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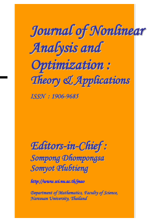

ON THE GENERALIZED VARIATIONAL-LIKE INEQUALITIES PROBLEMS FOR MULTIVALUED MAPPINGS

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ABSTRACT. In this paper, two classes of generalized variational-like inequalities problems for multivalued mappings are introduced and then by using KKM technique and Kakutani-Fan-Glicksberg fixed point theorem the solvability of them are investigated when the mappings are relaxed $\eta-\alpha$-monotone. One can consider this paper the topological vector space version of reference [15]..

KEYWORDS : Generalized multivalued variational like inequalities; KKM-mappings, η-hemicontinuity; η-Coercivity; relaxed η-α-monotone; Relaxed η-α-semimonotone mappings
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1. INTRODUCTION

The existence of solutions for variational inequality problems, complementarity problems, equilibrium problems and others is mainly dependent on the monotonicity of a map (see [1, 2, 3, 4, 6, 8, 10, 14, 19]). Recently, many authors, see [7, 8, 9, 10, 11] considered the quasimonotonicity in dealing with variational inequality problems. Verma [17, 18] studied and established some existence theorems for a solution of a class of nonlinear variational inequality problems with p-monotone and p-Lipschitz mappings in the setting of reflexive Banach spaces.

Inspired and motivated by several authors[1, 3, 4, 7, 9, 13, 20], we introduce two new concepts of relaxed η-α-semimonotonicity as well as two classes of variational-like inequalities with relaxed η-α-monotone mappings and relaxed η-α-semimonotone mappings. Using KKM-technique, we obtain the existence of a solution for variational-like inequalities problems with relaxed η-α-monotone mappings in the setting of reflexive Banach spaces. We also present the solvability of variational-like inequalities problems with η-α-semimonotone mappings for an arbitrary Banach space by applying of Kakutani-Fan fixed point theorem [5, 20].

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2. VARIATIONAL-LIKE INEQUALITIES WITH RELAXED η-α-MONOTONE MAPPINGS

Throughout this paper, unless otherwise specified, we always let E be a Hausdorff topological vector space with dual space $E^\star$, K a nonempty closed convex subset of E, T a multivalued mapping from K to $E^\star$, and $\eta : K \times K \longrightarrow K$ and $\alpha : E \longrightarrow \Re$(the real numbers) are mappings. Furthermore, we assume that $\alpha(0) = 0$ and $\lim_{t\longrightarrow 0^+} \frac{\alpha(tz)}{t} = 0$, for all $z \in K$. This means that α the directional derivative at θ(zero of E) at every direction $z \in K$ is zero. For examples of these mappings, one can consider all α which has the property $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$. We note that if we take $E = \Re$ then it is easy to see that the directional derivative of the mapping $\alpha(x) = |x|$ at θ in each direction $z \in E$ is zero but it dose not satisfy $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$.

Definition 2.1. A multivalued mapping $T : K \longrightarrow 2^{E^\star}$ ($2^{E^\star}$ denotes the set of all subsets of $E^\star$) is said to be relaxed η-α-monotone if there exist mappings $\eta : K \times K \longrightarrow K$ and $\alpha : E \longrightarrow \Re$ such that the following inequality holds,

$$\langle u - v,\ \eta(x, y)\rangle \ \geq\ \alpha(x - y),\ \text{for all } x, y \in K,\ u \in T(x),\ \text{and } v \in T(y). \qquad (2.1)$$

Remark:

(i) If $\eta(x, y) = x - y$, for all $x, y \in K$ then (2.1) becomes

$$\langle u - v,\ x - y\rangle \ \geq\ \alpha(x - y),\ \text{for all } u \in T(x), \text{and } v \in T(y), \qquad (2.1a)$$

and T is called relaxed α-monotone.

(ii) If $T : K \longrightarrow E^\star$ is a single valued mapping then (2.1) becomes

$$\langle Tx - Ty,\ \eta(x, y)\rangle \ \geq\ \alpha(x - y),\ \text{for all } x, y \in K, \qquad (2.1b)$$

and T is called relaxed η-α-monotone mapping (see [14]).

(iii) If $\eta(x, y) = x - y$, for all $x, y \in K$ and $\alpha(z) = k\|z\|^p$, where p and k are positive constants, then $(2.1b)$ reduces to

$$\langle Tx - Ty,\ x - y\rangle \ \geq\ K\|x - y\|^p,\ \text{for all } x, y \in K,$$

and T is called p-monotone (see [12, 20]).

Definition 2.1 Let X and Y be two topological spaces. A set-valued mapping $G : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U,\ G(t) \subseteq V;\ G$ is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(iii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U,\ G(t) \cap V \neq \emptyset;\ G$ is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(vi) **continuous** if G is both lower semi-continuous and upper semi-continuous.

Proposition 2.1 ([16]) Let X and Y be two topological spaces. A set-valued mapping $T : X \longrightarrow 2^Y$ is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ which converges to x there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \longrightarrow y$.

Definition 2.2 Let $T : K \longrightarrow 2^{E^\star}$ and $\eta : K \times K \longrightarrow K$ be the two mappings. We say that T is lower η-hemicontinuous whenever, for any $x, y \in K$, the mapping

$f : [0, 1] \longrightarrow 2^{(-\infty,+\infty)}$ defined by,

$$f(t) \ = \ \langle T(x + t(y - x)),\ \eta(y, x)\rangle$$

is lower semicontinuous at 0.

Remark that this definition is weaker than the corresponding definition given in [3].

Definition 2.3 ([6]) A mapping $F : K \longrightarrow 2^E$ is said to be a KKM-mapping, if for any $\{x_1, x_2, \ldots, x_n\} \subset K$, $\text{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$, where $2^E\backslash\{\emptyset\}$ denotes the family of all nonempty subsets of E.

Lemma 2.1([6]) Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} F(x)$ is compact. Then

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Theorem 2.1. Let $T : K \longrightarrow 2^{E^\star}$ be lower η-hemicontinuous and relaxed η-α-monotone mapping. Let $f : K \times K \longrightarrow R \cup \{+\infty\}$ be a proper function (that is $f \neq +\infty$) and $\eta : K \times K \longrightarrow E$ be a mapping. Assume that

(i) $\eta(x, x) = 0$, for all $x \in K$,
(ii) for any fixed $x \in K$ and $u \in Ty$, the mapping $y \longrightarrow \langle u, \eta(y, x)\rangle$ is convex,
(iii) for any fixed $x \in K$, the mapping $y \longrightarrow f(y, x)$ is convex.

Then the following two variational-like inequality problems are equivalent (that is, their solution sets are equal):

(i) Find $x \in K$ such that

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ 0, \text{ for all } y \in K \text{ and } u \in T(x). \tag{2.2}$$

(ii) Find $x \in K$ such that

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ \alpha(y - x), \text{ for all } y \in K \text{ and } v \in T(y). \tag{2.3}$$

Proof. Let $x \in K$ be a solution of (2.2). Since T is relaxed η-α-monotone, we have

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ \langle u, \eta(y, x)\rangle + \alpha(y - x) + f(y, x) - f(x, x)$$

for all $y \in K$, $v \in T(y)$. Then $x \in K$ is a solution of (2.3).

For vice versa, let $x \in K$ be a solution of (2.3). Assume that y is an arbitrary element of K and $u \in T(x)$. Since x is a solution of (2.3) then $f(x, x) < \infty$. Letting

$$y_t \ = \ (1 - t)x + ty,\ t \in [0, 1],$$

(note K is a convex set) then $y_t \in K$. Moreover y_t approaches to x when t converges to zero and so by Proposition 2.1 (note $u \in T(x)$ and T is lower $\eta-$ hemicontinuous) there is $v_t \in T(y_t)$, such that

$$\langle v_t, \eta(y, x)\rangle \longrightarrow \langle u,\ \eta(y, x)\rangle \text{ if } t \longrightarrow 0 \quad (*)$$

and hence (note that x is a solution of (2.3))

$$\langle v_t, \eta(y_t, x)\rangle + f(y_t, x) - f(x, x) \ \geq \ \alpha(y_t - x) \ = \ \alpha(t(y - x)). \tag{2.4}$$

By condition (iii) we get

$$f(y_t,x) - f(x,x) \;=\; f((1-t)x + ty, x) - f(x,x) \;\leq\; t(f(y,x) - f(x,x)) \qquad (2.5)$$

and also conditions (ii) and (i) imply that

$$\begin{aligned} \langle v_t, \eta(y_t,x)\rangle \;&=\; \langle v_t, \eta((1-t)x + ty, x)\rangle \\ &\leq\; (1-t)\langle v_t, \eta(x,x)\rangle + t\langle v_t, \eta(y,x)\rangle \\ &=\; t\langle v_t, \eta(y,x)\rangle\;. \end{aligned} \qquad (2.6)$$

It follows from (2.4)-(2.6) , for $t \in]0,1]$, that,

$$\langle v_t, \eta(y,x)\rangle + f(y,x) - f(x,x) \;\geq\; \frac{\alpha(t(y-x))}{t} = \frac{\alpha(t(y-x)) - \alpha(\theta)}{t}, \qquad (2.7)$$

for all $y \in K$ and $v_t \in T(y_t)$. Now the result follows by letting $t \longrightarrow 0$ in (2.7), using (*), and the fact that α has nonnegative directional derivative at zero in each direction. That is

$$\langle u, \eta(y,x)\rangle + f(y,x) - f(x,x) \;\geq\; 0, \text{ for all } y \in K \text{ and } u \in T(x).$$

Hence $x \in K$ is a solution of (2.2). This completes the proof.

We need the following theorem in the sequel.

Theorem 2.2. Let K be a nonempty closed convex subset of a topological vector space E and $E^\star$ the dual space of E. Let $T : K \longrightarrow 2^{E^\star}\backslash\{\emptyset\}, f : K \times K \longrightarrow R \cup \{+\infty\}$ and $\eta : K \times K \longrightarrow E$ be three mappings such that,

(i) $\eta(x,y) + \eta(y,x) = 0$, for all $x \in K$,

(ii) for any fixed $y \in K$, the mapping $x \longrightarrow \langle Tx, \eta(y,x)\rangle + f(y,x) - f(x,x)$ is lower semi-continuous,

(iii) for any fixed $y \in K$, the mappings $x \longrightarrow \eta(x,y)$ and $x \longrightarrow f(x,y)$ are concave and convex, respectively,

(iv) $\langle u_i - u_j, \eta(a_i,a_j)\rangle \geq 0$, for each finite subset $A = \{a_1, a_2, ..., a_n\}$ of K, $y \in coA$ and $u_i \in T(y)$,

(v) there exist a compact convex subset D of K and a compact subset B of K such that

$$\forall x \in K\backslash B\; \exists z \in D:\; \langle u,\, \eta(z,x)\rangle + f(z,x) - f(x,x) \;<\; 0, \text{ for some } u \in T(z).$$

Then the solution set of problem (2.2) is nonempty and compact.

Proof. Define set-valued mapping, $F : K \longrightarrow 2^E$ as follows:

$$F(y) \;=\; \{x \in K \,:\, \forall\, u \in T(x),\; \langle u,\, \eta(y,x)\rangle + f(y,x) - f(x,x) \;\geq\; 0\}.$$

We claim that F is a KKM mapping. If F is not a KKM-mapping, then there exist subset $\{y_1, y_2, \ldots, y_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} t_i = 1$,

$$z \;=\; \sum_{i=1}^{n} t_i\, y_i \;\notin\; \bigcup_{i=1}^{n} F(y_i),$$

and hence there exist $u_i \in T(y)$, for $i = 1, 2, ..., n$ such that

$$\langle u_i,\, \eta(y_i,z)\rangle + f(y_i,z) - f(z,z) < 0, \text{ for } i = 1, 2, \ldots, n,$$

and so

$$\sum_{i=1}^{n} t_i \langle u_i,\, \eta(y_i,z)\rangle + \sum_{i=1}^{n} t_i f(y_i,z) - f(z,z) < 0,$$

and by (iii) (f is convex in the first variable) we have

$$\sum_{i=1}^{n} t_i\langle u_i,\ \eta(y_i,z) < 0,$$

and by (i) (note $\eta(y_i,z)=-\eta(z,y_i)$ and $z=\sum_{j=1}^{n} t_j y_j$) we get

$$-\sum_{i=1}^{n} t_i\langle u_i,\ \eta(z,y_i) < 0,$$

and it follows from (iii) and (i) that

$$-\sum_{j=1}^{n}\sum_{i=1}^{n} t_i t_j\langle u_i,\ \eta(y_j,y_i) < 0,$$

and so by (i) (note $\eta(y_i,y_i)=0, \eta(y_i,y_j)=-\eta(y_j,y_i)$) we get

$$\sum_{i<j} t_i t_j\langle u_i-u_j,\ \eta(y_i,y_j)\rangle\ < 0,$$

and so $\langle u_i-u_j,\ \eta(y_i,y_j)\rangle\ < 0$, for some $i<j$,which is contradicted (by (iv)). This implies that F is a KKM-mapping. We claim that $F(y)$ is closed for all $y\in K$. Indeed, let $\{x_\alpha\}$ be a net in $F(y)$ which converges to $x\in K$. We have to show that $x\in F(y)$. To see this let $v\in T(x)$ be an arbitrary element. By (ii) through Proposition 2.1 there is net $\{v_\alpha\}$ in E^* with $v_\alpha\in T(x_\alpha)$ such that

$$\langle v_\alpha,\eta(y,x_\alpha)\rangle+f(y,x_\alpha)-f(x_\alpha,x_\alpha)\longrightarrow\langle v,\eta(y,x)\rangle+f(y,x)-f(x,x)\quad (I)$$

and since $x_\alpha\in F(y)$ we deduce from (I) that

$$\langle v,\eta(y,x)\rangle+f(y,x)-f(x,x)\geq 0,$$

and hence $x\in F(y)$. Also it follows from (v) that $\cap_{z\in D}F(z)\subseteq B$, and so F satisfies all the assumptions of Lemma 2.1 and then there exists $\overline{x}\in\bigcap_{y\in K}F(y)$. This means that $\overline{x}$ is a solution of problem 2.2. Furthermore the solution set of problem 2.2 equals to the intersection $\bigcap_{y\in K}F(y)$ which by using (v) is a subset of the compact set B and, note $\bigcap_{y\in K}F(y)$ is closed, so it is compact. This completes the proof of theorem. □

Remark. (i) It is clear that one can omit condition (v) in Theorem 2.2 when the set K is compact.

(ii) In [7], the authors, instead of condition (v) in Theorem 2.2, considered the following condition for a reflexive Banach space, which consists of finding $x_0\in K$ such that,

$$\frac{\langle u-u_0,\ \eta(x,x_0)\rangle-f(x_0,x)+f(x,x)}{\|\eta(x,x_0)\|}\longrightarrow+\infty\ ,\qquad (II)$$

whenever $\|x\|\longrightarrow\infty$, for all $u\in T(x)$, $u_0\in T(x_0)$.
They (II) called η-coercive. It is clear that (II) is a special case of condition (v) in Theorem 2.2. Because for each positive real number M there is another positive number N such that

$$\|x\|>N\Rightarrow\frac{\langle u-u_0,\ \eta(x,x_0)\rangle-f(x_0,x)+f(x,x)}{\|\eta(x,x_0)\|}>M.\qquad (III)$$

Now we can take $B=\{x:\|x\|\leq N\}$ and $D=\{x_0\}$ which are weakly compact (note E is a reflexive Banach space) and convex. Moreover by condition (i) of Theorem 2.2 $\eta(x,x_0)=-\eta(x_0,x)$ and by multiplying the relation (III) by -1 we get condition (v) in Theorem 2.2.

An special case of (II) has been given in [19] as follows ,

$$\frac{\langle u - u_0,\ \eta(x, x_0)\rangle + f(x) - f(x_0)}{\|\eta(x, x_0)\|} \longrightarrow +\infty\ ,$$

whenever $\|x\| \longrightarrow \infty$, for all $u \in T(x)$, $u_0 \in T(x_0)$.

By combining Theorems 2.1 and 2.2 one can deduce the next result.

Theorem 2.3. Let K be a nonempty closed convex subset of a topological vector space E and $E^\star$ the dual space of E. Let $T : K \longrightarrow 2^{E^\star}\setminus\{\emptyset\}$ be lower η-hemicontinuous and relaxed η-α-monotone and the conditions (i)-(v) of Theorem 2.2 and condition (ii) of Theorem 2.1 hold. Then the solution sets of problems (2.2) and (2.3) are equal and a nonempty compact subset of K.

We note that if T is a single valued mapping and f is a zero map, then the Theorems 2.1 and 2.2 are equivalent to the problems considered and studied by Bai et al [1].

3. VARIATIONAL-LIKE INEQUALITIES WITH RELAXED η-α-SEMIMONOTONE MAPPINGS

Throughout this section, let E be an arbitrary locally convex topological vector space(briefly, locally convex space) with its dual E^* and K a nonempty closed convex subset of E.

Definition 3.1. Let $A : K \times K \longrightarrow 2^{E^*}$, $\eta : K \times K \longrightarrow E$ and $\alpha : E \longrightarrow \Re$ be three mappings. The mapping A is called relaxed $\eta - \alpha$-semimonotone if the mapping $y \longrightarrow A(w, y)$ is relaxed $\eta - \alpha-$monotone, for each $w \in K$. In this section we consider the following problem of finding $x \in K$ such that

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0, \text{for all } y \in K \text{ and} u \in A(x, y). \quad (3.1)$$

where $f : K \times K \longrightarrow \Re$.

In order to prove our existence theorem we need the following result.

Theorem 3.1 (Kakutani-Fan-Glicksberg)([5]). Let X be a locally convex Hausdorff space, $D \subseteq X$ a nonempty, convex compact subset. Let $T : D \longrightarrow 2^D$ be upper semicontinuous with nonempty, closed convex $T(x)$, for all $x \in D$. Then T has a fixed point in D.

Theorem 3.2. Let E be a locally convex Hausdorff space, $K \subseteq E$ a nonempty closed convex set, $A : K \times K \longrightarrow 2^E$ a relaxed $\eta - \alpha$-semimonotone mapping, $f : KtimesK \longrightarrow \Re\bigcup\{+\infty\}$ a proper convex and weakly lower semicontinuous functional, and $\eta : K \times K \longrightarrow E$ a mapping. If for all $w \in K$, the mapping $y \in A(w, y)$ satisfies all the assumptions of Theorem 2.2 and the mapping, for all $w \in K$, $x \longrightarrow \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0$, for all $y \in K$ and $u \in A(w, y)$, is convex and upper semicontinuous, then problem (3.1) has a solution. Moreover the solution set of problem (3.1) is compact and convex.

Proof. By Theorem 2.2, for each $w \in coB$, the set

$$G(w) = \{x \in coB : \langle u, \eta(y,x)\rangle + f(y,x) - f(x,x) \geq 0, \text{ for all } y \in K \text{ and } u \in A(w,y)\}$$

is nonempty convex and compact subset of $B \subset K$. Now the mapping $G : coB \longrightarrow 2^{coB}$ defined by $w \longrightarrow G(w)$ fulfils all the conditions of Theorem 3.1 and hence there is $x \in coB \subset K$ such that $x \in G(x)$ and so x is a solution of problem 3.1 and so the solution set of the problem 3.1 is nonempty. It is clear that the solution set of problem (3.1) is equal to the intersection

$$\bigcap_{w \in K} G(w) \subseteq \bigcap_{x \in coB} G(w) \subset D$$

and since $G(w)$, for all $w \in K$ is closed and D is compact then the solution set problem (3.1) is compact and the convexity of the solution set is obvious from the assumptions. This completes the proof. □

Remark 3.1. If A is a single valued mapping and f is a zero map, then problem (3.1) is equivalent to the problem (3.1) considered and studied by Bai et al [1]. Note that Theorems 2.2 and 3.1 are topological vector space version of Theorems 2.1 and 2.6, respectively, in [3].

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