v \in X. \end{aligned} \tag{3.15}$$

Then (a) *the pair* (F,g) *has a coupled fixed point and*

(b) *the pair* (F,g) *has a unique coupled common fixed point provided it is* $w-$ *compatible.*

Proof. By (i), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\longrightarrow\infty} F(x_n,y_n) = \lim_{n\longrightarrow\infty} g(x_n) = gx \text{ and}$$

$$\lim_{n\longrightarrow\infty} F(y_n,x_n) = \lim_{n\longrightarrow\infty} g(y_n) = gy \quad \text{for some } x,y \in X.$$

Now, we prove that $gx = F(x,y)$ and $gy = F(y,x)$.

We assume that $d(gx, F(x,y)) > 0$ or $d(gy, F(y,x)) > 0$.
Now, we consider

$$\begin{aligned} &T(d(F(x_n,y_n),F(x,y)),\ d(gx_n,gx),\ d(gy_n,gy),\ d(gx_n,F(x_n,y_n)),\\ &\quad d(gy_n,F(y_n,x_n)),\ d(gx,F(x,y)), d(gy,F(y,x)),\ d(gx_n,F(x,y)),\\ &\quad d(gy_n,F(y,x)),\ d(gx,F(x_n,y_n)),\ d(gy,F(y_n,x_n))) \leq 0. \end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\begin{aligned} &T\big(d(gx,F(x,y)),0,0,0,0,\ d(gx,F(x,y)),\ d(gy,F(y,x)),\ d(gx,F(x,y)),\\ &\qquad d(gy,F(y,x)),0,0\big) \leq 0. \end{aligned}$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gx, F(x, y)) \leq f(max\{d(gx, F(x, y)),\ d(gy, F(y, x))\}). \tag{3.16}$$

Again we consider

$$T\big(d(F(y_n, x_n), F(y, x)),\ d(gy_n, gy),\ d(gx_n, gx),\ d(gy_n, F(y_n, x_n)),$$
$$d(gx_n, F(x_n, y_n)),\ d(gy, F(y, x)), d(gx, F(x, y)),\ d(gy_n, F(y, x)),$$
$$d(gx_n, F(x, y)),\ d(gy, F(y_n, x_n)),\ d(gx, F(x_n, y_n))) \leq 0.$$

On taking limits as $n \to \infty$, we get

$$T\big(d(gy, F(y, x)), 0, 0, 0, 0,\ d(gy, F(y, x)),\ d(gx, F(x, y)),\ d(gy, F(y, x)),$$
$$d(gx, F(x, y)), 0, 0\big) \leq 0.$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gy, F(y, x)) \leq f\big(max\{d(gx, F(x, y)),\ d(gy, F(y, x))\}\big). \tag{3.17}$$

From (3.16) and (3.17) we get

$$max\{d(gx, F(x, y)), d(gy, F(y, x))\} \leq f(max\{d(gx, F(x, y)), d(gy, F(y, x))\}),$$
$$< max\{d(gx, F(x, y)), d(gy, F(y, x))\},$$

a contradiction.

Hence $gx = F(x, y)$ and $gy = F(y, x)$.

Thus (x, y) is a coupled fixed point of F and g.

Let (x, y) and (x^*, y^*) be two coupled coincidence points of F and g.

Now, we prove that $gx = gx^*$ and $gy = gy^*$.

We assume that $d(gx, gx^*) > 0$ and $d(gy, gy^*) > 0$.

Now, we consider

$$T(d(F(x, y), F(x^*, y^*)),\ d(gx, gx^*),\ d(gy, gy^*),\ d(gx, F(x, y)),\ d(gy, F(y, x))$$
$$d(gx^*, F(x^*, y^*)),\ d(gy^*, F(y^*, x^*)),\ d(gx, F(x^*, y^*)),\ d(gy, F(y^*, x^*))$$
$$d(gx^*, F(x, y)),\ d(gy^*, F(y, x))) \leq 0.$$ Hence

$$T\big(d(gx, gx^*),\ d(gx, gx^*),\ d(gy, gy^*), 0, 0, 0, 0,\ d(gx, gx^*),\ d(gy, gy^*),$$
$$d(gx^*, gx),\ d(gy, gy^*),\ d(gx^*, gx), d(gy^*, gy)\big) \leq 0.$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gx, gx^*) \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big). \tag{3.18}$$

Similarly it follows that

$$d(gy, gy^*) \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big). \tag{3.19}$$

From (3.18) and (3.19) we have

$$max\{d(gx, gx^*),\ d(gy, gy^*)\} \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big)$$
$$< max\{d(gx, gx^*),\ d(gy, gy^*)\},$$

a contradiction.

Hence $gx = gx^*$ and $gy = gy^*$.

Now we prove that $gx = gy^*$ and $gy = gx^*$.

We assume that either $d(gx, gy^*) > 0$ or $d(gy, gx^*) > 0$.

Now, we consider

$$T(d(F(x, y), F(y^*, x^*)),\ d(gx, gy^*),\ d(gy, gx^*),\ d(gx, F(x, y)),\ d(gy, F(y, x)),$$
$$d(gy^*, F(y^*, x^*)),\ d(gx^*, F(x^*, y^*)),\ d(gx, F(y^*, x^*)),\ d(gy, F(x^*, y^*))$$

$$d(gy^*, F(x,y)),\ d(gx^*, F(y,x))) \leq 0.$$

$$T(d(gx,gy^*),\ d(gx,gy^*),\ d(gy,gx^*), 0,0,0,0,\ d(gx,gy^*), d(gy,gx^*),$$
$$d(gy^*,gx),\ d(gx^*,gy)) \leq 0.$$ Hence

$$T(d(gx,gy^*),\ d(gx,gx^*),\ d(gy,gy^*), 0,0,0,0,\ d(gx,gx^*),\ d(gy,gy^*),$$
$$d(gx^*,gx),\ d(gy,gy^*),\ d(gx^*,gx), d(gy^*,gy)) \leq 0.$$

Hence from condition (T_1) of Definition 2.4, we get

$$d(gx,gy^*) \leq f\big(max\{d(gx,gy^*),\ d(gy,gx^*)\}\big). \tag{3.20}$$

Similarly it follows that

$$d(gy,gx^*) \leq f\big(max\{d(gx,gy^*),\ d(gy,gx^*)\}\big). \tag{3.21}$$

From (3.20) and (3.21) we have

$$max\{d(gx,gy^*),\ d(gy,gx^*)\} \leq f(max\{d(gx,gy^*),\ d(gy,gx^*)\})$$
$$< max\{d(gx,gy^*),\ d(gy,gx^*)\},$$

a contradiction.

Hence $gx = gy^*$ and $gy = gx^*$.

Thus $gx = gx^* = gy = gy^*$. (3.22)

Let (x,y) be a coupled coincidence point and take $u = gx,\ v = gy$.

Hence $u = gx = F(x,y)$ and $v = gy = F(y,x)$.

Since the pair (F,g) is $w-$compatible, we have

$$gu = ggx = gF(x,y) = F(gx,gy) = F(u,v)$$

and

$$gv = ggy = gF(y,x) = F(gy,gx) = F(v,u)$$

so that (u,v) is a coupled coincidence point. Hence $gu = gx$ and $gy = gv$.

Thus $u = gx = gu = F(u,v)$ and $v = gy = gv = F(v,u)$.

Hence (u,v) is a coupled common fixed point.

Moreover from (3.22) we get $u = v$.

Now we suppose that (u_1,v_1) be another coupled common fixed point

$i.e.,\ u_1 = gu_1 = F(u_1,v_1)$ and $v_1 = gv_1 = F(v_1,u_1)$.

From (3.22) we get

$u_1 = gu_1 = gu = u$ and $v_1 = gv_1 = gv = v$.

Hence coupled fixed point is unique. □

Corollary 3.4. *Let (X,d) be a metric space, $F: X \times X \to X$ and $g: X \to X$ be two mappings such that*

(i) *F and g satisfy property (E.A),*

(ii) *g(X) is a closed subset of X,*

(iii) *there exists $T \in \Lambda$ such that*

$$T(d(F(x,y),F(u,v)),\ d(gx,gu), d(gy,gv), d(gx,F(x,y)), d(gy,F(y,x)), d(gu,F(u,v),$$
$$d(gv,F(v,u)), d(gx,F(u,v)),\ d(gy,F(v,u)), d(gu,F(x,y)),\ d(gv,F(y,x))) \leq 0$$

for all $x,y \in X$,

then (a) the pair (F,g) has a coupled fixed point

(b) the pair (F,g) has a unique coupled common fixed point provided it is $w-$ compatible.

Example 3.5. Let $X = [0,1)$ with the usual metric. We define $T: R_+^{11} \to R$ by $T(t_1,t_1,...,t_{11}) = t_1 - h\ max\{t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9,t_{10},t_{11}\}$, where $h = \frac{2}{3}$.

Clearly $T \in \Lambda$. Now, we define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = \begin{cases} \frac{2x-y}{3} & if\ x \geq y \\ 0 & if\ x < y; \end{cases} \qquad gx = \begin{cases} x & if\ x \in [0, \frac{1}{5}) \\ \frac{9}{10} & if\ x \in [\frac{1}{5}, 1). \end{cases}$$

Now, we choose the sequences $\{x_n\}$ and $\{y_n\}$ in X by
$x_n = \frac{1}{n+5}$ and $y_n = \frac{1}{2(n+2)}$, $n = 1, 2, 3...$, then

$$\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} g(x_n) = 0 = g0$$

and

$$\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} g(y_n) = 0 = g0\ .$$

We define $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{t}{9}$, $t \geq 0$; here we observe that $t - \varphi(t)$ is an increasing function. And at $(x, y) = (0, 0)$ we have $gF(x, y) = F(gx, gy)$. Here we note that F and g satisfy the inequality (3.15). Hence F and g satisfy all the hypotheses of Theorem 3.2. and $(0, 0)$ is a coupled common fixed point. Moreover $(0, 0)$ is unique.

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