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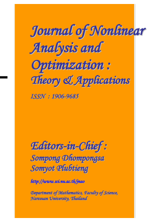

COINCIDENCE POINT THEOREMS IN HIGHER DIMENSION FOR NONLINEAR CONTRACTIONS

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ABSTRACT. In this manuscript, we introduce the concept of a coincidence point of N-order of $F : X^N \to X$ and $g : X \to X$ where $N \geq 2$ and X is an ordered set endowed with a metric d. We prove some coincidence point theorems of such mappings involving nonlinear contractions. The presented results are generalizations of the recent fixed point theorems due to Berzig and Samet [M. Berzig and B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (2012) 1319–1334]. Also, this work is an extension of M. Borcut [M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (2012) 7339–7346].

KEYWORDS : Coincidence point; Nonlinear contractions

1. INTRODUCTION

Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions, which imply the existence of a fixed point. Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered sets. The first result in this direction was given by Turinici [28], where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings [25] presented some applications of Turinici's theorem to matrix equations. Subsequently, many other results in ordered sets have been obtained, see [1]-[4],[11],[12], [17]-[19], [21]-[24].

In [15], Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and studied the problems of the uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to

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problems of the existence and uniqueness of solution for a periodic boundary value problem. In [20], Lakshmikantham and Ćirić introduced the concept of coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$, and proved some coupled coincidence point theorems for nonlinear contraction in partially ordered metric spaces.

We consider the following definitions and results which shall be required in the sequel.

Definition 1.1 ([20]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \quad \text{implies} \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \quad \text{implies} \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 1.2 ([20]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \;\text{ and }\; F(y, x) = gy.$$

Definition 1.3 ([20]). We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if

$$gF(x, y) = F(gx, gy).$$

Lakshmikantham and Ćirić [20] obtained the following result.

Theorem 1.1 ([20]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gv \preceq gy$. Assume that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either F is continuous or X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point.

Many generalizations and extensions of Theorem 1.1 exist in the literature, see [5],[6],[26],[27]. Recently, Berinde and Borcut [13] introduced the concept of tripled fixed point and established fixed point results for mappings having a monotone property and satisfying a contractive condition in ordered metric spaces. Later, Borcut [16] (see also [7]) established a tripled coincidence point theorem for a pair of mappings $F : X \times X \times X \to X$ and $g : X \to X$ satisfying a nonlinear contractive condition in ordered metric spaces. For other tripled fixed point results, see [8, 9, 10].

Definition 1.4 ([16]). Let $(X, \preceq)$ be a partially ordered set, and g a self map on X. The mapping $F : X \times X \times X \to X$ is said to has mixed g-monotone property if for any $x, y, z \in X$

$$x_1,\ x_2 \in X, \quad gx_1 \preceq gx_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z),$$
$$y_1,\ y_2 \in X, \quad gy_1 \preceq gy_2 \Longrightarrow F(x, y_1, z) \succeq F(x, y_2, z),$$
$$z_1,\ z_2 \in X, \quad gz_1 \preceq gz_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 1.5 ([16]). Let X be a non-empty set. Given $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled coincidence point of F and g if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \text{ and } F(z, y, x) = gz.$$

Definition 1.6 ([16]). Let X be a non-empty set. Let $F : X \times X \times X \to X$ and $g : X \to X$ are such that

$$g(F(x, y, z)) = F(gx, gy, gz)$$

whenever $x, y, z \in X$, then F and g are said to be commutative.

Consider also a class of function useful later.

Definition 1.7 (See ([20])). We denote by Θ the set of functions $\theta : [0, \infty) \longrightarrow [0, \infty)$ satisfying

(a) θ is non-decreasing,
(b) $\theta^{-1}(\{0\}) = \{0\}$,
(c) $\theta(t) < t$ for all $t > 0$,
(d) $\lim\limits_{r \longrightarrow t^+} \theta(r) < t$ for all $t > 0$.

Borcut [16] proved the following result.

Theorem 1.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F\colon X \times X \times X \to X$ and $g\colon X \to X$ are such that F has the mixed g-monotone property. Assume there is a function $\phi \in \Theta$ such that*

$$d(F(x, y, z), F(u, v, w)) \le \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}),$$

for any $x, y, z, u, v, w \in X$ for which $gx \succeq gu$, $gv \succeq gy$ and $gz \succeq gw$. Assume that $F(X \times X \times X) \subseteq g(X)$, g is continuous and commutes with F. Also suppose either F is continuous or X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz,$$

that is, F and g have a tripled coincidence point.

Throughout this paper, we will use the following notations:

$$\underbrace{X \times X \cdots X \times X}_{N \text{ terms}} = X^N \quad \text{where } N \text{ is a positive integer,}$$

and

$$(x_{\varphi(p)}, x_{\varphi(p+1)}, \dots, x_{\varphi(p+q)}) := x[\varphi(p : p + q)].$$

Now, following the concept of m-mixed monotone property introduced very recently by Berzig and Samet [14], we introduce:

Definition 1.8. Let $(X, \preceq)$ be an ordered set, N,m are positive integers, $0 \leq m \leq N$, $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings. We say that F has the m-g-mixed monotone property if $F(x_1, \dots, x_m, x_{m+1}, \dots, x_N)$ is monotone g-non-decreasing for the range of components from 1 to m and is monotone g-non-increasing for the range of components from $m+1$ to N, that is,

$$\underline{x}_i, \overline{x}_i \in X, \quad g(\underline{x}_i) \preceq g(\overline{x}_i) \quad \text{implies} \quad F(x_1, \dots, \underline{x}_i, \dots, x_N) \preceq F(x_1, \dots, \overline{x}_i, \dots, x_N), \text{ for } i = 1, \dots, m,$$

and

$$\underline{x}_i, \overline{x}_i \in X, \quad g(\underline{x}_i) \preceq g(\overline{x}_i) \quad \text{implies} \quad F(x_1, \dots, \underline{x}_i, \dots, x_N) \succeq F(x_1, \dots, \overline{x}_i, \dots, x_N), \text{ for } i = m+1, \dots, N,$$

for all $(x_1, \dots, x_N) \in X^N$.

Also, we introduce the concept of a coincidence point of N-order of $F : X^N \to X$ and $g : X \to X$ as follows:

Definition 1.9. Let $(X, \preceq)$ be an ordered set, N,m are positive integers, $0 \leq m \leq N$, $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings such that F has the m-g-mixed monotone property. An element $U = (x_1, x_2, \dots, x_N) \longrightarrow X^N$ is called a coincidence point of N-order of F and g if there exist $2N$ maps $\varphi_1, \dots, \varphi_m : \{1, \dots, m\} \longrightarrow \{1, \dots, m\}$, $\psi_1, \dots, \psi_m : \{m+1, \dots, N\} \longrightarrow \{m+1, \dots, N\}$, $\varphi_{m+1}, \dots, \varphi_N : \{1, \dots, m\} \longrightarrow \{m+1, \dots, N\}$, and $\psi_{m+1}, \dots, \psi_N : \{m+1, \dots, N\} \longrightarrow \{1, \dots, m\}$ such that

$$gx_i = F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]), \text{ for } i = 1, \dots, N. \tag{1.1}$$

Definition 1.10. Let X be a non-empty set. Let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings. We say F and g are commutative if

$$g(F(x_1, x_2, \dots, x_N)) = F(gx_1, gx_2, \dots, gx_N)$$

for all $(x_1, x_2, \dots, x_N) \in X^N$.

In this paper, we establish some coincidence point theorems of N-order for $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ satisfying a contractive condition in complete ordered metric spaces. The presented results extend and generalize many results in literature.

2. MAIN RESULTS

Let (X, d) be a metric space and N be a positive integer, $N \geq 1$. We endow the product set X^N with the metric $\overline{d} : X^N \longrightarrow [0, \infty)$, given by

$$\overline{d}((u_1, u_2, \dots, u_N), (v_1, v_2, \dots, v_N)) = \max_{1 \leq i \leq N} d(u_i, v_i), \tag{2.1}$$

which also will be denoted by d.

Our first result is the following.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. For N, m positive integers, $0 \leq m \leq N$, let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that F has the m-g-mixed monotone property. Suppose that there exists $\theta \in \Theta$ such that*

$$d(F(U), F(V)) \leq \theta \left(\max_{1 \leq i \leq N} d(gx_i, gy_i) \right), \tag{2.2}$$

for all $U = (x_1, \ldots, x_N), V = (y_1, \ldots, y_N) \in X^N$ *such that*

$$gx_i \preceq gy_i, \text{ for } i = 1, \ldots, m \quad \text{and} \quad gx_i \succeq gy_i, \text{ for } i = m+1, \ldots, N.$$

Suppose $F(X^N) \subseteq g(X)$*,* g *is continuous and commute with* F *and suppose either*

(a) F *is continues or*

(b) X *has the following property*

 (i) *if non-decreasing sequence* $\{x_n\} \longrightarrow x$*, then* $x_n \preceq x$ *for all* n*,*

 (ii) *if non-increasing sequence* $\{y_n\} \longrightarrow x$*, then* $y_n \succeq x$ *for all* n*.*

If there exists $U^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \in X^N$ *such that*

$$\begin{array}{l} gx_i^{(0)} \preceq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, m, \\ gx_i^{(0)} \succeq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = m+1, \ldots, N, \end{array} \tag{2.3}$$

where $\varphi_1, \ldots, \varphi_m : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$*,*$\psi_1, \ldots, \psi_m : \{m+1, \ldots, N\} \longrightarrow \{m+1, \ldots, N\}$*,* $\varphi_{m+1}, \ldots, \varphi_N : \{1, \ldots, m\} \longrightarrow \{m+1, \ldots, N\}$*, and* $\psi_{m+1}, \ldots, \psi_N : \{m+1, \ldots, N\} \longrightarrow \{1, \ldots, m\}$*, then there exists* $(x_1, x_2, \ldots, x_N) \longrightarrow X^N$ *satisfying (1.1), that is,* F *and* g *have a coincidence point of* N*-order.*

Proof. Let $U^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \in X^N$ satisfying (2.3). Since $F(X^N) \subseteq g(X)$, we can choose $U^{(1)} = (x_1^{(1)}, \ldots, x_N^{(1)})$ such that

$$gx_i^{(1)} = F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Again from $F(X^N) \subseteq g(X)$, we can choose $U^{(2)} = (x_1^{(2)}, \ldots, x_N^{(2)})$ such that

$$gx_i^{(2)} = F(x^{(1)}[\varphi_i(1:m)], x^{(1)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Continuing this process we can construct sequences $\{U^{(n)}\} = \{(x_1^{(n)}, \ldots, x_N^{(n)})\}$ in X^N such that

$$gx_i^{(n+1)} = F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N. \tag{2.4}$$

Since F has the m-g-mixed monotone property, we have

$$\begin{array}{l} gx_i^{(0)} \preceq gx_i^{(1)} \preceq gx_i^{(2)}, \text{ for } i = 1, \ldots, m. \\ gx_i^{(0)} \succeq gx_i^{(1)} \succeq gx_i^{(2)}, \text{ for } i = m+1, \ldots, N. \end{array}$$

Continuing this process, we can construct N sequences $\{gx_1^{(n)}\}, \ldots, \{gx_N^{(n)}\}$ in X such that

$$\begin{array}{rcl} gx_i^{(n)} &=& F(x^{(n-1)}[\varphi_i(1:m)], x^{(n-1)}[\psi_i(m+1:N)]) \preceq gx_i^{(n+1)} \\ &=& F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), i = 1, \ldots, m \\ gx_i^{(n)} &=& F(x^{(n-1)}[\varphi_i(1:m)], x^{(n-1)}[\psi_i(m+1:N)]) \succeq gx_i^{(n+1)} \\ &=& F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), i = m+1, \ldots, N. \end{array}$$

Assume $(gx_1^{(n+1)}, \ldots, gx_N^{(n+1)}) \neq (gx_1^{(n)}, \ldots, gx_N^{(n)})$ for all $n \geq 0$, that is, $(x_1^{(n)}, \ldots, x_N^{(n)})$ is not a coincidence point of N-order of F and g. For $n \geq 0$, let

$$t_n = \max_{1 \leq i \leq N} d(gx_i^{(n)}, gx_i^{(n+1)}).$$

By assumption, $t_n > 0$ for all $n \geq 0$. We shall prove that $\{t_n\}$ is a decreasing sequence. Since

$$gx_i^{(n)} \preceq gx_i^{(n+1)}, \text{ for } i = 1, \ldots, m, \quad \text{and} \quad gx_i^{(n)} \succeq gx_i^{(n+1)}, \text{ for } i = m+1, \ldots, N. \tag{2.5}$$

we have

$$d(gx_1^{(n)}, gx_1^{(n+1)}) = d(F(x^{(n-1)}[\varphi_1(1:m)], x^{(n-1)}[\psi_1(m+1:N)]), F(x^{(n)}[\varphi_1(1:m)], x^{(n)}[\psi_1(m+1:N)]))$$
$$\leq \theta\left(\max_{\substack{1\leq i\leq m\\ m+1\leq j\leq N}}\left\{d(gx_{\varphi_1(i)}^{(n-1)}, gx_{\varphi_1(i)}^{(n)}), d(gx_{\psi_1(j)}^{(n-1)}, gx_{\psi_1(j)}^{(n)})\right\}\right).$$

Since θ is non-decreasing and $\max_{\substack{1\leq i\leq m\\ m+1\leq j\leq N}}\left\{d(gx_{\varphi_1(i)}^{(n-1)}, gx_{\varphi_1(i)}^{(n)}), d(gx_{\psi_1(j)}^{(n-1)}, gx_{\psi_1(j)}^{(n)})\right\} \leq$ t_{n-1}, then

$$d(gx_1^{(n)}, gx_1^{(n+1)}) \leq \theta(t_{n-1}). \tag{2.6}$$

Similarly, we obtain

$$d(gx_i^{(n)}, gx_i^{(n+1)}) \leq \theta(t_{n-1}), \quad i = 2, \ldots, N. \tag{2.7}$$

Using (2.6) and (2.7) we obtain

$$0 < t_n = \max_{1\leq i\leq N} d(gx_i^{(n)}, gx_i^{(n+1)}) \leq \theta(t_{n-1}) < t_{n-1} \text{ ; since } \theta(t) < t; \text{ for all } t > 0.$$

Thus, a sequence $\{t_n\}$ is monotone decreasing. Therefore, there is some $t_+ > 0$ such that $\lim_{n\longrightarrow\infty} t_n = t_+$. We show that $t = 0$.

Suppose, on the contrary, that $t > 0$. Then, taking the limit as $n \longrightarrow \infty$ of both sides of $t_n \leq \theta(t_n)$ where $\theta \in \Theta$, we obtain

$$t = \lim_{n\longrightarrow\infty} t_n \leq \lim_{n\longrightarrow\infty} \theta(t_{n-1}) < \lim_{n\longrightarrow\infty} t_{n-1} = t$$

which is a contradiction. Thus $t = 0$, that is,

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} \max_{1\leq i\leq N} d(gx_i^{(n)}, gx_i^{(n+1)}) = 0. \tag{2.8}$$

Now we prove that $\left\{gx_i^{(n)}\right\}, i = 1, \ldots, N$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\left\{gx_i^{(n)}\right\}, i = 1, \ldots, N$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and two subsequences of integers $\{\sigma(k)\}$, $\{\mu(k)\}$, $\mu(k) > \sigma(k) \geq k$ with

$$r_k = \max_{1\leq i\leq N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \geq \varepsilon. \tag{2.9}$$

We may also assume

$$\max_{1\leq i\leq N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k)-1)}) < \varepsilon \tag{2.10}$$

by choosing $\sigma(k)$ to be the smallest number exceeding $\sigma(k)$ for which $r_k \geq \varepsilon$. By (2.10) and the triangle inequality, we get for $i = 1, \ldots, N$

$$\begin{aligned} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k)-1)}) + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\ &< \varepsilon + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}). \end{aligned}$$

Letting $k \to \infty$ in above inequality and using (2.8), we have

$$\lim_{k\to\infty} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \leq \varepsilon, \quad i = 1, \ldots, N. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) \\ &+ d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\ &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k))}) \end{aligned}$$

$$\begin{aligned} &+ \quad d(gx_i^{(\mu(k))}, gx_i^{(\mu(k)-1)}) + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\ &< \quad d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + \varepsilon + 2d(gx_i^{(\mu(k))}, gx_i^{(\mu(k)-1)}). \end{aligned}$$

Letting again $k \to \infty$ in above inequality and using (2.8), we have

$$\lim_{k\to\infty} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \leq \lim_{k\to\infty} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) \leq \varepsilon, \quad i = 1, \ldots, N. \tag{2.12}$$

By (2.9), (2.11) and (2.12), we may get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \max_{1\leq i\leq N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) = \varepsilon. \tag{2.13}$$

From (2.5), we have

$$gx_i^{(\sigma(k))} \preceq gx_i^{(\mu(k))}, \text{ for } i = 1, \ldots, m, \quad \text{and} \quad gx_i^{(\sigma(k))} \succeq gx_i^{(\mu(k))}, \text{ for } i = m+1, \ldots, N.$$

Now using this, (2.2), (2.4) and monotonicity of θ, we get

$$\begin{aligned} d(gx_1^{(\sigma(k))}, gx_1^{(\mu(k))}) &= d(F(x^{(\sigma(k)-1)}[\varphi_1(1:m)], x^{(\sigma(k)-1)}[\psi_1(m+1:N)]), \\ &\quad F(x^{(\mu(k)-1)}[\varphi_1(1:m(n))], x^{(\mu(k)-1)}[\psi_1(m+1:N)])) \\ &\leq \theta\left(\max_{1\leq i\leq N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)})\right). \end{aligned}$$

As a consequence, similarly we have

$$r_k = \max_{1\leq i\leq N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \leq \theta\left(\max_{1\leq i\leq N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)})\right), \text{ for } i = 2, \ldots, N. \tag{2.14}$$

Letting $k \longrightarrow \infty$ and using (2.13), we get

$$\varepsilon \leq \theta(\varepsilon) < \varepsilon, \tag{2.15}$$

which is a contradiction. Therefore, we proved that $\left\{gx_i^{(n)}\right\}, i = 1, \ldots, N$ are Cauchy sequences. Since X is complete, there exist $U = (x_1, \ldots, x_N) \in X$ such that,

$$\lim_{n\longrightarrow\infty} gx_i^{(n)} = x_i, \quad i = 1, \ldots, N. \tag{2.16}$$

Thus, by continuity of g, we get

$$\lim_{n\longrightarrow\infty} g(gx_i^{(n)}) = gx_i, \quad i = 1, \ldots, N. \tag{2.17}$$

From, (2.4) and commutativity of F and g, we have,

$$g(gx_i^{(n+1)}) = g(F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)])) \tag{2.18}$$

$$= F(gx_{\varphi_i(1)}^{(n)}, \ldots, gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)}, \ldots, gx_{\psi_i(N)}^{(n)}) \tag{2.19}$$

for $i = 1, \ldots, N$.

Suppose now that (a) holds. We take $n \longrightarrow \infty$ and using the continuity of F, we get

$$\begin{aligned} gx_i = \lim_{n\longrightarrow\infty} g(gx_i^{(n+1)}) &= \lim_{n\longrightarrow\infty} F(gx_{\varphi_i(1)}^{(n)}, \ldots, gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)}, \ldots, gx_{\psi_i(N)}^{(n)}) \\ &= F(\lim_{n\longrightarrow\infty} gx_{\varphi_i(1)}^{(n)}, \ldots, \lim_{n\longrightarrow\infty} gx_{\varphi_i(m)}^{(n)}, \lim_{n\longrightarrow\infty} gx_{\psi_i(m+1)}^{(n)}, \ldots, \lim_{n\longrightarrow\infty} gx_{\psi_i(N)}^{(n)}) \\ &= F(x_{\varphi_i(1)}, \ldots, x_{\varphi_i(m)}, x_{\psi_i(m+1)}, \ldots, x_{\psi_i(N)}) \\ &= F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]), \quad \text{for} \quad i = 1, \ldots, N. \end{aligned}$$

Thus, we proved that F and g have a coincidence point of N-order.

Suppose now that (b) holds. Since $\left\{g(x_i^{(n)})\right\}$ is monotone non-decreasing for $i = 1,\dots,m$ and non-increasing for $i = m+1,\dots,N$, and $gx_i^{(n)} \longrightarrow x_i$, $i = 1,\dots,N$, from (b) for all n, we have two cases. The first case is $gx_{\varphi(i)}^{(n)} \preceq x_{\varphi(i)}$, for $i = 1,\dots,m$ and $gx_{\psi(i)}^{(n)} \succeq x_{\psi(i)}$, for $i = m+1,\dots,N$. The second case is $gx_{\varphi(i)}^{(n)} \succeq x_{\varphi(i)}$, for $i = 1,\dots,m$ and $gx_{\psi(i)}^{(n)} \preceq x_{\psi(i)}$ for $i = m+1,\dots,N$. For both cases and by the triangle inequality, the monotonicity of θ, (2.2) and (2.18), we get

$$\begin{aligned}
&d(gx_i, F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]))\\
&\le d(gx_i, g(gx_i^{(n+1)})) + d(g(gx_i^{(n+1)}), F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]))\\
&\le d(gx_i, g(gx_i^{(n+1)})) + d(F(gx_{\varphi_i(1)}^{(n)},\dots,gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)},\dots,gx_{\psi_i(N)}^{(n)}),\\
&F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]))\\
&\le d(gx_i, g(gx_i^{(n+1)})) + \theta\left(\max_{\substack{1\le i\le m\\ m+1\le j\le N}} \left\{d(g(gx_{\varphi(i)}^{(n)}), gx_{\varphi(i)}^{(n)}), d(g(gx_{\psi(j)}^{(n)}), gx_{\psi(j)}^{(n)})\right\}\right)\\
&\le d(gx_i, g(gx_i^{(n+1)})) + \theta(\max_{1\le i\le N} d(g(gx_i^{(n)}), gx_i^{(n)})).
\end{aligned}$$

So letting $n \longrightarrow \infty$ yields $d(gx_i, F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])) \le 0$. Hence $gx_i = F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])$ for $i = 1,\dots,N$. Then, we proved that F and g have a coincidence point of N-order. This completes the proof of Theorem 2.1. □

Proposition 2.1. *Theorem 5 in [16] is a particular case of Theorem 2.1.*

Proof. Let $F : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings satisfying the hypotheses of Theorem 5 in [16]. For all $x_1, x_2, x_3 \in X$, define the mapping $G : X \times X \times X \longrightarrow X$ by

$$G(x_1, x_2, x_3) = F(x_1, x_3, x_2).$$

Since F has the g-mixed monotone property, the mapping G has the 2-g-mixed monotone property with $N = 3$. F is continuous, thus also G is continuous . Now, for all $X_1, X_2, X_3 \in X$ and $Y_1, Y_2, Y_3 \in X$ with $gX_1 \preceq gY_1, gX_2 \preceq gY_2$ and $gX_3 \succeq gY_3$, we have

$$\begin{aligned}
d(G(X_1, X_2, X_3), G(Y_1, Y_2, Y_3)) &= d(F(Y_1, Y_3, Y_2), F(X_1, X_3, X_2))\\
&\le \theta(\max\{d(X_1, Y_1); d(X_2, Y_2); d(X_3, Y_3)\}).
\end{aligned}$$

Moreover, from the hypotheses of Theorem 5 in [16], we know that there exist $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \in X$ such that $gx_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$, $gx_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)}, x_2^{(0)})$ and $gx_3^{(0)} \preceq F(x_3^{(0)}, x_2^{(0)}, x_1^{(0)})$. Denote $X_1^{(0)} = x_3^{(0)}, X_2^{(0)} = x_1^{(0)}$ and $X_3^{(0)} = x_2^{(0)}$, we have

$$gX_1^{(0)} \preceq F(X_1^{(0)}, X_3^{(0)}, X_2^{(0)}), gX_2^{(0)} \preceq F(X_2^{(0)}, X_3^{(0)}, X_1^{(0)}) \quad\text{and}\quad gX_3^{(0)} \succeq F(X_3^{(0)}, X_2^{(0)}, X_3^{(0)}).$$

This implies that

$$gX_1^{(0)} \preceq G(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}), gX_2^{(0)} \preceq G(X_2^{(0)}, X_1^{(0)}, X_3^{(0)}) \quad\text{and}\quad gX_3^{(0)} \succeq G(X_3^{(0)}, X_3^{(0)}, X_2^{(0)}).$$

Now, all the required hypotheses of Theorem 2.1 are satisfied with $N = 3$, $m = 2$, $\varphi_1(1) = 1$, $\varphi_1(2) = 2$, $\varphi_2(1) = 2$, $\varphi_2(2) = 1$ and $\psi_3(3) = 2$. Applying Theorem 2.1, we get that there exist $X_1, X_2, X_3 \in X$ such that

$$gX_1 = G(X_1, X_2, X_3), gX_2 = G(X_2, X_1, X_3) \quad\text{and}\quad gX_3 = G(X_3, X_3, X_2),$$

that is,

$$gX_1 = F(X_1, X_3, X_2), gX_2 = F(X_2, X_3, X_1) \quad \text{and} \quad gX_3 = F(X_3, X_2, X_3).$$

This implies that $(u_1, u_2, u_3) = (X_2, X_3, X_1)$ is a tripled coincidence point of F and g. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. For N, m positive integers, $0 \le m \le N$, let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that F has the m-g-mixed monotone property. Suppose that there exists $k \in [0, 1)$ such that*

$$d(F(U), F(V)) \le k \max_{1\le i\le N} d(gx_i, gy_i), \tag{2.20}$$

for all $U = (x_1, \dots, x_N), V = (y_1, \dots, y_N) \longrightarrow X^N$ such that

$$gx_i \preceq gy_i, \text{ for } i = 1, \dots, m \quad \text{and} \quad gx_i \succeq gy_i, \text{ for } i = m + 1, \dots, N.$$

Suppose $F(X^N) \subseteq g(X)$, g is continuous and commute with F and suppose either

(a) *F is continues or*
(b) *X has the following property*
  (i) *if non-decreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$ for all n.*
  (ii) *if non-increasing sequence $\{y_n\} \longrightarrow x$, then $y_n \succeq x$ for all n.*

If there exists $U^{(0)} = (x_1^{(0)}, \dots, x_N^{(0)}) \in X^N$ such that

$$\begin{aligned} gx_i^{(0)} &\preceq F(x^{(0)}[\varphi_i(1 : m)], x^{(0)}[\psi_i(m + 1 : N)]), \text{ for } i = 1, \dots, m,\\ gx_i^{(0)} &\succeq F(x^{(0)}[\varphi_i(1 : m)], x^{(0)}[\psi_i(m + 1 : N)]), \text{ for } i = m + 1, \dots, N. \end{aligned} \tag{2.21}$$

where $\varphi_1, \dots, \varphi_m : \{1, \dots, m\} \longrightarrow \{1, \dots, m\}$,$\psi_1, \dots, \psi_m : \{m + 1, \dots, N\} \longrightarrow \{m+1, \dots, N\}$, $\varphi_{m+1}, \dots, \varphi_N : \{1, \dots, m\} \longrightarrow \{m+1, \dots, N\}$, and $\psi_{m+1}, \dots, \psi_N : \{m+1, \dots, N\} \longrightarrow \{1, \dots, m\}$, then there exists $(x_1, x_2, \dots, x_N) \longrightarrow X^N$ satisfying (1.1).

Proof. Following the proof of Theorem 2.1, for $\theta(t) = kt$ with $k \in [0, 1)$, then there exists $(x_1, x_2, \dots, x_N) \in X^N$ satisfying (1.1). □

Remark 2.3. Corollary 2.2 generalizes Theorem 3.1 and Theorem 3.2 of Berzig and Samet [14] (when taking $\delta_i = k$ for each $i = 1, \dots, N$).

Theorem 2.2. *By adding to the hypotheses of Theorem 2.1 the condition: for every $U = (x_1, \dots, x_N), V = (y_1, \dots, y_N) \in X^N$, there exists a $W = (z_1, \dots, z_N) \in X^N$ such that $(F(z[\varphi_1(1 : m)], z[\psi_1(m+1 : N)]), \dots, F(z[\varphi_N(1 : m)], z[\psi_N(m+1 : N)]))$ is comparable to $(gx_1, \dots, gx_N)$ and $(gy_1, \dots, gy_N)$. Then F and g have a unique N-order coincidence point.*

Proof. If $U = (x_1, \dots, x_N)$ and $V = (y_1, \dots, y_N) \in X^N$ are two N-order coincidence points of F and g, then we show that

$$d((gx_1, \dots, gx_N), (gy_1, \dots, gy_N)) = 0.$$

Since U and V are two N-order coincidence points, we have

$$gx_i = F(x[\varphi_i(1 : m)], x[\psi_i(m + 1 : N)]), \text{ for } i = 1, \dots, N,$$

and

$$gy_i = F(y[\varphi_i(1 : m)], y[\psi_i(m + 1 : N)]), \text{ for } i = 1, \dots, N.$$

By assumption, there is $W = (z_1, \dots, z_N) \in X^N$ such that $(F(z[\varphi_1(1 : m)], z[\psi_1(m+1 : N)]), \dots, F(z[\varphi_N(1 : m)], z[\psi_N(m + 1 : N)]))$ is comparable to $(gx_1, \dots, gx_N)$

and $(gy_1, \ldots, gy_N)$. Set $z_i^{(0)} = z_i$, for $i = 1, \ldots, N$, we can choose $W^{(1)} = (z_1^{(1)}, \ldots, z_N^{(1)})$ such that

$$gz_i^{(1)} = F(z^{(0)}[\varphi_i(1:m)], z^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

For $n > 1$, continuing this process we can construct the sequences $\left\{gz_i^{(n)}\right\}, i = 1, \ldots, N$ such that

$$gz_i^{(n+1)} = F(z^{(n)}[\varphi_i(1:m)], z^{(n)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Further, set $x_i^{(0)} = x_i$, for $i = 1, \ldots, N$ and $y_i^{(0)} = y_i$, for $i = 1, \ldots, N$, on the same way, define the sequences $\left\{gx_i^{(n)}\right\}, i = 1, \ldots, N$ and $\left\{gy_i^{(n)}\right\}, i = 1, \ldots, N$.

Then it is easy to show that, for all $n \geq 1$,

$$gx_i^{(n)} = F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N,$$

and

$$gy_i^{(n)} = F(y[\varphi_i(1:m)], y[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Since

$$F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]) = gx_i^{(1)} = gx_i, \text{ for } i = 1, \ldots, N,$$

and

$$F(z[\varphi_i(1:m)], z[\psi_i(m+1:N)]) = gz_i^{(1)}, \text{ for } i = 1, \ldots, N,$$

are comparable, then $gx_i \preceq gz_i^{(1)}$ for $i = 1, \ldots, m$ and $gx_i \succeq gz_i^{(1)}$ for $i = m+1, \ldots, N$. It is easy to show that gx_i for $i = 1, \ldots, N$ are also comparable to $gz_i^{(n)}$ for $i = 1, \ldots, N$, that is, $gx_i \preceq gz_i^{(n)}$ for $i = 1, \ldots, m$ and $gx_i \succeq gz_i^{(n)}$ for $i = m+1, \ldots, N$ for all $n \geq 1$. Thus, from (2.2) and using the proof of Theorem 2.1 we have

$$\begin{aligned} d(gx_i, gz_i^{(n+1)}) &= d(F(x[\varphi_i(1:m)], x[\psi_k(m+1:N)]), F(z^{(n)}[\varphi_i(1:m)], z^{(n)}[\psi_i(m+1:N)])) \\ &\leq \theta\left(\max_{\substack{1\leq k\leq m \\ m+1\leq j\leq N}} \left\{d(gx_{\varphi_i(k)}, gz^{(n)}_{\varphi_i(k)}), d(gx_{\psi_i(j)}, gz^{(n)}_{\psi_i(j)})\right\}\right) \\ &\leq \theta\left(\max_{1\leq i\leq N}\left\{d(gx_{(i)}, gz^{(n)}_{(i)})\right\}\right), \quad \text{for } i = 1, \ldots, N. \end{aligned}$$

Thus, we obtain

$$\max_{1\leq i\leq N} d(gx_i, gz_i^{(n+1)}) \leq \theta\left(\max_{1\leq i\leq N}\left\{d(gx_i, gz_i^{(n)})\right\}\right) \leq \cdots \leq \theta^n\left(\max_{1\leq i\leq N}\left\{d(gx_i, gz_i^{(1)})\right\}\right). \tag{2.22}$$

But, it is known that the fact that $\theta \in \Theta$ implies

$$\lim_{n\to\infty} \theta^n(t) = 0 \quad \text{for all } t > 0.$$

By letting $n \longrightarrow \infty$ in (2.22), we obtain

$$\lim_{n\longrightarrow\infty} d(gx_i, gz_i^{(n+1)}) = 0, \quad \text{for } i = 1, \ldots, N. \tag{2.23}$$

Similarly, one can prove that

$$\lim_{n\longrightarrow\infty} d(gy_i, gz_i^{(n+1)}) = 0, \quad \text{for } i = 1, \ldots, N. \tag{2.24}$$

By the triangle inequality, (2.23) and (2.24) we have

$$d(gx_i, gy_i) \leq d(gx_i, gz_i^{(n+1)}) + d(gz_i^{(n+1)}, gy_i) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

which ends the proof. □

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