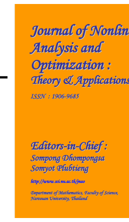
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 4, No. 1, (2013), 13-28
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF SELFMAPS SATISFYING GENERALIZED WEAKLY CONTRACTIVE CONDITION

G.V.R. BABU* AND K.T.KIDANE

Department of Mathematics, Andhra University, Visakhapatnam - 530 003, India

ABSTRACT. In this paper, we prove a common fixed point theorem for two pairs of selfmaps satisfying certain generalized weakly contractive condition. Also, we prove the same for two pairs of such selfmaps in which one pair is compatible, reciprocally continuous and the other pair is weakly compatible. Some existing results are drawn as corollaries from the main results of this paper. Examples are given in support of the main results of the paper.

KEYWORDS : Common fixed point; Generalized weakly contractive condition; Complete metric space, Compatible maps; Weakly compatible maps and reciprocally continuous maps
AMS Subject Classification: 47H10 54H25

1. INTRODUCTION

In 1977, Rhoades [10] compared various definitions of contractive mappings on a complete metric space which were used to generalize Banach contraction mapping principle. After 20 years, in 1997, weakly contractive maps were introduced by Alber and Guerre-Delabriere [1] in Hilbert spaces which generalize contraction maps, and established a fixed point theorem in Hilbert space setting. Rhoades [11] extended this idea to Banach spaces and proved the existence of fixed points of weakly contractive selfmaps in Banach space setting. Weakly contractive maps have been considered in several works by different researchers namely Alber, Guerre-Delabrier [1], Babu, Nageswara Rao and Alemayehu [2], Babu and Alemayehu [3], Choudhury, Konar, Rhoades and Metiya [4], Doric [5], Dutta and Choudhury [6] and Rhoades [11] and some references cited in these papers in order to establish the existence of fixed points.

Throughout this paper we denote
$\Phi = \{\phi : [0,\infty) \to [0,\infty) \mid \phi \text{ is lower semicontinuous and } \phi(t) = 0 \Leftrightarrow t = 0\}$.
$\Psi = \{\psi : [0,\infty) \to [0,\infty) \mid \psi \text{ is continuous, nondecreasing and } \psi(t) = 0 \Leftrightarrow t = 0\}$,

* *Corresponding author.*
Email address : gvr_babu@hotmail.com(G.V.R. Babu), kidanekoyas@yahoo.com(K.T.Kidane).
Article history : Received 14 February 2012. Accepted 3 September 2012.

Definition 1.1. (Rhoades [11]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weakly contractive if there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \text{ for all } x, y \in X.$$

Theorem 1.1. (*Rhoades* [11] *Let* (X, d) *be a complete metric space and* T *be a weakly contractive mapping. Then* T *has a unique fixed point in X.*

Definition 1.2. (Choudhury, Konar, Rhoades and Metiya [4]) Let (X, d) be a metric space and T be a selfmap of X. T is a generalized weakly contractive map if there exist maps $\psi \in \Psi$ and $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is continuous and$\phi(\mathrm{t}) = 0 \iff \mathrm{t} = 0$ such that
$d(Tx, Ty) \leq \psi(M(x, y)) - \phi(\max\{d(x, y),\ d(y, Ty)\})$ for all $x, y \in X$,
where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Definition 1.3. (Jungck [7]) Let f and g be selfmaps of a metric space (X, d). The pair (f, g) is said be a *compatible pair* on X, if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$.

Definition 1.4. (Jungck and Rhoades [8]) Let f and g be selfmaps of a metric space (X, d). The pair (f, g) is said be *weakly compatible* if they commute at their coincidence point, *i.e.*, $fgx = gfx$ whenever $gx = fx$, $x \in X$.

Here we note that every compatible pair is weakly compatible pair of maps but its converse need not be true [7].

Definition 1.5. (Pant [9]) Let f and g be selfmaps of a metric space (X, d). Then f and g are said to be *reciprocally continuous* if $\lim_{n\to\infty} fgx_n = ft$ and
$\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$.

Clearly if f and g are continuous then they are reciprocally continuous, but its converse need not be true (Pant [9]).

Theorem 1.2. (*Choudhury, Konar, Rhoades and Metiya* [4]) *Let* (X, d) *be a complete metric space and* T *a generalized weakly contractive mapping of* X. *Then T has a unique fixed point.*

Theorem 1.3. (*Choudhury, Konar, Rhoades and Metiya* [4]) *Let* (X, d) *be a complete metric space. Let* f *and* g *be selfmaps of* X. *Suppose that there exist maps* $\psi \in \Psi$ *and* $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is continuous and

$\phi(t) = 0$ if and only $\mathrm{t} = 0$ *such that*
$d(fx, gy) \leq \psi(M(x, y)) - \phi(m(x, y)\})$ *for all* $x, y \in X$, *where*

$M(x, y) =$ *max*$\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$
and
$m(x, y) =$ *max*$\{d(x, y),\ d(x, fx),\ d(y, gy)\}$.
Then f *and* g *have a unique common fixed point. Moreover, any fixed point of* f *is a fixed point of* g *and conversely.*

Definition 1.6. (Babu, Nageswara Rao and Alemayehu [2]) Let f and g be two selfmaps of a metric space (X, d). The pair (f, g) *is said to be a generalized weakly contractive pair* if there exists a function $\phi \in \Phi$ such that

$d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ for all x, y in X,
where
$M(x, y) = \max\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$.

Definition 1.7. (Babu, Nageswara Rao and Alemayehu [2]) Let f, g, S and T be selfmaps of a metric space (X, d). We say that the pair (f, g) *is* (S, T) *generalized weakly contractive* if there exists a function $\phi \in \Phi$ such that
$d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ for all x, y in X,
where
$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$.

Theorem 1.4. (*Babu, Nageswara Rao and Alemayehu* [2] *) Let f, g, S and T be selfmaps of a complete metric space (X, d) such that $fX \subseteq TX$ and $gX \subseteq SX$ and (f, g) is (S, T) generalized weakly contractive pair. If one of the ranges fX, gX, SX and TX is closed, then f, g, S and T have a unique common fixed point in X.*

Theorem 1.5. (*Babu, Nageswara Rao and Alemayehu* [2]) *Let f, g, S and T be selfmaps of a complete metric space (X, d) such that $fX \subseteq TX$ and $gX \subseteq SX$ and (f, g) is (S, T) generalized weakly contractive pair. Further assume that either*

(i) (f, S) is reciprocally continuous and compatible pair of maps and (g, T) a pair of weakly compatible maps

or

(ii) (g, T) is reciprocally continuous and compatible pair of maps and (f, S) a pair of weakly compatible maps.
Then f, g, S and T have a unique common fixed point in X.

Motivated by the works of Doric [5], Dutta and Choudhury [6], Choudhury, Konar, Rhoades and Metiya [4] we extend the concept of (ψ, ϕ)- weakly contractive maps to four maps.

Definition 1.8. Let f, g, S and T be four selfmaps of a metric space (X, d). If there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(m(x, y)) \text{ for all } x, y \text{ in } X \tag{A}$$

where
$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$
and
$m(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$

then we say that f, g, S and T satisfy *generalized* (ψ, ϕ)- *weakly contractive condition*.

In this paper, we prove a common fixed point theorem for two pairs of selfmaps satisfying generalized (ψ, ϕ)- weakly contractive condition. Also, we prove the same for two pairs of such selfmaps in which one pair is compatible, reciprocally continuous and the other pair is weakly compatible. Some existing results are drawn as corollaries from the main results of this paper. Examples are given in support of the main results of the paper.

2. A COMMON FIXED POINT OF TWO PAIRS OF WEAKLY CONTRACTIVE MAPS

Let f, g, S and T be selfmaps of a metric space (X, d) satisfying $fX \subseteq TX$ and $gX \subseteq SX$. Let $x_0 \in X$. Since $fX \subseteq TX$, we can choose $x_1 \in X$ such that

$fx_0 = Tx_1 = y_0$ (say).
Since $gX \subseteq SX$, corresponding to $x_1 \in X$ we can choose $x_2 \in X$ such that
$gx_1 = Sx_2 = y_1$ (say).
Continuing the same process we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, \ldots . \qquad (B)$$

The following proposition is useful in our subsequent discussion.

Proposition 2.1. *Let f, g, S and T be selfmaps of a metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ succh that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. Assume also that (f, S) and (g, T) are weakly compatible.*
Then $F(f, S) \neq \emptyset$ if and only if $F(g, T) \neq \emptyset$, where
$F(f, S)$={$x \in X : f(x) = S(x) = x$} and
$F(g, T)$={$x \in X : g(x) = T(x) = x$}.
In this case f, g, S and T have a unique common fixed point.

Proof. Assume that $F(f, S) \neq \emptyset$. Let $z \in F(f, S)$ and so

$$z = fz = Sz. \qquad (2.1)$$

Now, we show that $z \in F(g, T)$.
Since $fX \subseteq TX$ there exists $w \in X$ such that

$$fz = Tw. \qquad (2.2)$$

Then, from (2.1) and (2.2), we get

$$fz = Tw = Sz = z. \qquad (2.3)$$

Next we show that $gw = z$.
Now from (A) we have

$$\psi(d(z, gw)) = \psi(d(fz, gw)) \leq \psi(M(z, w)) - \phi(m(z, w)) \qquad (2.4)$$

where

$$\begin{aligned} M(z, w) &= max\{d(Sz, Tw),\ d(fz, Sz),\ d(gw, Tw), \frac{1}{2}[d(Sz, gw) + d(fz, Tw)]\} \\ &= max\{0,\ 0,\ d(gw, z),\ \frac{1}{2}d(z, gw)\} \\ &= d(z, gw). \end{aligned} \qquad (2.5)$$

and

$$\begin{aligned} m(z, w) &= max\{d(Sz, Tw),\ d(fz, Sz),\ d(gw, Tw)\} \\ &= max\{0,\ 0,\ d(gw, z)\} = d(z, gw). \end{aligned} \qquad (2.6)$$

On using (2.5) and (2.6) in (2.4), we have

$$\psi(d(z, gw)) \leq \psi(d(z, gw)) - \phi(d(z, gw))$$

which implies that $\phi(d(z, gw)) = 0$. Hence

$$z = gw. \qquad (2.7)$$

From (2.3) and (2.7) it follows that

$$gw = Tw = z. \qquad (2.8)$$

Since g and T are weakly compatible, we have by (2.8)

$$gz = gTw = Tgw = Tz.$$

Hence

$$gz = Tz. \tag{2.9}$$

Now, we show that $gz = z$.

From (A) we have

$$\psi(d(z, gz)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \phi(m(z, z)) \tag{2.10}$$

where

$$\begin{aligned} M(z, z) &= \{d(Sz, Tz),\ d(fz, Sz),\ d(gz, Tz),\ \frac{1}{2}[d(Sz, gz) + d(fz, Tz)]\} \\ &= max\{d(z, gz),\ 0,\ 0,\ \frac{1}{2}[d(z, gz) + d(z, gz)]\} \\ &= d(z, gz). \end{aligned} \tag{2.11}$$

Also, it is easy to see that

$$m(z, z) = d(z, gz). \tag{2.12}$$

Therefore, on using (2.11) and (2.12) in (2.10), we have

$$\psi(d(z, gz)) \leq \psi(d(z, gz)) - \phi(d(z, gz))$$

which implies that

$$\phi(d(z, gz)) = 0$$

i.e.,

$$z = gz. \tag{2.13}$$

Hence, from (2.9) and (2.13), we have $z = gz = Tz$.

Therefore

$$F(g, T) \neq \emptyset \tag{2.14}$$

Hence, from (2.1) and (2.14), we have

$$F(f, S) \subseteq F(g, T). \tag{2.15}$$

Conversely assume that $F(g, T) \neq \emptyset$.

Let $z \in F(g, T)$, then

$$gz = Tz = z. \tag{2.16}$$

Now, we show that $z \in F(f, S)$. Since $gX \subseteq SX$, there exists $u \in X$ such that

$$gz = Su. \tag{2.17}$$

Then, by (2.16) and (2.17), we have

$$gz = Su = Tz = z. \tag{2.18}$$

Next we show that $fu = z$.

From (A) we have

$$\psi(d(fu, z)) = \psi(d(fu, gz)) \leq \psi(M(u, z)) - \phi(m(u, z)) \tag{2.19}$$

where

$$\begin{aligned} M(u, z) &= max\{d(Su, Tz),\ d(fu, Su),\ d(gz, Tz),\ \frac{1}{2}[d(Su, gz) + d(fu, Tz)]\} \\ &= max\{0,\ d(fu, z),\ 0,\ \frac{1}{2}d(fu, z)\} \end{aligned}$$

$$= \quad d(fu, z). \tag{2.20}$$

Also we have

$$m(u, z) = d(fu, z). \tag{2.21}$$

Now on using (2.20) and (2.21) in (2.19), we have

$$\psi(d(fu, z)) \leq \psi(d(fu, z)) - \phi(d(fu, z))$$

which implies that $\phi(d(fu, z)) = 0$. Hence

$$fu = z.$$

Therefore from (2.18), it follows that

$$fu = Su = z.$$

Since f and S are weakly compatible we have

$$fz = fSu = Sfu = Sz,$$

so that

$$fz = Sz. \tag{2.22}$$

Now, we show that $fz = z$.

From (A) we have

$$\psi(d(fz, z)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \phi(m(z, z)) \tag{2.23}$$

where

$$\begin{aligned} M(z, z) &= max\{d(Sz, Tz),\ d(fz, Sz),\ d(gz, Tz), \frac{1}{2}[d(Sz, gz) + d(fz, Tz)]\} \\ &= d(fz, z). \end{aligned} \tag{2.24}$$

Also, it is easy to see that

$$m(z, z) = d(fz, z). \tag{2.25}$$

On using (2.25) and (2.24) in (2.23), we have

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \phi(d(fz, z))$$

which implies that $\phi(d(fz, z)) = 0$ so that

$$fz = z. \tag{2.26}$$

Hence from (2.22) and (2.26) we have $fz = Sz = z$. Therefore

$$F(f, S) \neq \emptyset. \tag{2.27}$$

Thus from (2.16) and (2.27) we get

$$F(g, T) \subseteq F(f, S). \tag{2.28}$$

Therefore from (2.15) and (2.28) we have $F(f, S) = F(g, T)$. □

Proposition 2.2. *Let f, g, S and T be selfmaps of a metric space (X, d) satisfying $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. Then for each $x_0 \in X$ the sequence $\{y_n\}$ defined by (B) is Cauchy in X.*

Proof. First we suppose that $y_n = y_{n+1}$ for some n.

If $n = 2m$ then

$$y_{2m} = y_{2m+1}.$$

Now, we have

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) & \\ &= max\{d(Sx_{2m+2}, Tx_{2m+1}),\ d(fx_{2m+2}, Sx_{2m+2}),\ d(gx_{2m+1}, Tx_{2m+1}), \\ &\quad \frac{1}{2}[d(Sx_{2m+2}, gx_{2m+1}) + d(fx_{2m+2}, Tx_{2m+1})]\} \\ &= max\{d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}),\ d(y_{2m+1}, y_{2m}), \\ &\quad \frac{1}{2}[d(y_{2m+1}, y_{2m+1}) + (d(y_{2m+2}, y_{2m})]\} \\ &= max\{0,\ d(y_{2m+2}, y_{2m+1}),\ 0\ , \frac{1}{2}[0 + (d(y_{2m+2}, y_{2m})]\} \\ &= max\{d(y_{2m+2}, y_{2m+1}),\ \frac{1}{2}d(y_{2m+2}, y_{2m})\} \\ &\leq max\{d(y_{2m+2}, y_{2m+1}),\ \frac{1}{2}[d(y_{2m+2}, y_{2m+1}) + d(y_{2m+1}, y_{2m})]\} \\ &= max\{d(y_{2m+2}, y_{2m+1}), \frac{1}{2}d(y_{2m+2}, y_{2m+1})\} \\ &= d(y_{2m+2}, y_{2m+1}), \end{aligned}$$

but we have

$$d(y_{2m+2}, y_{2m+1}) \leq M(x_{2m+2}, x_{2m+1}).$$

Hence we have

$$M(x_{2m+2}, x_{2m+1}) = d(y_{2m+2}, y_{2m+1}). \tag{2.29}$$

Also, we have

$$\begin{aligned} m(x, y) &= max\{d(Sx_{2m+2}, Tx_{2m+1}), d(fx_{2m+2}, Sx_{2m+2}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= max\{d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}),\ d(y_{2m+1}, y_{2m})\} \\ &= max\{0,\ d(y_{2m+2}, y_{2m+1}),\ 0)\} \\ &= d(y_{2m+2}, y_{2m+1}). \end{aligned} \tag{2.30}$$

Now, from (A) we have

$$\begin{aligned} \psi(d(y_{2m+2}, y_{2m+1})) &= \psi(d(fx_{2m+2}, gx_{2m+1})) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1}) - \phi(m(x_{2m+2}.x_{2m+1})) \end{aligned} \tag{2.31}$$

On using (2.29) and (2.30) in (2.31) we get

$$\psi(d(y_{2m+2}, y_{2m+1})) \leq \psi(d(y_{2m+2}, y_{2m+1}) - \phi(d(y_{2m+2}, y_{2m+1})),$$

which implies that $\phi(d(y_{2m+2}, y_{2m+1})) \leq 0$.

Hence $d(y_{2m+2}, y_{2m+1}) = 0$, i.e.,

$$y_{2m+2} = y_{2m+1}. \tag{2.32}$$

In a similar way it is easy to see that

$$y_{2m+3} = y_{2m+2}. \tag{2.33}$$

Hence, from (2.32) and (2.33), we have

$$y_{n+1} = y_{n+2}.$$

Now by applying mathematical induction it follows that

$$y_n = y_{n+k},$$

for all $k \geq 0$. Therefore, $\{y_m\}$ is a constant sequence for $m \geq n$ and hence it is a Cauchy sequence in X.

Now, we suppose that

$$y_n \neq y_{n+1}, \text{ for all } n. \tag{2.34}$$

Then from (A) we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(M(x_{2n+2}, x_{2n+1})) - \phi(m(x_{2n+2}, x_{2n+1})) \tag{2.35}$$

where

$$\begin{aligned}
M(x_{2n+2}, x_{2n+1}) &= max\{d(Sx_{2n+2},\ Tx_{2n+1}),\ d(fx_{2n+2}, Sx_{2n+2}),\ d(gx_{2n+1}, Tx_{2n+1}) \\
&\quad \frac{1}{2}[d(Sx_{2n+2}, gx_{2n+1}) + d(fx_{2n+2}, Tx_{2n+1})]\} \\
&= max\{d(y_{2n+1}, y_{2n}),\ d(y_{2n+2}, y_{2n+1}),\ d(y_{2n+1}, y_{2n}), \\
&\quad \frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]\} \\
&= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \frac{1}{2}d(y_{2n+2}, y_{2n})\} \\
&\leq max\{d(y_{2n+1}, y_{2n}),\ d(y_{2n+2}, y_{2n+1}),\ \frac{1}{2}[d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]\} \\
&\leq max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), max\{d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n})\}\} \\
&= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}.
\end{aligned} \tag{2.36}$$

Also we have

$$m(x_{2n+2}, x_{2n+1}) = max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}. \tag{2.37}$$

Hence from (2.36) and (2.37) we get

$$M(x_{2n+2}, x_{2n+1}) = m(x_{2n+2}, x_{2n+1})$$

If

$$max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} = d(y_{2n+2}, y_{2n+1}) \tag{2.38}$$

then using (2.38) in (2.35) we get

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+2}, y_{2n+1})) - \phi(d(y_{2n+2}, y_{2n+1}))$$

which implies that

$$\phi(d(y_{2n+2}, y_{2n+1})) \leq 0.$$

It follows that $y_{2n+2} = y_{2n+1}$, which is a contradiction with (2.34).

Therefore

$$max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} = d(y_{2n+1}, y_{2n})$$

and

$$\begin{aligned}
\psi(d(y_{2n+2}, y_{2n+1})) &\leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})) \\
&< \psi(d(y_{2n+1}, y_{2n})).
\end{aligned}$$

Since ψ is nondecreasing we have

$$d(y_{2n+2}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n}). \tag{2.39}$$

With a similar argument it follows that

$$d(y_{2n+3}, y_{2n+2}) \leq d(y_{2n+2}, y_{2n+1}). \tag{2.40}$$

Therefore, from (2.39) and (2.40) we have

$$d(y_{n+2}, y_{n+1}) \leq d(y_{n+1}, y_n), \text{ for } n = 0, 1, 2.3, \ldots.$$

Hence the sequence $\{d(y_{n+1}, y_n)\}$ is a nonincreasing sequence of nonnegative real numbers and hence it converges to some real number δ (say), $\delta \geq 0$.

Now, we show that $\delta = 0$. If possible, suppose that

$$\delta > 0. \tag{2.41}$$

Since $M(x_{2n+2}, x_{2n+1}) = m(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$ and from (2.35), we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})). \tag{2.42}$$

On taking upper limit as $n \to \infty$, using the continuity of ψ and lower semicontinuity of ϕ in (2.42) we get $\psi(\delta) \leq \psi(\delta) - \phi(\delta)$, a contradiction.

Therefore

$$\delta = 0.$$

Next, we show that the sequence $\{y_n\}$ is a Cauchy sequence in X. It suffices to show that $\{y_{2n}\}$ is a Cauchy sequence in X.

If possible, suppose that $\{y_{2n}\}$ is not a Cauchy sequence.

Then there exist $\epsilon > 0$ and sequences of even positive integers $\{2m_k\}$, $\{2n_k\}$ with $2m_k > 2n_k > k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon. \tag{2.43}$$

Let $2m_k$ be the least positive integer exceeding $2n_k$ and satisfying (2.43). Then it follows that

$d(y_{2m_k}, y_{2n_k}) \geq \epsilon$ and

$$d(y_{2m_k - 2}, y_{2n_k}) < \epsilon. \tag{2.44}$$

We now prove

(i) $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$, (ii) $\lim\limits_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon$,

(iii) $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \epsilon$, (iv) $\lim\limits_{k\to\infty} d(y_{2m_k+1}, y_{2n_k-1}) = \epsilon$.

Since the proof in each case is similar we prove (i).

Now from (2.43) we have

$$\epsilon \leq d(y_{2m_k}, y_{2n_k})$$

which implies that

$$\epsilon \leq \limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}).$$

By using the triangle inequality and (2.44) we have

$$\begin{aligned} d(y_{2m_k}, y_{2n_k}) &\leq d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\ &< \epsilon + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}). \end{aligned}$$

Therefore we have

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \leq \epsilon.$$

It follows that

$$\epsilon \leq \limsup_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) \leq \epsilon$$

so that

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon. \tag{2.45}$$

On the other hand, we have

$$\epsilon \leq \liminf_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) \leq \limsup_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) = \epsilon$$

so that

$$\liminf_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) = \epsilon. \tag{2.46}$$

Hence, from (2.45) and (2.46), we have

$$\epsilon = \liminf_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}).$$

Therefore $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k})$ exists and $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$.

Now we have

$$\begin{aligned} M(x_{2n_k}, x_{2m_k+1}) &= max\{d(Sx_{2n_k}, Tx_{2m_k+1}),\ d(fx_{2n_k}, Sx_{2n_k}),\ d(gx_{2m_k+1}, Tx_{2m_k+1}), \\ &\quad \frac{1}{2}[d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_K}, Tx_{2m_k+1})]\} \\ &= max\{d(y_{2n_k-1}, y_{2m_k}),\ d(y_{2n_k}, y_{2n_k-1}),\ d(y_{2m_k+1}, y_{2m_k}) \\ &\quad \frac{1}{2}[d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2n_k}, y_{2m_k})]\} \end{aligned}$$

On taking limits as $k \to \infty$ we get

$$\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = max\{\epsilon, 0, 0, \epsilon\} = \epsilon. \tag{2.47}$$

In a similarly way it is easy to see that

$$\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k+1}) = \epsilon. \tag{2.48}$$

Now putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (A) we obtain

$$\begin{aligned} \psi(d(y_{2n_k}, y_{2m_k+1})) &= \psi(d(fx_{2n_k}, gx_{2m_k+1})) \\ &\leq \psi(M(x_{2n_k}, x_{2m_k+1})) - \phi(m(x_{2n_k}, x_{2m_k+1})). \end{aligned}$$

On taking upper limit as $k \to \infty$ using (2.47), (2.48), the continuity of ψ and lower semicontinuity of ϕ in the last inequality we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence so that $\{y_n\}$ is a Cauchy sequence. □

Theorem 2.1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then for each $x_0 \in X$ the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z$ (say) and z is a unique common fixed point of f, g, S and T.*

Proof. By Proposition 2.2 the sequence $\{y_n\}$ is Cauchy in X. Since X is complete there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Thus

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{2.49}$$

Case (i): Suppose that SX is closed. Then z is in SX and hence there exists $u \in X$ such that

$$Su = z. \tag{2.50}$$

Now, we show that $fu = z$. Now we have

$$\begin{aligned} M(u, x_{2n+1}) &= max\{d(Su, Tx_{2n+1}),\ d(fu, Su),\ d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}[d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(u, x_{2n+1}) = d(fu, z) \tag{2.51}$$

Similarly it is easy to see that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(fu, z). \tag{2.52}$$

Using (A), we have

$$\psi(d(fu, gx_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \phi(m(u, x_{2n+1})). \tag{2.53}$$

On taking upper limit as $n \to \infty$ and using (2.51), (2.52), the continuity of ψ and lower semicontinuity of ϕ in (2.53), we get

$$\psi(d(fu, z)) \leq \psi(d(fu, z)) - \phi(d(fu, z))).$$

Hence it follows that $\phi(d(fu, z))) \leq 0$. Therefore

$$fu = z.$$

Hence from (2.50), we get

$$Su = fu = z.$$

Since f and S are weakly compatible we have $fz = fSu = Sfu = Sz$. Therefore

$$fz = Sz. \tag{2.54}$$

Now, we show $fz = z$. We have

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(Sz, Tx_{2n+1}), d(fz, Sz), d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}[d(Sz, gx_{2n+1}) + d(fz, Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(z, x_{2n+1}) = d(fz, z). \tag{2.55}$$

Also we have

$$\lim_{n\to\infty} m(z, x_{2n+1}) = d(fz, z). \tag{2.56}$$

Now, from (A) we have

$$\psi(d(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \phi(m(u, x_{2n+1})). \tag{2.57}$$

On taking upper limit as $n \to \infty$ using (2.55), (2.56), the continuity of ψ and lower semicontinuity of ϕ in (2.57) we get

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \phi(d(fz, z))$$

so that $\phi(d(fz, z)) \leq 0$.

Hence

$$fz = z.$$

Therefore from (2.54) we have $z = fz = Sz$. By Proposition 2.1, $F(g, T) \neq \emptyset$ with z in $F(g, T)$. Hence $z = fz = gz = Sz = Tz$.

Case (ii): Suppose that gX is closed.
In this case, $z \in gX \subseteq SX$ which implies that $z \in SX$ and hence the proof follows as in case (i).

For the cases TX is closed and fX is closed we follow the arguments similar to the cases of SX is closed and gX is closed respectively. □

By choosing ψ as the identity map on $[0, \infty)$ in Theorem 2.3 we get the following corollary.

Corollary 2.3. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there is $\phi \in \Phi$ such that*
$d(fx, gy) \leq M(x, y) - \phi(m(x, y))$ *for all* $x, y \in X$
where
$M(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$
and
$m(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$.
If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z$ (say) and z is a unique common fixed point of f, g, S and T.

Remark 2.4. Theorem 1.3 of (Choudhury, Konar, Rhoades and Metiya [4]) follows as a corollary to Theorem 2.1 by choosing $S = T = I_X$ (I_X, the identity mapping on X).

Remark 2.5. Theorem 1.4 (Babu, Nageswara Rao and Alemayehu[2]) follows as a corollary to Corollary 2.3 by choosing $\phi \in \Phi$ nondecreasing.

Now we give an example in support of Theorem 2.1.

Example 2.6. Let $X = [0, 1]$ with the usual metric and let f, g, S and T be self maps on X defined by

$$gx = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1, \end{cases} \qquad fx = \begin{cases} \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x = 1, \end{cases}$$

$$Sx = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \text{ and } \frac{3}{4} \leq \text{x} \leq 1, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 1, & \frac{1}{2} < x < \frac{3}{4}. \end{cases} \qquad Tx = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ \frac{1}{20}, & \frac{1}{2} < x \leq 1. \end{cases}$$

Define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$, $t \geq 0$ and $\phi(t) = \frac{1}{20}t^2$, $0 \leq t \leq \frac{19}{20}$ and $\phi(t) = \frac{1}{4}t$, $t > \frac{19}{20}$ then $\psi \in \Psi$ and $\phi \in \Phi$ and the maps f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition so that f, g, S and T satisfy all the hypotheses of Theorem 2.1 and f, g, S and T have a unique common fixed point $\frac{1}{2}$.

3. A COMMON FIXED POINT THEOREM WITH RECIPROCAL CONTINUITY

Theorem 3.1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition.*
Assume that either

(i) (f, S) *is reciprocally continuous and compatible pair of maps and* (g, T)

a pair of weakly compatible maps

or

(ii) (g, T) *is reciprocally continuous and compatible pair of maps and* (f, S)

a pair of weakly compatible maps.
Then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z$ (say), and z is a unique common fixed point of f, g, S and T.

Proof. By Proposition 2.2 the sequence $\{y_n\}$ is Cauchy in X. Since X is complete there exist $z \in X$ such that $\lim_{n\to\infty} y_n = z$.
Thus $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$
and $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$.

Suppose (i) holds.
Since (f, S) is reciprocally continuous it follows that
$\lim_{n\to\infty} fSx_{2n+2} = fz$ and $\lim_{n\to\infty} Sfx_{2n+2} = Sz$.
Since (f, S) is a compatible pair, we have $\lim_{n\to\infty} d(fSx_{2n+2}, Sfx_{2n+2}) = 0$. Hence we have $fz = Sz$. Since $fX \subseteq TX$ there exists $u \in X$ such that

$$fz = Tu.$$

Thus we have

$$fz = Tu = Sz. \tag{3.1}$$

Now, we show that $fz = gu$. Using (A) we have

$$\psi(d(fz, gu))) \leq \psi(M(z, u)) - \phi(m(z, u)), \tag{3.2}$$

where

$$\begin{aligned} M(z, u) &= \max\{d(Sz, Tu),\ d(fz, Sz),\ d(gu, Tu),\ \frac{1}{2}[d(Sz, gu) + d(fz, Tu)]\} \\ &= \max\{0,\ 0,\ d(gu, fz),\ \frac{1}{2}[d(fz, gu)]\} = d(fz, gu). \end{aligned} \tag{3.3}$$

Also it follows that

$$m(z, u) = d(fz, gu). \tag{3.4}$$

Therefore by using (3.3) and (3.4) in (3.2) we get

$$\psi(d(fz, gu)) \leq \psi(d(fz, gu)) - \phi(d(fz, gu)).$$

Hence it follows that $\phi(d(fz, gu)) \leq 0$. Therefore

$$fz = gu. \tag{3.5}$$

Therefore from (3.1) we have $fz = Sz = gu = Tu$. Since every compatible pair is weakly compatible, the pair (f, S) is weakly compatible. Hence from $fz = Sz$ we get that

$$ffz = Sfz. \tag{3.6}$$

Next, we show that $ffz = fz$. By using (A), we have

$$\begin{aligned}\psi(d(ffz, fz)) &= \psi(d(ffz, gu)) \\ &\leq \psi(M(fz, u)) - \phi(m(fz, u))\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}M(fz, u) &= \max\{d(Sfz, Tu),\ d(ffz, Sfz),\ d(gu, Tu), \\ &\quad \frac{1}{2}[d(Sfz, gu) + d(ffz, Tu)]\} \\ &= \max\{d(ffz, fz),\ 0,\ 0,\ \frac{1}{2}[d(ffz, fz) + d(ffz, fz)]\} \\ &= d(ffz, fz).\end{aligned} \tag{3.8}$$

Also we have

$$m(fz, u) = d(ffz, fz). \tag{3.9}$$

On using (3.8) and (3.9) in (3.7) we have

$$\psi(d(ffz, fz)) \leq \psi(d(ffz, fz)) - \phi(d(ffz, fz))$$

which implies that $\phi(d(ffz, fz)) \leq 0$. Hence

$$ffz = fz. \tag{3.10}$$

Therefore, from (3.6) and (3.10), we have

$$ffz = Sfz = fz. \tag{3.11}$$

Hence fz is a common fixed point of f and S. Since (g, T) is weakly compatible and $gu = Tu$ we have

$$gTu = Tgu. \tag{3.12}$$

Therefore, from (3.5)and (3.12),we have

$$gfz = Tfz. \tag{3.13}$$

Now, we show that $gfz = fz$.
By using (A) we have

$$\psi(d(fz, gfz)) \leq \psi(M(z, fz)) - \phi(m(z, fz)) \tag{3.14}$$

where

$$\begin{aligned}M(z, fz) &= \max\{d(Sz, Tfz),\ d(fz, Sz),\ d(gfz, Tfz) \\ &\quad \frac{1}{2}[d(Sz, gfz) + d(fz, Tfz)]\} \\ &= \max\{d(fz, gfz),\ 0,\ 0,\ \frac{1}{2}[d(fz, gfz) + d(fz, gfz)]\} \\ &= d(fz, gfz)\end{aligned} \tag{3.15}$$

Also we have

$$m(fz, u) = d(fz, gfz). \tag{3.16}$$

Now using (3.15) and (3.16) in (3.14) we have

$$\psi(d(fz, gfz)) \leq \psi(d(fz, gfz)) - \phi(d(fz, gfz))$$

which implies that $\phi(d(fz, gfz)) \leq 0$. Hence

$$fz = gfz. \tag{3.17}$$

Therefore

$$fz = gfz = Tfz. \tag{3.18}$$

From (3.11) and (3.18) we have

$$ffz = gfz = Sfz = Tfz = fz. \tag{3.19}$$

Hence fz is a common fixed point of f, g , S and T.
Finally we show that $fz = z$.
From (A), we have

$$\psi(d(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1}) - \phi(m(z, x_{2n+1}) \tag{3.20}$$

where

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(Sz, Tx_{2n+1}),\ d(fz, Sz),\ d(gx_{2n+1}, Tx_{2n+1}) \\ &\quad \frac{1}{2}[d(Sz, gx_{2n+1}) + d(fz, Tx_{2n+1})]\}. \end{aligned}$$

On letting $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} M(z, x_{2n+1}) &= \max\{d(fz, z),\ 0,\ 0, \frac{1}{2}[d(fz, z) + d(fz, z)]\} \\ &= d(fz, z). \end{aligned} \tag{3.21}$$

Also we have

$$\lim_{n\to\infty} m(z, x_{2n+1}) = d(fz, z). \tag{3.22}$$

Now, on taking limits as $n \to \infty$, using (3.21), (3.22) the continuity of ψ and lower semicontinuity ϕ in (3.20) we get

$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \phi(d(fz, z))$ which implies that $\phi(d(fz, z)) \leq 0$. Hence

$$fz = z. \tag{3.23}$$

Therefore from (3.19) and (3.23) we have $z = fz = gz = Sz = Tz$.

The proof of case (ii) is similar and hence is omitted. □

By choosing ψ as the identity map on $[0, \infty)$ in Theorem 3.1 we get the following corollary.

Corollary 3.1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there is $\phi \in \Phi$ such that*
$d(fx, gy) \leq M(x, y) - \phi(m(x, y))$ for all $x, y \in X$ where
$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$
and
$m(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$.
Assume that either

(i) (f, S) is reciprocally continuous and compatible pair of maps and (g, T) a pair of weakly compatible maps

or

(ii) (g,T) is reciprocally continuous and compatible pair of maps and (f,S) a pair of weakly compatible maps.
Then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z\ (say)$ and z is a unique common fixed point of f, g, S and T.

Remark 3.2. Theorem 1.5 (Babu, Nageswara Rao and Alemayehu [2]) follows as a corollary to Corollary 3.1 by choosing $\phi \in \Phi$ nondecreasing.

Now, we give an example in support of Theorem 3.1.

Example 3.3. Let $X = [0,1]$ with the usual metric and let f, g, S and T be selfmaps on X defined by

$$fx = \begin{cases} 0, & x = 0 \\ \frac{2}{3}, & 0 < x < 1 \\ \frac{3}{4}, & x = 1, \end{cases} \qquad gx = \begin{cases} \frac{1}{3}, & x = 0 \\ \frac{2}{3}, & 0 < x < 1 \\ \frac{5}{12}, & x = 1, \end{cases}$$

$$Sx = \begin{cases} 1, & x = 0 \\ \frac{1}{3}, & 0 < x < \frac{2}{3} \\ \frac{4}{3} - x, & \frac{2}{3} \le x < 1 \\ 0, & x = 1, \end{cases} \qquad Tx = \begin{cases} 1, & x = 0 \\ 1 - \frac{1}{2}x, & 0 < x < 1 \\ 0, & x = 1. \end{cases}$$

Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t^2$, $t \ge 0$ and $\phi(t) = \frac{1}{2}t^2$, if $0 \le t \le \frac{2}{3}$ and $\phi(t) = \frac{1}{3}t$, if $t > \frac{2}{3}$, then $\psi \in \Psi$ and $\phi \in \Phi$. Here we observe that (f,S) is reciprocally continuous, (f,S) is a compatible pair and (g,T) is a weakly compatible pair of maps and the maps f, g, S and T satisfy generalized (ψ,ϕ)-weakly contractive condition so that f, g, S and T satisfy all the hypotheses of Theorem 3.1 and f, g, S and T have a unique common fixed point $\frac{2}{3}$.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New results in Operatar theory and its applications (I. Gohberg, and Yu. Lyubich Eds), Operator Theory Adv. Appd., 98, Basel. 1997, 7-22.

[2] G. V. R. Babu, K. Nageswara Rao and G. N. Alemayehu, Common fixed points of two pairs of generalized weakly contractive maps, Advanced Studies in Contemporary Mathematics, 20(4)(2010), 575 -594.

[3] G.V.R. Babu, G. N. Alemayehu, Point of coincidence and common fixed points of a pair of generalized weakly contractive maps, Journal of Advanced Research in Pure Mathematics, 2(2)(2010), 89-106.

[4] B. S. Choudhury, P. Konar, B. E. Rhoades, N. Metiya, Fixed point theorems for generalized weak contractive mappings, Nonlinerar Analysis, 74(2011), 2116-2126.

[5] D. Doric, Common fixed point for generalized $(\psi,\ \phi)$ - weak contractions, Applied Mathematics Letters, 22(2009), 1896 - 1900.

[6] P. N. Dutta, B. S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory and Applications, 2008(2008), Article ID 406368.

[7] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci., 9(1986), 771-779.

[8] G. Jungck, and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. of Pure and Applied Math., 29(3)(1998), 227-238.

[9] R. P. Pant, Common fixed points for four selfmappings, Bull. Cal. Math. Soc., 90(1998), 281-286.

[10] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc., 226(1977), 257-290.

[11] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis, 47(4)(2001), 2683-2693.