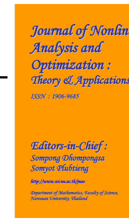

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AN ITERATIVE ALGORITHM FOR A SYSTEM OF GENERALIZED IMPLICIT NONCONVEX VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this paper, we consider a new system of generalized implicit nonconvex variational inequality problems in the setting of two different Hilbert spaces. Using projection method, we establish the equivalence between the system of generalized implicit nonconvex variational inequality problems and a system of nonconvex variational inequality inclusions. Using this equivalence formulation, we suggest an iterative algorithm and show that the sequences generated by this iterative algorithm converge strongly to a solution of the system of generalized implicit nonconvex variational inequality problems. The results presented in this paper can be viewed as an improvement and refinement of previously known results for nonconvex (convex) variational inequality problems.

KEYWORDS : System of generalized implicit nonconvex variational inequality problems; uniformly prox-regular set; projection method; iterative algorithm; strongly monotone mapping; relaxed-Lipschitz continuous mapping.
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1. INTRODUCTION

In 1985, Pang [1] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities and discussed the convergence of method of decomposition for system of variational inequalities. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent, see for applications [1, 2, 3, 4]. Since then many authors, see for example [3, 4, 5, 6, 7, 8]

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studied the existence theory of various classes of system of variational inequalities by exploiting fixed-point theorems and minimax theorems. Recently, a number of iterative algorithms based on projection method and its variant forms have been developed for solving various systems of variational inequalities, see for instance [9, 10, 11, 12].

It is well known that the projection method and its variant forms based on projection operator over convex set are important tools for studying of existence and iterative approximation of solutions of various classes (systems) of variational inequality problems in the convexity settings, but these may not be applicable in general, when the sets are nonconvex. To overcome the difficulties that rise from the nonconvexity of underlying sets, the properties of projection operators over uniformly prox-regular sets are used.

In recent years, Bounkhel *et al.* [13], Moudafi [14], Wen [15], Kazmi *et al.* [16], Noor [17, 18, 19], and the relevant references cited therein], Alimohammady *et al.* [20], Balooee *et al.* [21] suggested and analyzed iterative algorithms for solving some classes (systems) of nonconvex variational inequality problems in the setting of uniformly prox-regular sets.

On the other hand, to the best of our knowledge, the study of iterative algorithms for solving the systems of variational inequality problems considered in [9, 11] in nonconvex setting has not been done so far.

Motivated and inspired by research going on in this area, we introduce a system of generalized implicit nonconvex variational inequality problems (in short, SGINVIP) defined on the uniformly prox-regular sets in different two Hilbert spaces. SGINVIP is different from those considered in [13, 14, 15, 16, 17, 18, 19, 20, 21] and includes the new and known systems of nonconvex (convex) variational inequality problems as special cases. Using the properties of projection operator over uniformly prox-regular sets, we suggest an iterative algorithm for finding the approximate solution of SGINVIP. Further, we prove that SGINVIP has a solution and the approximate solution obtained by iterative algorithm converges strongly to the solution of SGINVIP. The method presented in this paper extend, unify and improves the methods presented in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

2. PRELIMINARIES

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed set in H, not necessarily convex.

First, we recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis, see [23, 24, 25, 26].

Definition 2.1. The *proximal normal cone* of K at $u \in K$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K(u + \alpha\xi)\},$$

where $\alpha > 0$ is a constant and P_K is projection operator of H onto K, that is,

$$P_K(u) = \{u^* \in K : d_K(u) = \|u - u^*\|\},$$

where $d_K(u)$ is the usual distance function to the subset K, that is,

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty closed subset of H. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \xi, v - u \rangle \ \leq \alpha \|v - u\|^2, \ \forall v \in K.$$

Definition 2.2. The *Clarke normal cone*, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \bar{\text{co}}[N_K^P(u)],$$

where $\bar{\text{co}}A$ means the closure of the convex hull of A.

Poliquin *et al.* [24] and Clarke *et al.* [25] have introduced and studied a class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important role in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

Definition 2.3. For a given $r \in (0, \infty]$, a subset K of H is said to be *normalized uniformly r-prox-regular* if and only if every nonzero proximal normal to K can be realized by any r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in \ N_K^P(u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, \ v - u \rangle \ \leq \ \frac{1}{2r} \ \|v - u\|^2, \ \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [23, 25]. It is clear that if $r = \infty$, then uniformly r-prox-regularity of K reduces to its convexity.

It is known that if K is a uniformly r-prox-regular set, the proximal normal cone $N_K^P(u)$ is closed as a set-valued mapping. Thus, we have $N_K^C(u) = N_K^P(u)$.

Now, let us state the following proposition which summarizes some important consequences of the uniformly prox-regularities:

Proposition 2.1. Let $r > 0$ and let K_r be a nonempty closed and uniformly r-prox-regular subset of H. Set $U_r = \{x \in H : d(x, K_r) < r\}$.

- (i) For all $x \in U_r, \ P_{K_r}(x) \neq \emptyset$;
- (ii) For all $r' \in (0, r), \ P_{K_r}$ is Lipschitz continuous with constant $\dfrac{r}{r - r'}$ on $U_{r'} = \{x \in H : d(x, K_r) < r'\}$.

3. System of generalized implicit nonconvex variational inequality problems

Throughout the rest part of the paper, we assume that, for each $i \in \{1, 2\}$, H_i is a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|_i$ and $\langle \cdot, \cdot \rangle_i$, respectively, and K_{i,r_i} is uniformly r_i-prox-regular subset of H_i.

For $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, assume that $A_i, \ C_i : H_i \longrightarrow H_j, \ B_i : H_j \longrightarrow H_i, \ N_i : H_j \times H_i \times H_j \longrightarrow H_i, \ g_i : H_i \longrightarrow H_i$ are single-valued mappings. For any constant $\rho_i > 0$ $(i = 1, 2)$, we consider the system of generalized implicit nonconvex variational inequality problems (SGINVIP): Find $(x_1, x_2) \in H_1 \times H_2$ such that $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ and

$$\langle \rho_i N_i(A_i x_i, B_i x_j, C_i x_i) + \rho_i x_i, y_i - g_i(x_i)\rangle_i + \frac{1}{2r_i}\|y_i - g_i(x_i)\|_i^2 \geq 0,\ \forall y_i \in K_{i,r_i}. \tag{3.1}$$

Some special cases of SGINVIP (3.1)

Case 1. For each $i \in \{1, 2\}$, if $g_i = I_i$, the identity operator, $N_1(A_1x_1, B_1x_2, C_1x_1) = G_1(x_1, x_2) - x_1,\ N_2(A_2x_2, B_2x_1, C_2x_2) = G_2(x_1, x_2) - x_2$ for all $x_i \in H_i$, where $G_i : H_1 \times H_2 \longrightarrow H_i$ is a nonlinear mapping then SGINVIP (3.1) reduces to the system of problems of finding $(x_1, x_2) \in K_{1,r_1} \times K_{2,r_2}$ such that

$$\langle \rho_i G_i(x_1, x_2), y_i - x_i\rangle_i + \frac{1}{2r_i}\|y_i - x_i\|_i^2 \geq 0,\ \forall y_i \in K_{i,r_i}, \tag{3.2}$$

which appears to be new.

Case 2. In Case 1, if $H_1 = H_2, K_{2,r_2} = K_{1,r_1}$ then SGINVIP (3.1) reduces to the nonconvex variational inequality problem of finding $x \in K_{1,r_1}$ such that

$$\langle \rho_1 G_1(x, x), y - x\rangle_1 + \frac{1}{2r_1}\|y - x\|_1^2 \geq 0,\ \forall y \in K_{1,r_1},$$

which appears to be new.

Case 3. In Case 1, for each $i \in \{1, 2\}$, if $r_i = \infty$, i.e., $K_{i,r_i} = K_i$, the convex set in H_i, then SGINVIP (3.1) reduces to the system of variational inequality problems of finding $(x_1, x_2) \in K_1 \times K_2$ such that

$$\langle G_i(x_1, x_2), y_i - x_i\rangle_i \geq 0,\ \forall y_i \in K_i \tag{3.3}$$

which has been studied by Ansari *et al.* [5] and Verma [9].

The following definitions are needed in the proof of main result.

Definition 3.1. A nonlinear mapping $g_1 : H_1 \longrightarrow H_1$ is said to be k_1-*strongly monotone* if there exists a constant $k_1 > 0$ such that

$$\langle g_1(x_1) - g_1(y_1), x_1 - y_1\rangle_1 \ \geq\ k_1\|x_1 - y_1\|_1^2,\ \forall x_1, y_1 \in H_1.$$

Definition 3.2. Let $N_1 : H_2 \times H_1 \times H_2 \longrightarrow H_1,\ A_1, C_1 : H_1 \longrightarrow H_2,\ B_1 : H_2 \longrightarrow H_1$ be nonlinear mappings. Then N_1 is said to be

(i) δ_1-*strongly monotone* with respect to A_1 in the first argument if there exists a constant $\delta_1 > 0$ such that

$$\langle N_1(A_1u, x_1, x_2) - N_1(A_1v, x_1, x_2), u - v\rangle_1 \ \geq\ \delta_1\|u - v\|_1^2,$$

$\forall u, v, x_1 \in H_1,\ x_2 \in H_2$;

(ii) σ_1-*relaxed Lipschitz continuous* with respect to C_1 in the third argument if there exists a constant $\sigma_1 > 0$ such that

$$\langle N_1(x_2, x_1, C_1u) - N_1(x_2, x_1, C_1v), u - v\rangle_1 \leq -\sigma_1\|u - v\|_1^2,$$
$$\forall u, v, x_1 \in H_1,\ x_2 \in H_2;$$

(iii) $L_{(N_1,1)}$-*Lipschitz continuous* in the first argument if there exists a constant $L_{(N_1,1)} > 0$ such that

$$\|N_1(u, x_1, x_2) - N_1(v, x_1, x_2)\|_1 \leq L_{(N_1,1)}\|u - v\|_1,$$
$$\forall x_1 \in H_1,\ u, v, x_2 \in H_2.$$

Similarly, we can define the Lipschitz continuity of N_1 in the second and third arguments. First, we prove the following technical lemmas.

Lemma 3.1. SGINVIP (3.1) is equivalent to the following system of generalized implicit nonconvex variational inclusions: Find $(x_1, x_2) \in H_1 \times H_2$ such that $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ and

$$\mathbf{0}_i \in x_i + N_i(A_i x_i, B_i x_j, C_i x_i) + \rho_i^{-1} N^P_{K_{i,r_i}}(g_i(x_i)), \tag{3.4}$$

for $i = 1, 2$, where $N^P_{K_{i,r_i}}(u)$ denotes the proximal normal cone of K_{i,r_i} at u in the sense of nonconvex analysis (See Definition 2.1), and $\mathbf{0}_i$ is the zero vector of H_i.

Proof. Let $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ be a solution of SGINVIP (3.1). If $N_i(A_i x_i, B_i x_j, C_i x_i) + x_i = \mathbf{0}_i$, then evidently the inclusuion (3.4) follows. If $N_i(A_i x_i, B_i x_j, C_i x_i) + x_i \neq \mathbf{0}_i$, then from (3.1) and Lemma 2.1, we get the inclusion (3.4). Conversely, let $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ be a solution of system (3.4) then it follows from Definition 2.3 that $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1).

Lemma 3.2. $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1) if and only if $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ satisfies the system of relations

$$g_i(x_i) = P_{K_{i,r_i}}[g_i(x_i) - \rho_i(x_i + N_i(A_i x_i, B_i x_j, C_i x_i))], \tag{3.5}$$

for $i = 1, 2$, where $P_{K_{i,r_i}}$ is the projection operator of H_i onto the uniformly r_i-prox-regular set K_{i,r_i}.

Proof. The result follows immediately from Lemma 3.1 and from the fact that $P_{K_{i,r_i}} = (I_i + N^P_{K_{i,r_i}})^{-1}$.

We can rewrite the equations (3.5) as follows:

$$g_i(x_i) = P_{K_{i,r_i}}(w_i), \ w_i = g_i(x_i) - \rho_i(x_i + N_i(A_i x_i, B_i x_j, C_i x_i)). \tag{3.6}$$

The alternative formulation (3.6) enables us to suggest the following iterative algorithm for solving SGINVIP (3.1).

Iterative algorithm 3.1. For given $(w_1^0, w_2^0) \in H_1 \times H_2$, compute the iterative sequences $\{w_1^n\}, \{w_2^n\}, \{x_1^n\}$ and $\{x_2^n\}$ defined by the iterative schemes:

$$g_i(x_i^n) = P_{K_{i,r_i}}(w_i^n), \tag{3.7}$$

$$w_i^{n+1} = (1 - \alpha^n) w_i^n + \alpha^n [g_i(x_i^n) - \rho_i(x_i^n + N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n))], \tag{3.8}$$

for all $n = 0, 1, 2, ...$, and for each $i \in \{1, 2\}$ with $j \in \{1, 2\} \setminus \{i\}$, where $\alpha^n \in (0, 1)$ for $n > 0$ and $\alpha^0 = 1$ and $\sum\limits_{n=1}^{\infty} \alpha^n = \infty$ and $\rho_1, \rho_2 > 0$ are constants.

In Case I, Iterative algorithm 3.1 reduces to the following iterative algorithm for solving the system (3.2).

Iterative algorithm 3.2. For given $(w_1^0, w_2^0) \in H_1 \times H_2$, compute the iterative sequences $\{w_1^n\}, \{w_2^n\}, \{x_1^n\}$ and $\{x_2^n\}$ defined by the iterative schemes:

$$x_i^n = P_{K_{i,r_i}}(w_i^n),$$

$$w_i^{n+1} = (1 - \alpha^n) w_i^n + \alpha^n [x_i^n - \rho_i G_i(x_1^n, x_2^n)], \tag{3.9}$$

for all $n = 0, 1, 2, ...$ and for each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, where $\alpha^n \in (0, 1)$ for $n > 0$ and $\alpha^0 = 1$ and $\sum_{n=1}^{\infty} \alpha^n = \infty$ and $\rho_1, \rho_2 > 0$ are constants.

Now, we prove the existence and iterative approximation of solutions for SGINVIP (3.1).

Theorem 3.1. For each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, let the projection operator $P_{K_{i,r_i}}$ be $(\frac{r_i}{r_i - r'_i})$-Lipschitz continuous; let $A_i,\ C_i : H_i \longrightarrow H_j$ and $B_i : H_j \longrightarrow H_i$ be L_{A_i}-Lipschitz continuous, L_{C_i}-Lipschitz continuous and L_{B_i}-Lipschitz continuous, respectively. Let $g_i : H_i \longrightarrow H_i$ be k_i-strongly monotone and continuous; let $N_i : H_j \times H_i \times H_j \longrightarrow H_i$ be δ_i-strongly monotone with respect to A_i in the first argument, τ_i-relaxed Lipschitz continuous with respect to C_i in the third argument, and $L_{(N_i,p)}$-Lipschitz continuous in the p^{th} argument, where $p = 1, 2, 3$. If the constant ρ_i satisfy the following conditions:

$$M_i - \triangle_i < \rho_i < \min\{M_i + \triangle_i, \Psi_i\}, \tag{3.10}$$

where

$$\begin{aligned}
M_i &= \frac{b_i k_i - a_i e_i}{b_i(1 - e_i^2)};\ \triangle_i = \frac{\sqrt{(b_i k_i - a_i e_i)^2 - b_i^2(1 - e_i^2)(1 - a_i^2)}}{b_i^2(1 - e_i^2)};\\
\Psi_i &< \frac{1}{b_j d_i};\ a_i = \frac{1}{\mu_i} - \phi_i;\ \phi_i = b_i \rho_j d_j;\\
b_i k_i &> a_i e_i + b_i\sqrt{(1 - e_i^2)(1 - a_i^2)};\ d_i = L_{(N_i,2)} L_{B_i};\\
\mu_i &= \frac{r_i}{r_i - r'_i};\ b_i = \frac{1}{\sqrt{2k_i + 3}};\\
e_i &= \sqrt{(1 - 2\delta_i + L^2_{(N_i,1)} L^2_{A_i})} + \sqrt{(1 - 2\sigma_i + L^2_{(N_i,3)} L^2_{C_i})};\\
&\frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3} \in [-1, 0);\ r'_i \in (0, r_i);\ r_i \in (0, \infty].
\end{aligned}$$

Then the sequences $\{x_i^n\}$ and $\{w_i^n\}$ generated by Iterative algorithm 3.1 converge strongly to x_i and w_i, respectively, where (x_1, x_2) with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1).

Proof. From Iterative algorithm 3.1, we have

$$\begin{aligned}
\|w_i^{n+1} - w_i^n\|_i \leq\ & (1 - \alpha^n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1} - x_i^n)\|_i\\
& + \alpha_n \rho_i \|N_i(A_i x_i^{n+1}, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n)\|_i\\
\leq\ & (1 - \alpha_n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n\|g_i(x_i^{n+1}) - g_i(x_i^n)) - \rho_i(x_i^{n+1} - x_i^n)\|_i\\
& + \alpha^n \rho_i \big[\|[N_i(A_i x_i^{n+1}, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^{n+1})\\
& - (x_i^{n+1} - x_i^n)\|_i\\
& + \|N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^n) + (x_i^{n+1} - x_i^n)\|_i\\
& + \|N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^n)] - N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n)\|_i\big].
\end{aligned} \tag{3.11}$$

Since A_i, B_i, C_i are L_{A_i}-, L_{B_i}-, L_{C_i}-Lipschitz continuous, N_i is δ_i-strongly monotone with respect to A_i in the first argument, σ_i-relaxed Lipschitz continuous with respect to C_i, and is $L_{(N_i,1)}$-, $L_{(N_i,2)}$-, $L_{(N_i,3)}$-Lipschitz continuous in the first, second and third arguments, respectively, one can obtain:

$$\|N_i(A_ix_i^{n+1}, B_ix_j^{n+1}, C_ix_i^{n+1}) - N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^{n+1}) - (x_i^{n+1} - x_i^n)\|_i$$
$$\leq \sqrt{(1 - 2\delta_i + L^2_{(N_i,1)}L_{A_i}}\ \|x_i^{n+1} - x_i^n\|_i, \tag{3.12}$$
$$\|N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^{n+1}) - N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^n) + (x_i^{n+1} - x_i^n)\|_i$$
$$\leq \sqrt{(1 - 2\sigma_i + L^2_{(N_i,3)}L_{C_i}}\ \|x_i^{n+1} - x_i^n\|_i, \tag{3.13}$$

and

$$\|N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^n)] - N_i(A_ix_i^n, B_ix_j^n, C_ix_i^n)\|_i \leq L_{(N_i,2)}L_{B_i}\|x_i^{n+1} - x_i^n\|_i. \tag{3.14}$$

Since g_i is k_i-strongly monotone and $P_{K_{i,r_i}}$ be $(\frac{r_i}{r_i - r_i'})$-Lipschitz continuous, then using (3.7), we have

$$\begin{aligned}
\|x_i^{n+1} - x_i^n\|_i^2 &= \|x_i^{n+1} - x_i^n - (g_i(x_i^{n+1}) - g_i(x_i^n)) + (g_i(x_i^{n+1}) - g_i(x_i^n))\|_i^2 \\
&\leq \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 \\
&\quad -2\langle g_i(x_i^{n+1}) - g_i(x_i^n) + x_i^{n+1} - x_i^n, x_i^{n+1} - x_i^n\rangle_i \\
&= \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 - 2\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1} - x_i^n\rangle_i \\
&\quad -2\langle x_i^{n+1} - x_i^n, x_i^{n+1} - x_i^n\rangle_i \\
&\leq \left(\frac{r_i}{r_i - r_i'}\right)^2 \|w_i^{n+1} - w_i^n\|_i^2 - (2k_i + 2)\|x_i^{n+1} - x_i^n\|_i^2,
\end{aligned}$$

or

$$\|x_i^{n+1} - x_i^n\|_i \leq \left(\frac{\mu_i}{\sqrt{2k_i + 3}}\right) \|w_i^{n+1} - w_i^n\|_i, \tag{3.15}$$

where $\mu_i = \left(\dfrac{r_i}{r_i - r_i'}\right)$.

Next, we estimate

$$\begin{aligned}
&\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho(x_i^{n+1} - x_i^n)\|_i^2 \\
\leq\ & \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 \\
& -2\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1} - x_i^n\rangle_i + \rho_i^2\|x_i^{n+1} - x_i^n\|_i^2 \qquad (3.16) \\
\leq\ & \mu_i^2\|w_i^{n+1} - w_i^n\|_i^2 + (\rho_i^2 - 2\rho_i k_i)\|x_i^{n+1} - x_i^n\|_i^2.
\end{aligned}$$

From (3.15) and (3.16), we have

$$\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1} - x_i^n)\|_i \leq \mu_i\sqrt{1 + \frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3}}\ \|w_i^{n+1} - w_i^n\|_i. \tag{3.17}$$

Further, from (3.11)-(3.15) and (3.17), we have

$$\begin{aligned}
\|w_i^{n+2} - w_i^{n+1}\|_i &\leq (1 - \alpha^n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n\mu_i\sqrt{1 + \frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3}}\|w_i^{n+1} - w_i^n\|_i \\
&\quad +\alpha^n\rho_i\left[\frac{\mu_i}{\sqrt{2k_i + 3}}\left\{\sqrt{(1 - 2\delta_i + L^2_{(N_i,1)}L_{A_i}} + \sqrt{1 - 2\sigma_i + L^2_{(N_i,3)}L_{C_i}}\right\}\right.
\end{aligned}$$

$$\times\|w_i^{n+1}-w_i^n\|_i+\frac{\mu_j}{\sqrt{2k_j+3}}(L_{(N_i,2)}L_{B_i})\Bigg]\|w_j^{n+1}-w_j^n\|_j. \qquad (3.18)$$

Define $\|.\|_\star$ on $H_1\times H_2$ by $\|(y_1,y_2)\|_\star=\sum_{i=1}^{2}\|y_i\|_i$ for any $(y_1,y_2)\in H_1\times H_2$. We note that $H_1\times H_2$ is a Hilbert space with induced norm $\|.\|_\star$. It follows from (3.18) that

$$\|(w_1^{n+2},w_2^{n+2})-(w_1^{n+1},w_2^{n+1})\|_\star=\sum_{i=1}^{2}\|w_i^{n+2}-w_i^{n+1}\|_i,$$

$$\leq[1-\alpha^n(1-\theta)]\|(w_1^{n+1},w_2^{n+1})-(w_1^n,w_2^n)\|_\star, \qquad (3.19)$$

where $\theta=\max\{\theta_1,\theta_2\};\ \theta_i=\mu_i[p_i+b_i(\rho_ie_i+\rho_jd_j)]$;

$$p_i=\sqrt{1+\frac{\rho_i^2-2\rho_ik_i}{2k_i+3}};\ b_i=\frac{1}{\sqrt{2k_i+3}};$$
$$d_i=L_{(N_i,2)}L_{A_i};\ e_i=\sqrt{(1-2\delta_i+L^2_{(N_i,1)}L_{A_i}}+\sqrt{(1-2\sigma_i+L^2_{(N_i,3)}L_{C_i}}.$$

From conditions (3.10), we have $0<\theta<1$, and hence, using the similar lines of proof of Theorem 4.3 [22], there exists an integer $n^0>0$ and a number $\alpha\in(0,1)$ such that $(1-\alpha^n(1-\theta))\leq(1-\alpha(1-\theta))$ for all $n>n^0$. Therefore, from (3.19), we have

$$\|(w_1^{n+1},w_2^{n+1})-(w_1^n,w_2^n)\|_\star\leq(1-\alpha(1-\theta))^{n-n^0}\|(w_1^{n^0+1},w_2^{n^0+1})-(w_1^{n^0},w_2^{n^0})\|_\star.$$

Hence for any $m\geq n\geq n^0$, it follows that

$$\|(w_1^m,w_2^m)-(w_1^n,w_2^n)\|_\star\leq\sum_{i=n}^{m-1}\|(w_1^{i+1},w_2^{i+1})-(w_1^{n^0},w_2^{n^0})\|_\star$$

$$\leq\sum_{i=1}^{m-1}(1-\alpha(1-\theta))^{i-n^0}\|(w_1^{n^0+1},w_2^{n^0+1})-(w_1^{n^0},w_2^{n^0})\|_\star. \qquad (3.20)$$

Since $0<(1-\alpha(1-\theta))<1$, it follows from (3.20) that $\|(w_1^m,w_2^m)-(w_1^n,w_2^n)\|_\star\leq\sum_{i=n}^{m-1}\|w_i^m-w_i^n\|\longrightarrow 0$ as $n\longrightarrow\infty$, and hence for each $i\in\{1,2\}$, $\{w_i^n\}$ is a Cauchy sequence in H_i. Assume $w_i^n\longrightarrow w_i$ in H_i as $n\longrightarrow\infty$. We observe from (3.15) that $\{x_i^n\}$ is a Cauchy sequence and hence assume that $x_i^n\longrightarrow x_i$ in H_i as $n\longrightarrow\infty$.

Further, from the continuity of $N_i,A_i,B_i,C_i,g_i,P_{K_{i,r_i}}$ and Iterative algorithm 3.1, we observe that

$$g_i(x_i)=P_{K_{i,r_i}}[g_i(x_i)-\rho_i(x_i+N_i(A_ix_i,B_ix_j,C_ix_i))].$$

Hence it follows from Lemma 3.2 that $(x_1,x_2)\in H_1\times H_2$ with $(g_1(x_1),g_2(x_2))\in K_{1,r_1}\times K_{2,r_2}$ is a solution of SGINVIP (3.1). This completes the proof.

Remark 3.1.

(i) The method presented in this paper unifies the methods considered in [13, 14, 15, 16, 17, 18, 19, 20, 21] to the system of nonconvex variational inequality problems defined on the product of two different Hilbert spaces.

(ii) The method presented in this paper improves the methods considered in [19, 20, 21] in the sense that the continuity of g is required instead of the Lipschitz continuity.

(iii) One needs further research effort to extend the method presented for solving the system of nonconvex variational inequality problems involving set-valued mappings.

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