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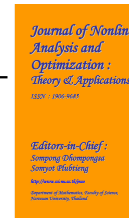


# CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF GENERALIZED $I$-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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**ABSTRACT.** In this paper, we consider strong convergence of implicit iteration process to common fixed point for generalized $I$-asymptotically nonexpansive mappings. The main results extend to finite family of generalized $I$-asymptotically nonexpansive mappings in a Banach space.

**KEYWORDS** : $I$-asymptotically nonexpansive; Implicit iteration process; Common fixed point; Convergence theorem.
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## 1. INTRODUCTION

Let $K$ be a nonempty subset of a real normed linear space $X$ and $T : K \to K$ be a mapping. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping $T$.

We introduce the following definitions and statements which will be used in our main results(see references therein):

A mapping $T : K \longrightarrow K$ is called *nonexpansive* provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$. $T$ is called *asymptotically nonexpansive* mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n \longrightarrow \infty} \lambda_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [4]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point.

$T$ is called *quasi-nonexpansive* mapping provided

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$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

$T$ is called *asymptotically quasi-nonexpansive* mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n \longrightarrow \infty} \lambda_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda_n)\|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

**Remark 1.1.** From above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive.

We introduce the following definitions and statements which will be used in our main results(see [9]-[11]).

Let us recall some notions.

Let $T, I : K \longrightarrow K$. Then $T$ is called *I-nonexpansive* on $K$ if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

for all $x, y \in K$.

$T$ is called *I- asymptotically nonexpansive* on $K$ if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n \longrightarrow \infty} \lambda'_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda'_n)\|I^n x - I^n y\|$$

for all $x, y \in K$ and $n \geq 1$.

$T$ is called *I- asymptotically quasi-nonexpansive* on $K$ if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n \longrightarrow \infty} \lambda'_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda'_n)\|I^n x - p\|$$

for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n = 1, 2, ....$

**Remark 1.2.** From the above definitions it follows that if $F(T) \cap F(I)$ is nonempty, a $I$-nonexpansive mapping must be $I$-quasi-nonexpansive, and linear $I$-quasi-nonexpansive mappings are $I$-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous $I$-quasi-nonexpansive mappings which are not $I$-nonexpansive.

Now, we give the definition of the generalized asymptotically quasi-nonexpansive mapping as follows:

**Definition 1.3.** [7] Let $X$ be a real normed linear space and $K$ a nonempty subset of $X$. A mapping $T : K \longrightarrow K$ is called *generalized asymptotically quasi-nonexpansive* mapping if $F(T) \neq \emptyset$ and there exist sequences of real numbers $\{u_n\}$, $\{\varphi_n\}$ with $\lim_{n \longrightarrow \infty} u_n = 0 = \lim_{n \longrightarrow \infty} \varphi_n$ such that

$$\|T^n x - p\| \leq \|x - p\| + u_n\|x - p\| + \varphi_n$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

If, in Definition 1.3, $\varphi_n = 0$ for all $n \geq 1$ then $T$ becomes asymptotically quasi-nonexpansive mapping and hence the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

Now we give generalized $I$-asymptotically quasi-nonexpansive mappings as follows:

**Definition 1.4.** Let $X$ be a real normed linear space and $K$ a nonempty subset of $X$. A mapping $T : K \longrightarrow K$ is called *generalized I-asymptotically quasi-nonexpansive* mapping if $F(T) \cap F(I) \neq \emptyset$ and there exist sequences of real numbers $\{u'_n\}$, $\{\varphi'_n\}$ with $\lim_{n \longrightarrow \infty} u'_n = 0 = \lim_{n \longrightarrow \infty} \varphi'_n$ such that

$$\|T^n x - p\| \leq \|I^n x - p\| + u'_n \|I^n x - p\| + \varphi'_n$$

for all $x \in K$, $p \in F(T) \cap F(I)$ and $n \geq 1$.

Also, if, in Definition 1.4, $\varphi'_n = 0$ for all $n \geq 1$ then $T$ becomes $I$-asymptotically quasi-nonexpansive mapping and hence the class of generalized $I$-asymptotically quasi-nonexpansive mappings includes the class of $I$-asymptotically quasi-nonexpansive mappings.

Recently, concerning the convergence problems of an implicit(or non-implicit) iterative process to a common fixed point for finite family of asymptotically nonexpansive mappings( or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been obtained by a number of authors (see, the references therein).

Xu and Ori [13], in 2001, introduced an implicit iteration process for a finite family of nonexpansive mappings. Let $K$ be a nonempty closed convex subset of $\mathcal{H}$ Hilbert space. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of $K$ such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i$, $i = 1, ..., N$. An implicit iteration process for finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ are defined as follows, with $\{\alpha_n\} \subset (0,1)$, and an initial point $x_0 \in K$, the sequence $\{x_n\}_{n \geq 1}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1 \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2 \\
&\;\,. \\
&\;\,. \\
&\;\,. \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_{N+1} x_{N+1} \\
&\;\,. \\
&\;\,. \\
&\;\,.
\end{aligned}$$

The process is expressed in the following form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad n \geq 1 \tag{1.1}$$

where $T_n = T_{n(mod N)}$.

Xu and Ori [13] proved the weak convergence of the sequence $\{x_n\}$ defined implicity by (1.1) to a common fixed point of the finite family of nonexpansive mappings defined in Hilbert space. Zhou and Chang [14], in 2002, studied the weak and strong convergence of implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Liu [5], in 2002, and Chidume - Shahzad [2], in 2005, proved the strong convergence of an implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Sun [8], in 2003, extended an implicit iteration process for a finite family of nonexpansive mappings due to Xu and Ori [13] to the case of asymptotically quasi-nonexpansive mappings in a setting of Banach spaces. Chang and et al.[1], in 2003, studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. Guo and Cho [3], in 2008, studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Shahzad and Zegeye[7], in 2007, studied the strong convergence of implicit iteration process to a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces. Recently, in [11], the weak convergence theorem for $I$-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [9] the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of $I$-asymptotically nonexpansive mappings were studied. More recently, Temir [10], studied the weak and strong convergence of the explicit iterative process of generalized $I$-asymptotically quasi-nonexpansive mappings to common fixed point in Banach space.

In this paper, we consider the following implicit iterative process with new type of conception which combines notions such as generalized asymptotically nonexpansive mapping and generalized $I$-asymptotically nonexpansive mapping. Let $K$ be a nonempty subset of $X$ Banach space.

Let $\{T_i\}_{i=1}^N$ be finite family of generalized $I_i$- asymptotically nonexpansive self-mappings and $\{I_i\}_{i=1}^N$ be finite family of generalized asymptotically nonexpansive self-mappings on $K$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. Then, an initial point $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{gathered}
x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1[\beta_1 x_1 + (1-\beta_1)I_1 x_1],\\
x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2[\beta_2 x_2 + (1-\beta_2)I_2 x_2],\\
.\\
.\\
.\\
x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N[\beta_N x_N + (1-\beta_N)I_N x_N],\\
x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2[\beta_{N+1} x_{N+1} + (1-\beta_{N+1})I_1^2 x_{N+1}],\\
.\\
.\\
.\\
x_{2N} = \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2[\beta_{2N} x_{2N} + (1-\beta_{2N})I_N^2 x_{2N}],\\
x_{2N+1} = \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3[\beta_{2N+1} x_{2N+1} + (1-\beta_{2N+1})I_1^3 x_{2N+1}]\\
.\\
.\\
.
\end{gathered}$$

Let $x_0 \in K$ be any given point, the implicitly iterative sequence $x_n$ generated by (1.2) should be written in the following compact form:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) I_{i(n)}^{k(n)} x_n; \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n, \end{cases} \tag{1.2}$$

$\forall n \geq 1$, where $n = (k(n)-1)N + i(n), i(n) \in \{1,2,...,N\}, k(n) \geq 1$.

If we take $\{I_i\}_{i=1}^N$ identity mappings and $\forall n \geq 1, \beta_n = 0$ then the compact form induces (1.1) implicit iteration process defined in Xu and Ori [13].

The aim of this paper is to prove the strong convergence of implicit iterative sequence $\{x_n\}_{n\geq 1}$ defined by (1.2) to common fixed point for finite family of generalized $I_i$-asymptotically nonexpansive mappings in Banach space. We consider also $\{I_i\}_{i=1}^N$ be finite family of generalized asymptotically nonexpansive self-mappings of $K$ subset of Banach space. Our results will thus improve and generalize corresponding results of [13], [7], [10] and [9].

## 2. PRELIMINARIES AND NOTATIONS

In order to prove the main results of this paper, we need the following statements:

**Lemma 2.1.** [12] *Let $\{a_n\}$, $\{b_n\}$ and $\{\kappa_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $a_{n+1} \leq (1+\kappa_n)a_n + b_n$, where $\sum\limits_{n=0}^{\infty} \kappa_n < \infty$ and $\sum\limits_{n=0}^{\infty} b_n < \infty$. Then $\lim\limits_{n\longrightarrow\infty} a_n$ exists.*

**Lemma 2.2.** [6] *Let $K$ be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $\{\alpha_n\}$ a sequence $[\delta, 1-\delta]$, for some $\delta \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that*

$$\limsup_{n\longrightarrow\infty} \|x_n\| \leq d,$$

$$\limsup_{n\longrightarrow\infty} \|y_n\| \leq d$$

*and*

$$\limsup_{n\longrightarrow\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = d$$

*holds for some $d \geq 0$.Then*

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0.$$

Let $X$ be a uniformly convex Banach space and $K$ a nonempty, closed and subset of $X$. Let $\{T_i : i \in \{1, ..., N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0,\infty)$ and $\theta_{in},\ \varphi_{in} \to 0$ as $n \to \infty$ such that $\|T_i^k x - T_i^k y\| \leq (1+\theta_{in})\|I_i^k x - I_i^k y\| + \varphi_{in}$ for all $x, y \in K$ and $n \geq 1$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized asymptotically nonexpansive mappings of $K$ with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ and $\tau_{in},\ \psi_{in} \to 0$ as $n \to \infty$ such that $\|I_i^k x - I_i^k y\| \leq \tau_{in}\|x-y\| + \psi_{in}$ for all $x, y \in K$, for each $i = 1, ...N$ and $n \geq 1$ .

Letting $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1, ..., N\}$, $\nu_n \subset [0,\infty)$, with $\lim\limits_{k\longrightarrow\infty} \nu_n = 0$, $\sum\limits_{n=1}^{\infty} \nu_n < \infty$, also $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1, ..., N\}$, $\phi_n \subset [0,\infty)$, with $\lim\limits_{n\longrightarrow\infty} \phi_n = 0$, $\sum\limits_{n=1}^{\infty} \phi_n < \infty$. Then there exist nonnegative real sequences $\{\nu_n\}$ and $\{\phi_n\}$ with $\nu_n,\ \ \phi_n \to 0$ as $n \to \infty$ such that

$$\|T_i^k x - T_i^k y\| \leq (1+\theta_{in})\|I_i^k x - I_i^k y\| + \varphi_{in} \leq (1+\nu_n)^2\|x-y\| + (2+\nu_n)\phi_n,$$

$$\|I_i^k x - I_i^k y\| \leq (1+\tau_{in})\|x-y\| + \psi_{in} \leq (1+\nu_n)\|x-y\| + \phi_n$$

for all $x, y \in K$,for each $i = 1, ...N$ and $n \geq 1$.

Let denote the distance of $x$ to set $F \subset K$, i.e., $d(x, F) = \inf\{\|x-p\| : p \in F\}$. A mapping $T : K \to K$ is said to *semi-compact* if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$

such that $\{x_{n_i}\} \to p \in K$.

The mappings $T, I : K \to K$ are said to satisfying condition *(A)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Ix\|) \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F = F(T) \cap F(I)\}$.

A family $\{T_i : i \in \{1,...,N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings on $K$ with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$ are said to satisfy condition *(B)* on $K$ if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ and all $x \in K$ such that $\max_{1 \leq \ell \leq N}\{\frac{1}{2}(\|x - T_\ell x\| + \|x - I_\ell x\|)\} \geq f(d(x,F))$ for at least one $T_\ell$ and $I_\ell$, $\ell = \{1,...,N\}$.

## 3. STRONG CONVERGENCE OF IMPLICIT ITERATION FOR GENERALIZED $I$-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Let $X$ be a uniformly convex Banach space, $K$ be a nonempty closed convex subset of $X$, $\{T_i : i \in \{1,...,N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ with sequences of real numbers $\{\theta_{in}\}, \{\varphi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \theta_{in} < \infty$, $\sum_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings of $K$ with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum_{n=1}^{\infty} \psi_{in} < \infty$. Letting $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, with $\lim_{k \longrightarrow \infty} \nu_n = 0$, $\sum_{n=1}^{\infty} \nu_n < \infty$, also $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, with $\lim_{n \longrightarrow \infty} \phi_n = 0$, $\sum_{n=1}^{\infty} \phi_n < \infty$. Let $z \in K$ be fixed and $\alpha, \beta \in (0,1)$.

Define $W : K \to K$

$$W(x) = \alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta x + (1-\beta) I_{i(n)}^{k(n)} x]. \tag{3.1}$$

Then

$$\begin{aligned}
\|W(x) - W(y)\| &= \|\alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta x + (1-\beta) I_{i(n)}^{k(n)} x] \\
&\quad -[\alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta y + (1-\beta) I_{i(n)}^{k(n)} y]\| \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \|\beta(x-y) \\
&\quad +(1-\beta)(I_{i(n)}^{k(n)} x - I_{i(n)}^{k(n)} y)\| + (2+\nu_n)\phi_n\Big] \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \beta \|x-y\| \\
&\quad +(1+\nu_n)^2 (1-\beta) \|I_{i(n)}^{k(n)} x - I_{i(n)}^{k(n)} y\| + (2+\nu_n)\phi_n\Big] \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \beta \|x-y\| + (1+\nu_n)^3 (1-\beta)\|x-y\| \\
&\quad +(1+\nu_n)^2 (1-\beta)\phi_n + (2+\nu_n)\phi_n\Big]
\end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha)\Big[(1+\nu_n)^3\|x-y\| + (1+\nu_n)^2(1-\beta)\phi_n + (2+\nu_n)\phi_n\Big] \\ &\to (1-\alpha)\|x-y\| \ \ (n\to\infty). \end{aligned}$$

Thus, there exists a positive integer $N_0$ such that $\|W(x)-W(y)\| \leq (1-\alpha)\|x-y\|$ for all $n \geq N_0$. Since $1-\alpha < 1$, then $W$ is a contraction. By Banach contraction mapping principal, there exists a unique fixed point in $K$ satisfiying the equation (3.1). This implies the implicit iterative process (1.2) is well defined.

**Lemma 3.1.** *Let $X$ be a uniformly convex Banach space, $K$ be a nonempty closed convex subset of $X$, $\{T_i : i \in \{1, ..., N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\theta_{in} < \infty$, $\sum_{n=1}^{\infty}\varphi_{in} < \infty$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized asymptotically nonexpansive mappings of $K$ with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\tau_{in} < \infty$ and $\sum_{n=1}^{\infty}\psi_{in} < \infty$. Let be $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$.*

(1) $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\nu_n \subset [0,\infty)$, *with* $\lim_{n\longrightarrow\infty} \nu_n = 0$, *also* $\sum_{n=1}^{\infty}\nu_n < \infty$,

(2) $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\phi_n \subset [0,\infty)$, *with* $\lim_{n\longrightarrow\infty} \phi_n = 0$, *also* $\sum_{n=1}^{\infty}\phi_n < \infty$,

(3) $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$.

*Then the implicitly iterative sequence $\{x_n\}$ is generated by (1.2) converges to a common fixed point in $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$ if and only if*

$$\liminf_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0.$$

*Proof.* The necessity is obvious and so it is omitted.

Now, we prove the sufficiency. From (1.2), we have for any $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$,

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n - p\| \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\|T_{i(n)}^{k(n)} y_n - p\| \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\theta_{in})\|I_{i(n)}^{k(n)} y_n - p\| + \varphi_{in}\Big) \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)\|I_{i(n)}^{k(n)} y_n - p\| + \phi_n\Big) \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)(1+\tau_{in})\|y_n - p\| + (1+\tau_{in})\psi_{in}\Big) \\ &\quad +(1-\alpha_n)\phi_n \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n - p\| + (1+\nu_n)\psi_{in}\Big) \\ &\quad +(1-\alpha_n)\phi_n \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n - p\| + (1+\nu_n)\phi_n\Big) \\ &\quad +(1-\alpha_n)\phi_n \end{aligned}$$

$$\leq \alpha_n\|x_{n-1}-p\|+(1-\alpha_n)\Big((1+\nu_n)^2\|y_n-p\|+(2+\nu_n)\phi_n\Big)$$

which implies that

$$\|x_n-p\|\leq \alpha_n\|x_{n-1}-p\|+(1-\alpha_n)\Big((1+\nu_n)^2\|y_n-p\|+(2+\nu_n)\phi_n\Big). \tag{3.2}$$

$$\begin{aligned}\|y_n-p\| &= \|\beta_n x_n+(1-\beta_n)I_{i(n)}^{k(n)}x_n-p\|\\ &\leq \beta_n\|x_n-p\|+(1-\beta_n)\|I_{i(n)}^{k(n)}x_n-p\|\\ &\leq \beta_n\|x_n-p\|+(1-\beta_n)\Big((1+\tau_{in})\|x_n-p\|+\psi_{in}\Big)\\ &\leq \beta_n\|x_n-p\|+(1-\beta_n)\Big((1+\nu_n)\|x_n-p\|+\phi_n\Big)\\ &\leq (1+\nu_n)\|x_n-p\|+(1-\beta_n)\phi_n. \end{aligned}\tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned}\|x_n-p\| \leq\ & \alpha_n\|x_{n-1}-p\|\\ &+(1-\alpha_n)\Big((1+\nu_n)^3\|x_n-p\|+(1-\beta_n)(1+\nu_n)^2\phi_n+(2+\nu_n)\phi_n\Big),\end{aligned}$$

which implies that

$$\begin{aligned}1-[(1-\alpha_n)(1+\nu_n)^3]\|x_n-p\| \leq\ & \alpha_n\|x_{n-1}-p\|\\ &+(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^2\phi_n+(2+\nu_n)\phi_n\Big).\end{aligned}$$

Then we get

$$\begin{aligned}\|x_n-p\| &\leq \frac{\alpha_n}{1-[(1-\alpha_n)(1+\nu_n)^3]}\|x_{n-1}-p\|\\ &\quad +(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^3\phi_n+(2+\nu_n)\phi_n\Big)\\ &= \Big[1+\frac{(1-\alpha_n)[(1+\nu_n)^3-1]}{1-[(1-\alpha_n)(1+\nu_n)^3]}\Big]\|x_{n-1}-p\|\\ &\quad +\frac{(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^2\phi_n+(2+\nu_n)\phi_n\Big)}{1-[(1-\alpha_n)(1+\nu_n)^3]}.\end{aligned}\tag{3.4}$$

We assume that $(1+\nu_n)\leq\sqrt[3]{(1+\frac{\delta}{2(1-\delta)})}$ for some $n\geq n_0$ and $\lambda<\frac{1}{\delta}$. Then we can write $1-[(1-\alpha_n)(1+\nu_n)^3]\geq\frac{\delta}{2}$, $\forall n\geq 1$. Then (3.4) becomes

$$\begin{aligned}\|x_n-p\| &\leq \Big[1+\frac{2(1-\delta)[(\lambda^2+\lambda+1)(1+\nu_n-1)]}{\delta}\Big]\|x_{n-1}-p\|\\ &\quad + 2\frac{(1-\delta)\Big((1-\delta)(1+\frac{\delta}{2(1-\delta)})\phi_n+(2+\nu_n)\phi_n\Big)}{\delta}\\ &\leq (1+\kappa_{ik})\|x_{n-1}-p\|+2\frac{(1-\delta)(\phi_n+(2+\nu_n)\phi_n)}{\delta},\end{aligned}\tag{3.5}$$

where $\kappa_n=\Big[\frac{2(1-\delta)[(\lambda^2+\lambda+1)(\nu_n)]}{\delta}\Big]$ and $\Psi_{in}=2\frac{(1-\delta)(3+\nu_n)\phi_n}{\delta}$. Moreover, from the condition (1) and (2), since $\sum_{n=1}^{\infty}\nu_n<\infty$ and $\sum_{n=1}^{\infty}\phi_n<\infty$, it follows that $\sum_{n=1}^{\infty}\kappa_n<\infty$

and $\sum\limits_{n=1}^{\infty} \Psi_{in} < \infty$. Thus we obtain

$$\|x_n - p\| \le (1+\kappa_n)\|x_{n-1} - p\| + \Psi_{in}. \tag{3.6}$$

By Lemma 2.1, $\lim\limits_{n\longrightarrow\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F} = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i)$. By assumption $\liminf\limits_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$, we obtain

$$\lim_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $K$. Notice that $1+z \le \exp(z)$ for all $z > 0$. From (3.6), for any $p \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\le \exp\Big(\sum_{j=n}^{n+m-1} \kappa_j\Big)\|x_n - p\| + \exp\Big(\sum_{j=n}^{n+m-1} \kappa_j\Big)\Big(\sum_{j=n}^{n+m-1} \Psi_j\Big) \\
&\le M\|x_n - p\| + M\Big(\sum_{j=1}^{\infty} \Psi_j\Big)
\end{aligned}$$

for all natural numbers m, n, where $M = \exp\{\sum\limits_{j=1}^{\infty} \kappa_j\} < +\infty$. Since $\lim\limits_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$, for any given $\epsilon > 0$, there exists a positive integer $N_0$ such that for all $n \ge N_0$, $d(x_n, \mathcal{F}) < \frac{\epsilon}{4M}$ and $\sum\limits_{n=N_0}^{\infty} \Psi_n < \frac{\epsilon}{4M}$. There exists $p_1 \in \mathcal{F} = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i)$ such that $\|x_{N_0} - p_1\| < \frac{\epsilon}{4M}$.
Hence, for all $n \ge N_0$ and $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\le M\|x_{N_0} - p_1\| + M\Big(\sum_{n=N_0}^{\infty} \Psi_n\Big) + M\|x_{N_0} - p_1\| + M\Big(\sum_{n=N_0}^{\infty} \Psi_n\Big) \\
&\le 2M\Big(\|x_{N_0} - p_1\| + \Big(\sum_{n=N_0}^{\infty} \Psi_n\Big)\Big) \\
&\le 2M\Big(\frac{\epsilon}{4M} + \frac{\epsilon}{4M}\Big) = \epsilon
\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in $K$.
Thus, the completeness of $X$ implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point $p$.

Then $p \in K$, because $K$ is closed subset of $X$. The set $\mathcal{F} = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i)$ is closed. $\lim\limits_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$ gives that $d(p, \mathcal{F}) = 0$.

Thus $p \in \mathcal{F} = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i)$. This completes the proof.

□

**Lemma 3.2.** *Let $X$ be a uniformly convex Banach space, $K$ be a nonempty closed convex subset of $X$, $\{T_i : i \in \{1, ..., N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} \theta_{in} < \infty$, $\sum\limits_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized* *asymptotically nonexpansive mappings of $K$ with sequences $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum_{n=1}^{\infty} \psi_{in} < \infty$. Let be $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$.*

(1) $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ *for all* $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, *with* $\lim_{n\longrightarrow\infty} \nu_n = 0$, *also* $\sum_{n=1}^{\infty} \nu_n < \infty$,

(2) $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ *for all* $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, *with* $\lim_{n\longrightarrow\infty} \phi_n = 0$, *also* $\sum_{n=1}^{\infty} \phi_n < \infty$,

(3) $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$.

*Suppose that for any given $x \in K$, the sequence $\{x_n\}$ is generated by (1.2). Then*

$$\lim_{n\longrightarrow\infty} \|T_\ell x_n - x_n\| = \lim_{n\longrightarrow\infty} \|I_\ell x_n - x_n\| = 0, \forall \ell = 1,2,...,N.$$

*Proof.* By Lemma 3.1, we can assume that $\lim_{n\longrightarrow\infty} \|x_n - p\| = d$ for all $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$ .

Taking $\limsup_{n\longrightarrow\infty}$ on both sides in (3.3) inequality,

$$\limsup_{n\longrightarrow\infty} \|y_n - p\| \leq d. \tag{3.7}$$

Since $\{I_\ell : \ell \in \{1,...,N\}\}$ is N generalized asymptotically nonexpansive self-mappings of $K$, we can get that,

$$\|T_{i(n)}^{k(n)} y_n - p\| \leq (1+\nu_n)\|I_{i(n)}^{k(n)} y_n - p\| + \phi_n \leq (1+\nu_n)^2\|y_n - p\| + (2+\nu_n)\phi_n,$$

which on taking $\limsup_{n\longrightarrow\infty}$ and using (3.7) gives

$$\limsup_{n\longrightarrow\infty} \|T_{i(n)}^{k(n)} y_n - p\| \leq d.$$

Further,

$$\lim_{n\longrightarrow\infty} \|x_n - p\| = d$$

means that

$$\lim_{n\longrightarrow\infty} \|\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n - p\| = d,$$

$$\lim_{n\longrightarrow\infty} \|\alpha_n (x_{n-1} - p) + (1-\alpha_n)(T_{i(n)}^{k(n)} y_n - p)\| = d.$$

It follows from Lemma 2.2

$$\lim_{n\longrightarrow\infty} \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| = 0. \tag{3.8}$$

Moreover,

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|(1-\alpha_n)[T_{i(n)}^{k(n)} y_n - x_{n-1}]\| \\ &\leq (1-\alpha_n)\|T_{i(n)}^{k(n)} y_n - x_{n-1}\|. \end{aligned}$$

Thus , from (3.8) we have

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n-1}\| = 0 \tag{3.9}$$

and

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n+j}\| = 0, \forall j = 1,...,N. \tag{3.10}$$

Now,

$$\begin{aligned}\|x_{n-1}-p\| &\leq \|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+\|T_{i(n)}^{k(n)}y_n-p\| \\ &\leq \|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+(1+\nu_n)^2\|y_n-p\|+(2+\nu_n)\phi_n\end{aligned}$$

which on taking $\lim_{n\longrightarrow\infty}$ implies

$$\begin{aligned}d=\lim_{n\longrightarrow\infty}\|x_{n-1}-p\| &\leq \limsup_{n\longrightarrow\infty}(\|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+(1+\nu_n)^2\|y_n-p\| \\ &\quad +(2+\nu_n)\phi_n) \\ &\leq \limsup_{n\longrightarrow\infty}\|y_n-p\|\leq d.\end{aligned}$$

Then we have

$$\limsup_{n\longrightarrow\infty}\|y_n-p\|=d.$$

Next,

$$\|I_{i(n)}^{k(n)}x_n-p\| \quad\leq\quad (1+\nu_n)\|x_n-p\|+\phi_n.$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, we have

$$\lim_{n\longrightarrow\infty}\|I_{i(n)}^{k(n)}x_n-p\|\leq\lim_{n\longrightarrow\infty}\|x_n-p\|=d.$$

Further,

$$\lim_{n\longrightarrow\infty}\|\beta_n(x_n-p)+(1-\beta_n)(I_{i(n)}^{k(n)}x_n-p)\|=\lim_{n\longrightarrow\infty}\|y_n-p\|=d.$$

By Lemma 2.2, we have

$$\lim_{n\longrightarrow\infty}\|I_{i(n)}^{k(n)}x_n-x_n\|=0. \tag{3.11}$$

We have also,

$$\begin{aligned}\|T_{i(n)}^{k(n)}x_n-x_n\| &\leq \|T_{i(n)}^{k(n)}x_n-T_{i(n)}^{k(n)}y_n\|+\|T_{i(n)}^{k(n)}y_n-x_n\| \\ &\leq (1+\nu_n)^2\|x_n-y_n\|+\|T_{i(n)}^{k(n)}y_n-x_n\|+(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2\|x_n-[\beta_nx_n+(1-\beta_n)I_{i(n)}^{k(n)}x_n]\| \\ &\quad +\|T_{i(n)}^{k(n)}y_n-x_n\|+(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2\|(1-\beta_n)(x_n-I_{i(n)}^{k(n)}x_n)\|+\|T_{i(n)}^{k(n)}y_n-x_n\| \\ &\quad +(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2(1-\beta_n)\|I_{i(n)}^{k(n)}x_n-x_n\| \\ &\quad +\|T_{i(n)}^{k(n)}y_n-x_{n-1}+x_{n-1}-x_n\|+(2+\nu_n)\phi_n \\ &\leq (1+\nu_n)^2(1-\beta_n)\|I_{i(n)}^{k(n)}x_n-x_n\|+\|T_{i(n)}^{k(n)}y_n-x_{n-1}\| \\ &\quad +\|x_n-x_{n-1}\|+(2+\nu_n)\phi_n.\end{aligned}$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, from (3.8), (3.9) and (3.11), we obtain

$$\lim_{n\longrightarrow\infty}\|T_{i(n)}^{k(n)}x_n - x_n\| = 0. \tag{3.12}$$

Now we prove that

$$\lim_{n\longrightarrow\infty}\|T_\ell x_n - x_n\| = \lim_{n\longrightarrow\infty}\|I_\ell x_n - x_n\| = 0, \forall \ell = 1, 2, ..., N$$

holds. In fact, since for each $n > N$,$n = (n-N)(modN)$ and $n = (k(n)-1)N+i(n)$, hence $n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N)$, that is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$.
From (3.11),

$$\begin{aligned}
\|x_n - I_n x_n\| &\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + \|I_{i(n)}^{k(n)}x_n - I_n x_n\| \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)\|I_{i(n)}^{k(n)-1}x_n - x_n\| + \phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)(\|I_{i(n)}^{k(n)-1}x_n - I_{i(n-N)}^{k(n)-1}x_{n-N}\| \\
&\quad +\|I_{i(n-N)}^{k(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|) + \phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad +(1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (1+\nu_n)\|x_{n-N} - x_n\| \\
&\quad +(2+\nu_n)\phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad +(1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (2+\nu_n)\phi_n \to 0 \ \ (n\to\infty).
\end{aligned}$$

This implies that

$$\lim_{n\longrightarrow\infty}\|I_n x_n - x_n\| = 0. \tag{3.13}$$

Then we also have from (3.11) and (3.12)

$$\begin{aligned}
\|x_n - T_n x_n\| &\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + \|T_{i(n)}^{k(n)}x_n - T_n x_n\| \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)\|I_{i(n)}^{k(n)}x_n - I_n x_n\| + \phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)(\|I_{i(n)}^{k(n)-1}x_n - I_{i(n-N)}^{k(n)-1}x_{n-N}\| \\
&\quad +\|I_{i(n-N)}^{k(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|) + \phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad +(1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (1+\nu_n)\|x_{n-N} - x_n\| \\
&\quad +(2+\nu_n)\phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + \Big[(1+\nu_n)^2 + (1+\nu_n)\Big]\|x_n - x_{n-N}\| \\
&\quad +(1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (2+\nu_n)\phi_n \to 0 \ \ (n\to\infty).
\end{aligned}$$

This implies that

$$\lim_{n\longrightarrow\infty}\|T_n x_n - x_n\| = 0. \tag{3.14}$$

Now for all $\ell = \{1, ..., N\}$.

$\|x_n - T_{n+\ell}x_n\| \le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - T_{n+\ell}x_{n+\ell}\| + \|T_{n+\ell}x_{n+\ell} - T_{n+\ell}x_n\|.$

Taking $\lim\limits_{n\longrightarrow\infty}$ on both sides in the above inequality, then we get

$$\lim_{n\longrightarrow\infty} \|x_n - T_{n+\ell}x_n\| = 0$$

for all $\ell = \{1, ..., N\}$.
Thus, we have

$$\lim_{n\longrightarrow\infty} \|x_n - T_\ell x_n\| = 0. \tag{3.15}$$

$\|x_n - I_{n+\ell}x_n\| \le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - I_{n+\ell}x_{n+\ell}\| + \|I_{n+\ell}x_{n+\ell} - I_{n+\ell}x_n\|.$

Taking $\lim\limits_{n\longrightarrow\infty}$ on both sides in the above inequality, then we get

$$\lim_{n\longrightarrow\infty} \|x_n - I_{n+\ell}x_n\| = 0$$

for all $\ell = \{1, ..., N\}$.
Thus , we have

$$\lim_{n\longrightarrow\infty} \|x_n - I_\ell x_n\| = 0. \tag{3.16}$$

Then the proof is completed. □

**Theorem 3.3.** *Let $X$ be a uniformly convex Banach space, $K$ be a nonempty closed convex subset of $X$, $\{T_i : i \in \{1, ..., N\}\}$ be N generalized $I_i$-asymptotically nonexpansive self-mappings of $K$ with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} \theta_{in} < \infty$, $\sum\limits_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized asymptotically nonexpansive mappings of $K$ with sequences $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum\limits_{n=1}^{\infty} \psi_{in} < \infty$.*

(1) $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\nu_n \subset [0, \infty)$, *with* $\lim\limits_{n\longrightarrow\infty} \nu_n = 0$, *also* $\sum\limits_{n=1}^{\infty} \nu_n < \infty$,

(2) $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\phi_n \subset [0, \infty)$, *with* $\lim\limits_{n\longrightarrow\infty} \phi_n = 0$, *also* $\sum\limits_{n=1}^{\infty} \phi_n < \infty$,

(3) $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [\delta, 1 - \delta]$ *for some* $\delta \in (0, 1)$.

*Let be $\mathcal{F} = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that one of the mappings $\{T_i : i \in \{1, ..., N\}\}$ and one of the mappings $\{I_i : i \in \{1, ..., N\}\}$ are semi-compact or satisfy condition (B). Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of $\{T_i : i \in \{1, ..., N\}\}$ and $\{I_i : i \in \{1, ..., N\}\}$.*

*Proof.* Without loss of generality, we can assume that $\{T_1\}$ and $\{I_1\}$ are semi-compact or satisfy condition *(B)*. It follows from (3.15) and (3.16) in Lemma 3.2 $\lim\limits_{n\longrightarrow\infty} \|x_n - T_1x_n\| = 0 = \lim\limits_{n\longrightarrow\infty} \|x_n - I_1x_n\| = 0$ By semi-compactness of $\{T_1\}$ and

$\{I_1\}$ , there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to p \in K$ strongly as $j \longrightarrow \infty$. From (3.15) and (3.16) in Lemma 3.2

$$\lim_{n\longrightarrow\infty} \|x_n - T_\ell x_n\| = \|p - T_\ell p\|$$

for all $\ell \in \{1, ..., N\}$, and

$$\lim_{n\longrightarrow\infty} \|x_n - I_\ell x_n\| = \|p - I_\ell p\|.$$

for all $\ell \in \{1, ..., N\}$. This implies that $p \in \mathcal{F}$. Since $\liminf_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$, Lemma 3.1 guarantees that $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. If $\{T_1\}$ and $\{I_1\}$ satisfy condition *(B)*, then we have $\liminf_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$. From Lemma 3.1, we have that $\{x_n\}$ converges to a common fixed point in $\mathcal{F}$. This completes the proof. □

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