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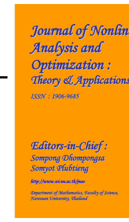

NEW CONVERGENCE ANALYSIS FOR COUNTABLE FAMILY OF RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we construct a new iterative scheme by hybrid methods and prove strong convergence theorem for approximation of a common fixed point of a countable family of relatively quasi-nonexpansive mappings in a uniformly smooth and strictly convex real Banach space with Kadec-Klee property using the properties of generalized f-projection operator. Our results extend many known recent results in the literature.

KEYWORDS : Relatively quasi-nonexpansive mappings; Generalized f-projection operator; Hybrid method; Banach spaces
AMS Subject Classification: 47H06 47H09 47J05 47J25

1. INTRODUCTION

Let E be a real Banach space with dual E^* and C be nonempty, closed and convex subset of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader can consult [1-3] for more details): If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E; $J(x) \neq \emptyset$, $x \in E$; if E is reflexive, then J is a mapping from E onto E^* and if E is smooth, then J is single valued. Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \ \forall x, y \in E. \tag{1.1}$$

From (1.1), we have $(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E.$
Let T be a mapping from C into E. A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence

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$\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is ***relatively nonexpansive*** (see, for example, [4-8]) if the following conditions are satisfied: $F(T) \neq \emptyset$; $\phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ p \in F(T)$ and $F(T) = \widehat{F}(T)$. If T satisfies $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ p \in F(T)$, then T is said to be ***relatively quasi-nonexpansive***. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have studied the methods of approximating the fixed points of relatively quasi-nonexpansive mappings (see, for example, [9-11] the references contained therein). Clearly, in Hilbert space H, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same, for $\phi(x, y) = ||x - y||^2,\ \forall x, y \in H$ and this implies that $\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p||,\ \forall x \in C,\ p \in F(T)$. The examples of relatively quasi-nonexpansive mappings are given in [10].

We next give an example of a mapping that is relatively quasi-nonexpansive but not relatively nonexpansive.

Example 1.1. Let $E = \ell^2$ and

$$\begin{cases} x_0 = (1, 0, 0, 0, \ldots) \\ x_1 = (1, 1, 0, 0, \ldots) \\ x_2 = (1, 0, 1, 0, 0, \ldots) \\ x_3 = (1, 0, 0, 1, 0, 0, \ldots) \\ \ldots \\ x_n = (1, 0, 0, 0, \ldots, 0, 1, 0, 0, \ldots) \\ \ldots \end{cases}$$

Clearly, $\{x_n\}$ converges weakly to x_0. Define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \frac{n}{n+1} x_n, \text{ if } x = x_n (\exists n \geq 1), \\ -x, \text{ if } x \neq x_n (\forall n \geq 1). \end{cases}$$

We can see that $F(T) = \{0\} \neq \emptyset$ and

$$||Tx - 0|| = ||Tx|| \leq ||x|| = ||x - 0||,\ \forall x \in E.$$

Furthermore, since ℓ^2 is a Hilbert space, we obtain

$$\phi(Tx, 0) = ||Tx - 0||^2 = ||Tx||^2 \leq ||x||^2 = ||x - 0||^2 = \phi(x, 0),\ \forall x \in E.$$

It then follows that T is a relatively quasi-nonexpansive mapping. We next show that T is not a relatively nonexpansive mapping. Since $\{x_n\}$ converges weakly to x_0, then there exists $M > 0$ such that $||x_n|| \leq M,\ \forall n \geq 1$. We observe that

$$||Tx_n - x_n|| = \left|\left|\frac{n}{n+1} x_n - x_n\right|\right| = \frac{1}{n+1} ||x_n|| \leq \frac{1}{n+1} M \to 0,\ n \to \infty,$$

but $x_0 \notin F(T)$. Thus, $F(T) \neq \widehat{F}(T)$ even though $F(T) \neq \emptyset$ and $||Tx_n - x_n|| \to 0,\ n \to \infty$. Hence, T is not a relatively nonexpansive mapping.

The above Example 1.1 shows that the class of relatively nonexpansive mappings is properly contained in the class of relatively quasi-nonexpansive mappings.
In [7], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex

real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{w \in C : \phi(w, y_n) \le \phi(w, x_n)\}, \\ W_n = \{w \in C : \langle x_n - w, Jx_0 - Jx_n \rangle, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \ n \ge 0. \end{cases}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)}x_0$, where $F(T) \neq \emptyset$.

In [12], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings: $x_0 \in C$,

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n) \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n) \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0 \rangle)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \le 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $(0,1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on uniformly smooth and uniformly convex Banach E. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common fixed point of T and S in a uniformly smooth and uniformly convex Banach space.

Recently, Li *et al.* [13] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C, \ C_0 = C$

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \le G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \ge 0, \end{cases}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T in a uniformly smooth real Banach space which is also uniformly convex.

Motivated by the above mentioned results and the on-going research, it is our purpose in this paper to prove strong convergence theorem for a countable family of relatively quasi-nonexpansive mappings in a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property using the properties of generalized f-projection operator. Our results extend the results of Matsushita and Takahashi [7], Plubtieng and Ungchittrakool [12], Li *et al.* [13] and many other recent known results in the literature.

2. PRELIMINARIES

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [14], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y \in C} \phi(y, x), \ \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [3, 14-17]). If

E is a Hilbert space, then Π_C is the metric projection of H onto C. Next, we recall the concept of generalized f-projector operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = ||\xi||^2 - 2\langle \xi, \varphi\rangle + ||\varphi||^2 + 2\rho f(\xi), \tag{2.1}$$

where $\xi \in C,\ \varphi \in E^*,\ \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:
(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.1. (Wu and Huang [18]) Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \Big\{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\Big\},\ \forall \varphi \in E^*.$$

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.1) gives

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx\rangle + ||x||^2 + 2\rho f(\xi). \tag{2.2}$$

Definition 2.2. Let E be a real Banach space and C a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \Big\{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\Big\},\ \forall x \in E.$$

Obviously, the definition of T is a relatively-quasi nonexpansive mapping is equivalent to: $F(T) \neq \emptyset$ and $G(p, JTx) \leq G(p, Jx),\ \forall x \in C,\ p \in F(T)$.

Lemma 2.3. *(Li et al. [13]) Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha,\ \forall x \in E.$$

Lemma 2.4. *(Li et al. [13]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:*
(i) *$\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$;*
(ii) *for all $x \in E,\ \hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} + \rho f(y) - \rho f(x) \geq 0,\ \forall y \in C;$$

(iii) *if E is strictly convex, then $\Pi_C^f x$ is a single valued mapping.*

Lemma 2.5. *(Li et al. [13]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx),\ \forall y \in C.$$

We recall that a Banach space E has *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$ as $n \to \infty$. We note that every uniformly convex Banach space has the Kadec-Klee property. For more details on Kadec-Klee property, the reader is referred to [2, 16].

Lemma 2.6. *(Kim et al. [19]) Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which also has Kadec-Klee property. Let T be a closed relatively-quasi nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.7. *(Kim et al. [19]) Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R},\ g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big|\Big| \sum_{n=1}^{\infty} \lambda_n x_n \Big|\Big|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

For the rest of this paper, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p shall be denoted by $x_n \to p$ as $n \to \infty$, $\{x_n\}_{n=0}^{\infty}$ converges weakly to p shall be denoted by $x_n \rightharpoonup p$.

Lemma 2.8. *(Li et al. [13]) Let E be a Banach space and $y \in E$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup x \in \operatorname{int}(D(f))$ and $\lim\limits_{n\to\infty} G(x_n, Jy) = G(x, Jy)$, then $\lim\limits_{n\to\infty} ||x_n|| = ||x||$.*

3. MAIN RESULTS

Theorem 3.1. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{\infty}$ is an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \operatorname{int}(D(f))$ and suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C,\ C_0 = C$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i x_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0,\ n \geq 0, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf\limits_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0,\ i = 1, 2, 3, ...,\ \sum_{i=0}^{\infty} \alpha_{ni} = 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

Proof. We first show that $C_n,\ \forall n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that C_n is closed and convex for each $n \geq 1$. Since $G(z, Jy_n) \leq G(z, Jx_n)$ is equivalent to

$$2\Big(\langle z, Jx_n\rangle - \langle z, Jy_n\rangle\Big) \leq ||x_n||^2 - ||y_n||^2.$$

This implies that C_n is closed and convex $\forall n \geq 0$. This shows that $\Pi_{C_{n+1}}^f x_0$ is well defined for all $n \geq 0$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.3, we see that there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^*\rangle + \alpha, \ \forall y \in E.$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho u^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha. \quad (3.2) \end{aligned}$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (3.2) that

$$G(x^*, Jx_0) \geq G(x_n, Jx_0) \geq (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha$$

for each $x^* \in C_n$. This implies that $\{x_n\}_{n=0}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_n$. It then follows Lemma 2.5 that

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \quad (3.3)$$

It is obvious that

$$\phi(x_{n+1}, x_n) \geq (||x_{n+1}|| - ||x_n||)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists.

We next show that $F \subset C_n, \ \forall n \geq 0$. For $n = 0$, we have $F \subset C = C_0$. Let $x^* \in F$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 2.7, we have for any positive integer $j > 0$ that

$$\begin{aligned} G(x^*, Jy_n) &= G(x^*, (\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_ix_n)) \\ &= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + ||\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_ix_n||^2 + 2\rho f(x^*) \\ &\leq ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^\infty \alpha_{ni}||JT_ix_n||^2 \\ &\quad -\alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) + 2\rho f(x^*) \\ &= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^\infty \alpha_{ni}||JT_ix_n||^2 \\ &\quad -\alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) + 2\rho f(x^*) \\ &\leq G(x^*, Jx_n) - \alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \quad (3.4) \\ &\leq G(x^*, Jx_n). \end{aligned}$$

So, $x^* \in C_n$. This implies that $F \subset C_n, \ \forall n \geq 0$.

Now since $\{x_n\}_{n=0}^\infty$ is bounded in C and E is reflexive, we may assume that $x_n \rightharpoonup p$ and since C_n is closed and convex for each $n \geq 0$, it is easy to see that $p \in C_n$ for each $n \geq 0$. Again since $x_n = \Pi^f_{C_n} x_0$, from the definition of $\Pi^f_{C_n}$, we obtain

$$G(x_n, Jx_0) \leq G(p, Jx_0),\ \forall n \geq 0.$$

Since

$$\begin{aligned}\liminf_{n\to\infty} G(x_n, Jx_0) &= \liminf_{n\to\infty}\Big\{||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n)\Big\}\\ &\geq ||p||^2 - 2\langle p, Jx_0\rangle + ||x_0||^2 + 2\rho f(p) = G(p, Jx_0)\end{aligned}$$

then, we obtain

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(p, Jx_0).$$

This implies that $\lim_{n\to\infty} G(x_n, Jx_0) = G(p, Jx_0)$. By Lemma 2.8, we obtain $\lim_{n\to\infty} ||x_n|| = ||p||$. In view of Kadec-Klee property of E, we have that $\lim_{n\to\infty} x_n = p$.

We next show that $p \in \cap_{i=1}^{\infty} F(T_i)$. By the fact that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_n$, we obtain

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

Now, (3.3) implies that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0 = \lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

It then yields that $\lim_{n\to\infty}(||x_{n+1}|| - ||y_n||) = 0$. Since $\lim_{n\to\infty} ||x_{n+1}|| = ||p||$, we have

$$\lim_{n\to\infty} ||y_n|| = ||p|| \text{ and } \lim_{n\to\infty} ||Jy_n|| = ||Jp||.$$

This implies that $\{||Jy_n||\}_{n=0}^{\infty}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can then assume that $Jy_n \rightharpoonup f_0 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = f_0$. Since

$$\begin{aligned}\phi(x_{n+1}, y_n) &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Jy_n\rangle + ||y_n||^2\\ &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Jy_n\rangle + ||Jy_n||^2.\end{aligned} \tag{3.6}$$

Taking the limit inferior of both sides of (3.6) and in view of weak lower semicontinuity of $||.||$, we have

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, f_0\rangle + ||f_0||^2 = ||p||^2 - 2\langle p, Jx\rangle + ||Jx||^2\\ &= ||p||^2 - 2\langle p, Jx\rangle + ||x||^2 = \phi(p, x),\end{aligned}$$

that is, $p = x$. This implies that $f_0 = Jp$ and so $Jy_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty} ||Jy_n|| = ||Jp||$ and Kadec-Klee property of E^* that $Jy_n \to Jp$. Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $y_n \rightharpoonup p$. It then follows from $\lim_{n\to\infty} ||y_n|| = ||p||$ and Kadec-Klee property of E that $\lim_{n\to\infty} y_n = p$. Hence,

$$\lim_{n\to\infty} ||x_n - y_n|| = 0 = \lim_{n\to\infty} ||Jx_n - Jy_n|| = 0.$$

It then follows from (3.4) that

$$\alpha_{n0}\alpha_{nj} g(||Jx_n - JT_j x_n||) \leq G(x^*, Jx_n) - G(x^*, Jy_n).$$

From $\lim_{n\to\infty} ||x_n - y_n|| = 0$ and $\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0$, we can easily show that

$$G(x^*, Jx_n) - G(x^*, Jy_n) \to 0,\ n \to \infty.$$

Using the condition $\liminf_{n\to\infty}\alpha_{n0}\alpha_{nj} > 0$, we have for any $j \geq 1$ that

$$\lim_{n\to\infty} g(||Jx_n - JT_jx_n||) = 0.$$

By property of g, we have $\lim_{n\to\infty} ||Jx_n - JT_jx_n|| = 0,\ j \geq 1$. Since $x_n \to p$ and J is uniformly continuous, we have $Jx_n \to Jp$. Now, from $\lim_{n\to\infty} ||Jx_n - JT_jx_n|| = 0$, we obtain $\lim_{n\to\infty} JT_jx_n = Jp$. Furthermore, since J^{-1} is hemi-continuous, it follows that $T_jx_n \rightharpoonup p$. On the other hand,

$$\Big|||T_jx_n|| - ||p||\Big| = \Big|||JT_jx_n|| - ||Jp||\Big| \leq ||JT_jx_n - Jp|| \to 0, \text{ as } n \to \infty.$$

By $T_jx_n \rightharpoonup p$, $\lim_{n\to\infty} ||T_jx_n|| = ||p||$ and Kadec-Klee property of E, we obtain that $T_jx_n \to p,$ as $n \to \infty,\ j \geq 1$. Hence, we have

$$\lim_{n\to\infty} ||x_n - T_jx_n|| = 0,\ j \geq 1. \tag{3.7}$$

Since $T_i,\ i \geq 1$ is closed and $x_n \to p$, we have $p \in F = \cap_{i=1}^{\infty} F(T_i)$.

Finally, we show that $p = \Pi_F^f x_0$. Since $F = \cap_{i=1}^{\infty} F(T_i)$ is a closed and convex set, from Lemma 2.4, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_n}^f x_0$ and $w \in F \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0),\ \forall n \geq 0.$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0).$$

From the definition of $\Pi_F^f x_0$ and $p \in F$, we see that $p = w$. This completes the proof. □

Take $f(x) = 0$ for all $x \in E$ in Theorem 3.1, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then we obtain the following corollary.

Corollary 3.1. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{\infty}$ is an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C,\ C_0 = C$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni}JT_ix_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 0, \end{cases} \tag{3.8}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty}\alpha_{n0}\alpha_{ni} > 0,\ i = 1, 2, 3, ...,\ \sum_{i=0}^{\infty} \alpha_{ni} = 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Corollary 3.2. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{N}$ is a finite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a convex* *and lower semicontinuous mapping with* $C \subset \operatorname{int}(D(f))$ *and suppose* $\{x_n\}_{n=0}^{\infty}$ *is iteratively generated by* $x_0 \in C,\ C_0 = C,$

$$\begin{cases} y_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{N}\alpha_{ni}JT_ix_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \le G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}}x_0,\ n \ge 0, \end{cases} \tag{3.9}$$

where J *is the duality mapping on* E. *Suppose* $\{\alpha_{ni}\}_{n=1}^{\infty}$ *for each* $i = 0, 1, 2, ..., N$ *is a sequence in* $(0, 1)$ *such that* $\liminf_{n\to\infty}\alpha_{n0}\alpha_{ni} > 0,\ i = 1, 2, 3, ..., N,\ \sum_{i=0}^{N}\alpha_{ni} = 1.$ *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi^f_F x_0$.

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