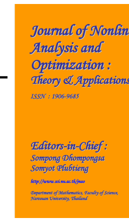

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SOME SUBORDINATION RESULTS ASSOCIATED WITH GENERALIZED RUSCHEWEYH DERIVATIVES

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ABSTRACT. In this paper, we consider an unified class of functions of complex order associated with generalized Ruscheweyh derivative. We obtain a necessary and sufficient condition for functions to be in this class.

KEYWORDS : Starlike functions of complex order; Convex functions of complex order; Subordination; generalized Ruscheweyh derivative.
AMS Subject Classification: 30C45.

1. INTRODUCTION

Let A be the class of all analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U.

A function $f \in A$ is subordinate to an univalent function $g \in A$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let Ω be the family of analytic functions $w(z)$ in the unit disk U satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in U$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(w(z))$.

Let $\phi(z)$ be an analytic function with positive real part on U and $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Ma and Minda [3] introduced and studied the class $S^*(\phi)$, consists of functions $f \in S$ for which

$$\tfrac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in U).$$

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Recently, Ravichandran et al.[5] defined classes related to the class of starlike functions of complex order as follows:

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}(\tfrac{zf'(z)}{f(z)} - 1) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [3], Shanmugam and Sivasubramanian [7] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\tfrac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha zf'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the conditions mentioned in Definition 1.1. Kamali et al.[2] introduced and studied a new class of functions $f \in T$ for which

$$Re(\tfrac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)}) > \beta,\ 0 \leq \alpha < 1, 0 \leq \beta < 1.$$

Shanmugum et. al.[8] remarked that the class of functions T is the familiar class of functions introduced and studied by Silverman [10]. In a later investigation, this particular class introduced by Kamali and Akbulut was generalized by Shanmugum et al. [9]. Shanmugum et. al.[8] introduced a more general class of complex order $M[b, \alpha](\phi)$ defined as follows:

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M[b, \alpha](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}(\tfrac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1) \prec \phi(z),\ 0 \leq \alpha < 1.$$

Clearly,

$$M[b, 0](\phi) \equiv C_b(\phi).$$

where the class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)} \prec \phi(z).$$

In the present paper we shall need a recent generalization of the Ruscheweyh derivative which was introduced in [1].

Let $f \in A$, $\lambda \geq 0$ and $m \in \mathbb{R}$, $m > -1$, then we consider

$$D_\lambda^m f(z) = \tfrac{z}{(1-z)^{m+1}} * D_\lambda f(z), \quad z \in U,$$

where $D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad z \in U.$

If $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U$ we obtain the power series expansion of the form

$$D_\lambda^m f(z) = z + \textstyle\sum_{n=2}^{\infty} [1 + (n-1)\lambda] \tfrac{(m+1)_{n-1}}{(1)_{n-1}} a_n z^n, \quad z \in U,$$

where $(a)_n$ is the Pochhammar symbol, given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)(a+2)...(a+n-1) & \text{for } n \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

In the case $m \in \mathbb{N} = \{1, 2, 3, ...\}$, we have

$$D_\lambda^m f(z) = \frac{z(z^{m-1}D_\lambda f(z))^{(m)}}{m!}, \quad z \in U,$$

and for $\lambda = 0$ we obtain the mth Ruscheweyh derivative introduced in [6], $D_0^m = D^m$,

Since

$$\begin{aligned} \frac{(m+1)_{n-1}}{(1)_{n-1}} &= \frac{\Gamma(m+1+n-1)\Gamma(1)}{\Gamma(m+1)\Gamma(1+n-1)} \\ &= \frac{\Gamma(m+n)}{\Gamma(m+1)\Gamma(n)} \\ &= \frac{(m+n-1)!}{m!(n-1)!}. \end{aligned}$$

And

$$\begin{aligned} \sigma(m,n) &= \binom{m+n-1}{m} \\ &= \frac{(m+n-1)!}{m!(n-1)!}. \end{aligned}$$

So, we get

$$\sigma(m,n) = \frac{(m+1)_{n-1}}{(1)_{n-1}}$$

Hence

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]\sigma(m,n)a_n z^n, \quad z \in U,$$

So in this paper we introduce a more general class of complex order $H[b,\alpha,m,\lambda](\phi)$ which we define below.

Definition 1.3. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\alpha,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b}\left(\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right) \prec \phi(z),$$
$$0 \leq \alpha < 1, \lambda \geq o, m > -1.$$

Clearly,

$$H[b,\alpha,m,\lambda](\phi) \equiv M[b,\alpha](\phi)$$

where $m = \lambda = 0$, [8].

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $H[b,\alpha,m,\lambda](\phi)$ which we have defined. The motivation of this paper is to generalize the results obtained by Shanmugum et. al.[8].

2. PRELIMINARIES AND NOTATIONS

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [5] *Let ϕ be a convex function defined on U, $\phi(0) = 1$. Define $F(z)$ by*

$$F(z) = z\, exp\left(\int_0^z \frac{\phi(x)-1}{x}dx\right). \tag{2.1}$$

Let $q(z) = 1 + c_1 z + ...$ *be analytic in* U. *Then*

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z), \tag{2.2}$$

if and only if for all $|s| \le 1$ *and* $|t| \le 1$, *we have*

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.3}$$

Lemma 2.2 ([4], Corrolary 3.4h.1,p.135). *Let* $q(z)$ *be univalent in* U *and let* $\varphi(z)$ *be analytic in a domain containing* $q(U)$. *If* $\frac{zq'(z)}{\varphi(q(z))}$ *is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

implies that $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

3. MAIN RESULTS

3.1. **Subordination Results.** Applying Lemma 2.1 we have

Theorem 3.1. *Let* $\phi(z)$ *and* $F(z)$ *be as in Lemma 2.1. The function* $f \in H[b, \alpha, m, \lambda](\phi)$ *if and only if for all* $|s| \le 1$ *and* $|t| \le 1$, *we have*

$$(\frac{s[\alpha z^2 (D_\lambda^m f)''(tz) + z(D_\lambda^m f)'(tz)]}{t[\alpha z^2 (D_\lambda^m f)''(sz) + z(D_\lambda^m f)'(sz)]})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.1}$$

Proof. Define the function $p(z)$ by

$$p(z) = (\frac{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z})^{\frac{1}{b}}. \tag{3.2}$$

By taking logarithmic derivative of $p(z)$ given by (3.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3 (D_\lambda^m f)'''(z) + (1+2\alpha) z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\}. \tag{3.3}$$

Now, by definition 1.3

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\{\frac{\alpha z^3 (D_\lambda^m f)'''(z) + (1+2\alpha) z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z), \quad (0 \le \alpha < 1). \tag{3.4}$$

Then applying Lemma 2.1 we get the result.
This completes the proof of Theorem 3.1. □

Putting $\lambda = 0$ in Theorem 3.1. Then we have the Ruscheweyh derivative and we get the following new result:

Corollary 3.1. *Let* $\phi(z)$ *and* $F(z)$ *be as in Lemma 2.1. The function* $f \in H[b, \alpha, m,](\phi)$ *if and only if for all* $|s| \le 1$ *and* $|t| \le 1$, *we have*

$$(\frac{s[\alpha z^2 (D^m f)''(tz) + z(D^m f)'(tz)]}{t[\alpha z^2 (D^m f)''(sz) + z(D^m f)'(sz)]})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.5}$$

And putting $m = \lambda = 0$ in Theorem 3.1 gives Theorem 2.1 [8]. Then we have.

Corollary 3.2. *Let* $\phi(z)$ *and* $F(z)$ *be as in Lemma 2.1. The function* $f \in H[b, \alpha,](\phi)$ *if and only if for all* $|s| \le 1$ *and* $|t| \le 1$, *we have*

$$(\frac{s[\alpha z^2 f''(tz) + zf'(tz)]}{t[\alpha z^2 f''(sz) + zf'(sz)]})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.6}$$

And putting $\alpha = m = \lambda = 0$ in Theorem 3.1 gives a result in Definition 1.1 [5]. Then we have

Corollary 3.3. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$(\frac{sf'(tz)}{tf'(sz)})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.7}$$

Theorem 3.2. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha,m,\lambda](\phi)$. Then*

$$\frac{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.8}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) = (\frac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z})^{\frac{1}{b}},\ q(z) = \frac{F(z)}{z}$$

Then a computation yields

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\}, \tag{3.9}$$

now, by Definition 1.3 we have

$$1+\frac{zp'(z)}{p(z)} = 1+\frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z), \tag{3.10}$$

and

$$\frac{zq'(z)}{q(z)} = \frac{zF'(z)}{F(z)} - 1 = \phi(z) - 1.$$

Since $f \in H[b,\alpha,m,\lambda](\phi)$, we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z)=(1+2\alpha)z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z) - 1 = \frac{zq'(z)}{q(z)}.$$

so

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

Now in Lemma 2.2 putting $\varphi(p(z)) = \frac{1}{p(z)}$ and $\varphi(q(z)) = \frac{1}{q(z))}$ we get that
$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$ implies that $p(z) \prec q(z)$
and $(p(z))^b \prec (q(z))^b$
Hence
$\frac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b$.
This completes the proof of Theorem 3.2. □
□

Putting $\lambda = 0$ in Theorem 3.2. Then we have the Ruscheweyh derivative and we get the following new result.

Corollary 3.4. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha,m](\phi)$. Then*

$$\frac{\alpha z^2(D^m f)''(z) + z(D^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.11}$$

And putting $m = \lambda = 0$ in Theorem 3.2 gives Theorem 2.3 [8]. Then we have.

Corollary 3.5. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha](\phi)$. Then*

$$\frac{\alpha z^2 f''(z)+zf'(z)}{z} \prec (\frac{F(z)}{z})^b$$

Also putting $\alpha = m = \lambda = 0$ in Theorem 3.2 we have the following new result.

Corollary 3.6. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b](\phi)$. Then*

$$f'(z) \prec (\frac{F(z)}{z})^b$$

3.2. **Coefficients Estimates.** This section is about the class β-convex functions involving complex order defined as follows.

Definition 3.7. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\beta,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} \prec \phi(z),\ 0 \leq \beta \leq 1, \lambda \geq o, m > -1.$$

We note that, for $m = \lambda = 0$ we get $H[b,\beta,m,\lambda](\phi) \equiv M_{\beta,b}(\phi)$, [8].

To prove our main result of this section, we need the following:

Lemma 3.8. [5] *If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part. Then*

$$|c_2 - \mu c_1^2| \leq 2max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\ \ p(z) = \frac{1+z}{1-z}.$$

Our main result is the following

Theorem 3.3. *Let $0 \leq \beta \leq 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + ..., z \in U$, where $B_n' s$ are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $H[b,\beta,m,\lambda](\phi)$. Then*

$$|a_3 - \mu a_2^2| \leq$$
$$\frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} max\{1, |\frac{B_2}{B_1} + \frac{bB_1}{(1+\beta)^2}(1+3\beta-\mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})c_1^2|\}.$$

Proof. If $f(z) \in H[b,\beta,m,\lambda](\phi)$, then there is a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} = \phi(w(z)). \tag{3.12}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \dots . \tag{3.13}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$.
Define the function $p(z)$ by

$$\begin{aligned} p(z) &= 1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} \\ &= 1 + b_1 z + b_2 z^2 + \dots . \end{aligned} \tag{3.14}$$

From (3.13) we get

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}[c_1z+(c_2-\frac{c_1^2}{2})z^2+(c_3-\frac{c_1^3}{4}-c_1c_2)z^3+...]\ . \tag{3.15}$$

Since

$$\phi(z)=1+B_1z+B_2z^2+...,z\in U,$$

so, we get

$$\phi(\frac{p_1(z)-1}{p_1(z)+1})=1+\frac{1}{2}B_1c_1z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}]z^2+...\ . \tag{3.16}$$

Using (3.12),(3.14) and (3.15) we have

$$p(z)=\phi(\frac{p_1(z)-1}{p_1(z)+1}), \tag{3.17}$$

hence

$$1+b_1z+b_2z^2+...=1+\frac{1}{2}B_1c_1z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}]z^2+...\ . \tag{3.18}$$

Equating the coefficients in (3.18) we get

$$b_1=\frac{1}{2}B_1c_1, \tag{3.19}$$

$$b_2=\frac{1}{2}(B_1(c_2-\frac{1}{2}c_1^2))+\frac{1}{4}B_2c_1^2. \tag{3.20}$$

For $f(z)$ in (1.1) we obtain from (3.14) that

$$\begin{gathered}1+\frac{1}{b}\{(m+1)(1+\lambda)(1+\beta)a_2z+[(m+1)(m+2)(1+2\lambda)(1+2\beta)a_3-\\(m+1)^2(1+\lambda)^2(1+3\lambda)a_2^2]z^2+...\}=1+b_1z+b_2z^2+....\end{gathered} \tag{3.21}$$

Equating the coefficients in (3.21) we get

$$a_2=\frac{bb_1}{(m+1)(1+\lambda)(1+\beta)}, \tag{3.22}$$

$$a_3=\frac{bb_2+(m+1)^2(1+\lambda)^2(1+3\beta)a_2^2}{(m+1)(m+2)(1+2\lambda)(1+2\beta)}. \tag{3.23}$$

By applying (3.19) and (3.20) in (3.22) and (3.23) respectively we obtain

$$a_2=\frac{bB_1c_1}{2(m+1)(1+\lambda)(1+\beta)}, \tag{3.24}$$

$$\begin{gathered}a3=\frac{bB_1c_2}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)}+\\\frac{c_1^2}{4(m+1)(m+2)(1+2\lambda)}[\frac{1+3\beta}{(1+\beta)^2}b^2B_1^2-b(B_1-B_2)].\end{gathered} \tag{3.25}$$

Now, we have

$$a_3-\mu a_2^2=\frac{bB_1}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)}[c_2-\nu c_1^2], \tag{3.26}$$

where

$$\nu=\tfrac{1}{2}[1-\tfrac{B_2}{B_1}-\tfrac{bB_1}{(1+\beta)^2}(1+3\beta-\mu\tfrac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

Then, applying lemma 3.8 on (3.26) we obtain

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)}|c_2 - \nu c_1^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} 2max\{1, |2\nu - 1|\},$$

for

$$\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} - \frac{bB_1}{(1+\beta)^2}(1 + 3\beta - \mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

This completes the proof of Theorem 3.3. □

□

Putting $m = \lambda = 0$ in Theorem 3.3 gives Theorem 3.3 in [8]. Then we have

Corollary 3.9. *Let* $0 \leq \beta \leq 1$. *Further let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., z \in U$, *where* $B_n's$ *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by (1.1) belongs to* $H[b, \beta](\phi)$.*Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+2\beta)} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu + \beta(3 - 4\mu))\frac{bB_1}{(1+\beta)^2}|\}.$$

Putting $m = \lambda = \beta = 0$ in Theorem 3.3, gives a result obtained in [5]. Then we have

Corollary 3.10. *Let* $0 \leq \beta \leq 1$. *Further let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., z \in U$, *where* $B_n's$ *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by (1.1) belongs to* $H[b](\phi)$. *Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu)bB_1|\}.$$

References

1. Al-Shaqsi,K., Darus, M., On certain subclasses of analytic univalent functions with negative coefficients, Appl. Math. Sci. Vol. 1(2007), no. 3, 1121-1128.
2. M. Kamali and S. Akbulut, On a subclass of certain convex functions with negative coefficients, J.Appl.Math.Comput. 145(2002), 341-350.
3. W.C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157-169, Internat. Press, Cambridge, MA.
4. S.S. Miller and P.T. Mocanu, Differential subordinations theory and applications,Dekker, New York, 2000.
5. V. Ravichandran, Yasar Polatoglu, Metin Bolcal and Arsu Sen, Certain subclasses of starlike and convex functions of complex order, Hacettepe J.Math. Stat.,34(2005), 9-15.
6. Ruscheweyh St.,New criterion for univalent functions, Proc. Amer. Math. Soc., 49(1975), 109-115.
7. T.N.Shanmugam and S.Sivasubramanian, On the Fekete-Szegö problem for some subclasses of analytic functions, J Inequal. Pure Appl. Math., 6(3),(2005), Article 71,6 [ONLINE:http://jipam.vu.edu.au/article.php?sid=544].
8. T.N.Shanmugum, S.Sivasubramanian and B.A.Frasin, Certain subclasses of analytic functions involving complex order, Faculty of Sciences and Mathematics, University of Niš, Serbia, Available at:http://www.pmf.ni.ac.yu/filomat. Filomat 22:2(2008),115-122.
9. T.N.Shanmugum, Sivasubramanian and M. Kamali, On the unified class of k-uniformly convex functions associated with generalized derivative with missing coefficients, J. Approx.theory Appl., 1(2)(2005), 141-155.
10. H.Silverman, Univalent functions with negative coefficients, Proc.Amer. Math.Soc., 51(1975), 109-116.