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SOME DOUBLE SEQUENCE SPACES IN n-NORMED SPACES USING IDEAL CONVERGENCE AND A SEQUENCE OF ORLICZ FUNCTIONS

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ABSTRACT. In the present paper we introduce some double sequence spaces using ideal convergence and a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ in n-normed spaces and examine some topological properties of the resulting sequence spaces.

KEYWORDS: Paranorm space; I-convergence; Difference sequence spaces; Orlicz function; Musielak-Orlicz function; n-normed spaces.
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1. INTRODUCTION AND PRELIMINARIES

The initial works on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [13], Moricz [20], Moricz and Rhoades [21], Tripathy ([38, 39]), Başarır and Sonalcan [2] and many others. Hardy [13] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [41] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [25] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [23] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals

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of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Recently Başar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Now, recently Raj and Sharma [33] have introduced double sequence spaces of entire functions. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [28]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

The concept of 2-normed spaces was initially developed by Gähler [8] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan ([10, 11]) and Gunawan and Mashadi [12] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;
(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \ \ \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
(iv) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called a n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|x_1, x_2, \cdots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{\substack{k \to \infty \\ p \to \infty}} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notion of difference sequence spaces was introduced by Kızmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and

$c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \begin{pmatrix} r \\ v \end{pmatrix} x_{k+v}.$$

Taking $r = 1$, we get the spaces which were introduced and studied by Kızmaz [14]. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|x\| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1 \Big\}.$$

Also, it was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. The Δ_2- condition is equivalent to $M(Lx) \le LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \ge 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \le \lambda_r + 1,\ \ \lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k,\ \ I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ see [16]. If $\lambda_r = r$, then the (V, λ)-summability is reduced to $(C, 1)$-summability see ([36, 37]).
The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \le \lambda_{m,n} + 1,\ \ \lambda_{m,n+1} \le \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \le \lambda_{m,n+1} - \lambda_{m+1,n+1},\ \ \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \Big\{ (k, l) : m - \lambda_{m,n} + 1 \le k \le m,\ \ n - \lambda_{m,n} + 1 \le l \le n \Big\}.$$

Let X be a linear metric space. A function $p :\ X \to \mathbb{R}$ is called paranorm, if

(i) $p(x) \geq 0$ for all $x \in X$,
(ii) $p(-x) = p(x)$ for all $x \in X$,
(iii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [40], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [18, 19, 24, 27, 29–31, 34]) and reference therein.
A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
The notion of ideal convergence was introduced first by P. Kostyrko [15] as a generalization of statistical convergence which was further studied in topological spaces (see [5]). More applications of ideals can be seen in ([5, 6]).
Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces (see [9, 35]). In particular Sahiner [35] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm. Raj and Sharma [32] have introduced some sequence spaces of ideal convergence in 2-normed spaces.
We continue in this direction and by using a sequence of Orlicz functions, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to a sequence of Orlicz functions in n-normed spaces.

Let $(X, \|\cdot, \cdots, \cdot\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \Big\{ n \in \mathbb{N} : \|x_n - x\| \geq \epsilon \Big\}$ has natural density zero for each $\epsilon > 0$.
A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

(i) $\phi \in \mathcal{I}$;
(ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
(iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [9]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{ n \in \mathbb{N} : \|x_n - x\| \geq \epsilon \Big\}$ belongs to $\mathcal{I}$ (see [15]).
Let $\Lambda = (\lambda_{m,n})$ be non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \geq \lambda_n + 1$, $\lambda_1 = 0$ and let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $(X, \|\cdot, \cdots, \cdot\|)$ is a n-normed space. Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers. By $S''(n-X)$ we denote the space of all sequences defined over $(X, \|\cdot, \cdots, \cdot\|)$. Now we define the following sequence spaces in this paper :

$$(W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) =$$

$$\Big\{ x \in S''(n-X) : \forall\, \epsilon > 0 \, \Big\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\| \frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \| \Big) \Big]^{p_{k,l}} \geq \epsilon \Big\} \in I$$

$$\text{for some } \ L, \rho > 0 \ \text{ and } \ z_1, \cdots, z_{n-1} \in X \Big\},$$

$(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\forall\ \epsilon>0\ \Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some }\ \rho>0\ \text{ and }\ z_1,\cdots,z_{n-1}\in X\Big\},$$

$(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\exists\ K>0\ \text{ such that }\ \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq K\ \text{ for some }\ \rho>0\ \text{ and }\ z_1,\cdots,z_{n-1}\in X\Big\},$$

$(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\exists\ K>0\ \text{ such that }\ \Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\geq K\Big\}\in I\ \text{ for some }\ \rho>0\ \text{ and }\ z_1,\cdots,z_{n-1}\in X\Big\}.$$

The following inequality will be used throughout the paper. If $0\leq p_{k,l}\leq\sup p_{k,l}=H$, $D=\max(1,2^{H-1})$ then

$$|a_{k,l}+b_{k,l}|^{p_{k,l}}\leq D\{|a_{k,l}|^{p_{k,l}}+|b_{k,l}|^{p_{k,l}}\}\tag{1.1}$$

for all k,l and $a_{k,l},b_{k,l}\in\mathbb{C}$. Also $|a|^{p_{k,l}}\leq\max(1,|a|^H)$ for all $a\in\mathbb{C}$.
The main aim of this paper is to study some topological properties and some inclusion relation between above defined sequence spaces.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions, $p=(p_{k,l})$ be a bounded sequence of positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $(W^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ are linear spaces.*

Proof. Let $x=(x_{k,l}),y=(y_{k,l})\in(W^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $\alpha,\beta\in\mathbb{R}$. So

$$\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some }\ L,\ \rho_1>0,\ \text{and }\ z_1,\cdots,z_{n-1}\in X\Big\}$$

and

$$\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some }\ L,\ \rho_2>0\text{ and }\ z_1,\cdots,z_{n-1}\in X\Big\}.$$

Since $\|\cdot,\cdots,\cdot\|$ is a n-norm and $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions the following inequality holds:

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r(\alpha x_{k,l}+\beta y_{k,l})-L}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq\quad D\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$
\begin{aligned}
&+ \quad D\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\
&\leq \quad DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\
&+ \quad DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}},
\end{aligned}
$$

where $F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get $\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r(\alpha x_{k,l}+\beta y_{k,l})-L}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq \epsilon\Big\}$

$$
\begin{aligned}
&\subseteq \quad \Big\{m,n\in\mathbb{N}:DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\frac{\epsilon}{2}\Big\}\\
&\cup \quad \Big\{m,n\in\mathbb{N}:DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\frac{\epsilon}{2}\Big\}.
\end{aligned}
$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions and $p=(p_{k,l})$ be a bounded sequence of positive real numbers. For any fixed $m,n\in\mathbb{N}$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ is a paranormed space with the paranorm defined by*

$$
g_{m,n}(x)=\inf\Big\{\rho^{\frac{p_{m,n}}{H}}:\rho>0 \quad \textit{is such that} \quad \sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}} \leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\}.
$$

Proof. It is clear that $g_{m,n}(x)=g_{m,n}(-x)$. Since $M_{k,l}(0)=0$, we get $\inf\{\rho^{\frac{p_{m,n}}{H}}\}=0$ for $x=0$ therefore, $g_{m,n}(0)=0$. Let us take $x=(x_{k,l})$ and $y=(y_{k,l})$ in $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$. Let

$$
B(x)=\Big\{\rho>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\},
$$

$$
B(y)=\Big\{\rho>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\}.
$$

Let $\rho_1\in B(x)$ and $\rho_2\in B(y)$. Then if $\rho=\rho_1+\rho_2$, we have

$$
\begin{aligned}
&\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}M_{k,l}\Big(\|\frac{\Delta^r(x_{k,l}+y_{k,l})}{\rho},z_1,\cdots,z_{n-1}\|\Big)\\
&\qquad \leq \quad \frac{\rho_1}{\rho_1+\rho_2}\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)
\end{aligned}
$$

$$+ \quad \frac{\rho_2}{\rho_1+\rho_2} \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big).$$

Thus $\sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r (x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\|\Big)^{p_{k,l}} \leq 1$ and

$$\begin{aligned} g_{m,n}(x+y) &\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_{m,n}}{H}} : \rho_1 \in B(x),\ \rho_2 \in B(y)\Big\} \\ &\leq \inf\Big\{\rho_1^{\frac{p_{m,n}}{H}} : \rho_1 \in B(x)\Big\} + \inf\Big\{\rho_2^{\frac{p_{m,n}}{H}} : \rho_2 \in B(y)\Big\} \\ &= g_{m,n}(x) + g_{m,n}(y). \end{aligned}$$

Let $\sigma^s \to \sigma$ where $\sigma, \sigma^s \in \mathbb{C}$ and let $g_{m,n}(x^s_{k,l} - x) \to 0$ as $s \to \infty$.
We have to show that $g_{m,n}(\sigma^s x^s_{k,l} - \sigma x) \to 0$ as $s \to \infty$. Let

$$B(x^s) = \Big\{\rho_s > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x^s_{k,l}}{\rho_s}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \leq 1, \forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(x^s - x) = \Big\{\rho_s' > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x^s_{k,l} - x_{k,l}}{\rho_s'}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \leq 1, \\ \forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

If $\rho_s \in B(x^s)$ and $\rho_s' \in B(x^s - x)$ then we observe that

$$\begin{aligned} &\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\|\frac{\Delta^r (\sigma^s x^s_{k,l} - \sigma x_{k,l})}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \cdots, z_{n-1}\| \\ &\leq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r (\sigma^s x^s_{k,l} - \sigma x^s_{k,l})}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \cdots, z_{n-1}\| \\ &+ \|\frac{\Delta^r (\sigma x^s_{k,l} - \sigma x_{k,l})}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \cdots, z_{n-1}\|\Big) \\ &\leq \frac{|\sigma^s - \sigma|\rho_s}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r (x^s_{k,l})}{\rho_s}, z_1, \cdots, z_{n-1}\|\Big) \\ &+ \frac{|\sigma|\rho_s'}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r (x^s_{k,l} - x_{k,l})}{\rho_s'}, z_1, \cdots, z_{n-1}\|\Big). \end{aligned}$$

From the above inequality, it follows that

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big(M_{k,l}\Big(\|\frac{\Delta^r (\sigma^s x^s_{k,l} - \sigma x_{k,l})}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \cdots, z_{n-1}\|\Big)\Big)^{p_{k,l}} \leq 1$$

and consequently,

$$\begin{aligned} g_{m,n}(\sigma^s x^s - \sigma x) &\leq \inf\Big\{\Big(\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|\Big)^{\frac{p_{m,n}}{H}} : \rho_s \in B(x^s), \rho_s' \in B(x^s - x)\Big\} \\ &\leq (|\sigma^s - \sigma|)^{\frac{p_{m,n}}{H}} \inf\Big\{\rho^{\frac{p_{m,n}}{H}} : \rho_s \in B(x^s)\Big\} \\ &+ (|\sigma|)^{\frac{p_{m,n}}{H}} \inf\Big\{(\rho_s')^{\frac{p_{m,n}}{H}} : \rho_s' \in B(x^s - x)\Big\} \\ &\longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions which satisfies Δ_2-condition. Then* $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *and the inclusions are strict.*

Proof. The inclusion $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ is obvious. We have only show that $(W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Let $(x_{k,l}) \in (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Then

$$\begin{aligned}
&\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&= \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} + L - L}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} - L}{2\rho}, z_1, \cdots, z_{n-1}\| + \|\frac{L}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq DG\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&+ DG\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}},
\end{aligned}$$

where $G = \max\big\{1, \left(\frac{1}{2}\right)^H\big\}$. Thus from Δ_2-condition, we have $x \in (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ and this completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are sequences of Orlicz functions. Then we have*
(i) $(W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subseteq (W_0^I)_2\Big(\lambda, \mathcal{M} \circ \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *provided* $(p_{k,l})$ *is such that* $H_0 = \inf p_{k,l} > 0$.
(ii) $(W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \cap (W_0^I)_2\Big(\lambda, \mathcal{M}'', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subseteq (W_0^I)_2\Big(\lambda, \mathcal{M}' + \mathcal{M}'', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$.

Proof. (i) For given $\epsilon > 0$, first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$. Now using the continuity of $M_{k,l}$. Choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_{k,l}(t) < \epsilon_0$. Let $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Now from the definition

$$B(\delta) = \Big\{m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \geq \delta^H\Big\} \in I.$$

Thus if $m, n \notin B(\delta)$ then

$$\begin{aligned}
&\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \delta^H \\
&\Longrightarrow \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \lambda_{m,n}\delta^H \\
&\Longrightarrow \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \delta^H \text{ for all } k, l \in I_{m,n} \\
&\Longrightarrow \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] < \delta \text{ for all } k, l \in I_{m,n}.
\end{aligned}$$

Hence from above using the continuity of $\mathcal{M} = (M_{k,l})$ we must have

$$M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big) < \epsilon_0 \ \forall \ \ k,l \in I_{m,n}$$

which consequently implies that

$$\begin{aligned}\sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} &< \lambda_{m,n} \max\{\epsilon_0^H, \epsilon_0^{H_0}\} \\ &< \lambda_{m,n}\epsilon \\ \Longrightarrow \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} &< \epsilon.\end{aligned}$$

This shows that

$$\Big\{m,n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} \geq \epsilon\Big\} \subset B(\delta)$$

and so belongs to I. This proves the result.

(ii) Let $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \cap (W_0^I)_2\Big(\lambda, \mathcal{M}''_{k,l}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Then the fact

$$\begin{aligned}&\frac{1}{\lambda_{m,n}}\Big[(M'_{k,l} + M''_{k,l})\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\ &\qquad \leq D\frac{1}{\lambda_{m,n}}\Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\ &\qquad +D\frac{1}{\lambda_{m,n}}\Big[M''_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\end{aligned}$$

gives the result. □

Theorem 2.5. *The sequence spaces* $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *and* $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *are solid.*

Proof. Let $(x_k) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$, let $(\alpha_{k,l})$ be a sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$. Then we have

$$\Big\{m,n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{(\Delta^r \alpha_{k,l} x_{k,l})}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big\} \subset$$

$$\Big\{m,n \in \mathbb{N} : \frac{C}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \geq \epsilon\Big\} \in I,$$

where $C = \max\{1, |\alpha_{k,l}|^H\}$. Hence $(\alpha_{k,l} x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ for all sequences of scalars $\alpha_{k,l}$ with $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$ whenever $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$.

Similarly, we can prove that $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ is also solid. □

Theorem 2.6. *The sequence spaces* $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *and* $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *are monotone.*

Proof. It is easy to prove so we omit the details. □

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