

A RELATED FIXED POINT THEOREM IN n COMPLETE FUZZY METRIC SPACES

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ABSTRACT. We prove a related fixed point theorem for n mappings in n complete fuzzy metric spaces using an implicit relation which generalizes results of Aliouche and Fisher [1] and Rao et al. [13].

KEYWORDS : Fuzzy metric space; implicit relation; related fixed point

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1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by L. Zadeh [16] in 1965. George and Veeramani [9] modified the concept of fuzzy metric spaces introduced by [11] in order to define the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and E -infinity theory which were studied by El- Naschie [4, 5, 6, 7, 8] and [15]. They showed also that every metric space induces a fuzzy metric space.

Recently, Aliouche and Fisher [1], Aliouche et.al [2] and Rao et.al [13] proved some related fixed point theorems in metric spaces and fuzzy metric spaces.

Inspired by a work of Popa [12], we prove a related fixed point theorem in n complete fuzzy metric spaces using an implicit relation because it includes several contractive conditions.

Definition 1.1 ([14]). A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t -norm if it satisfies the following conditions:

- (i) $*$ is associative and commutative,
- (ii) $*$ is continuous,
- (iii) $a * 1 = a$ for all $a \in [0, 1]$,
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

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Examples of a continuous t -norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 ([9]). The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t -norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t .

Let $(X, M, *)$ be a fuzzy metric space.

1) For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

2) A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$.

Let τ denote the family of all open subsets of X . Then τ is called the topology on X induced by the fuzzy metric M . This topology is Hausdorff and first countable, see [9].

Example 1.3 ([9]). Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. Define for each $t \in (0, \infty)$ and all $x, y \in X$

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

Then (X, M) is a fuzzy metric space. It is called the standard fuzzy metric induced by the metric d .

Definition 1.4 ([9]). Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \rightarrow 1$ as $n, m \rightarrow \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5 ([10]). For all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7 ([10]). M is a continuous function on $X^2 \times (0, \infty)$.

We denote by Ψ the set of all function $\psi : [0, 1]^4 \rightarrow [0, 1]$ such that

- (i) ψ is upper semi continuous in each coordinate variable,
- (ii) ψ is decreasing in 3rd and 4th variable,
- (iii) if either $\psi(u, v, 1, u) \geq 0$ or $\psi(u, 1, 1, v) \geq 0$ or $\psi(u, 1, v, 1) \geq 0$ for all $u, v \in [0, 1]$, then $u \geq v$.

Example 1.8. $\psi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}$

Example 1.9. $\psi(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\})$,

where $\phi :]0, 1] \rightarrow]0, 1]$ is a increasing and continuous function with $\phi(t) > t$ for $0 < t < 1$. For example $\phi(t) = \sqrt{t}$ or $\phi(t) = t^h$ for $0 < h < 1$.

We need the following lemma of [3].

Lemma 1.10. Let $\{x_n\}$ be a sequence in fuzzy metric space $(X, M, *)$ with $M(x, y, t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x, y \in X$. If there exists a number $k \in]0, 1[$ such that

$$M(x_{n+1}, x_n, kt) \geq M(x_n, x_{n-1}, t).$$

Then $\{x_n\}$ is a Cauchy sequence in X .

2. MAIN RESULTS

Theorem 2.1. Let $(X_i, M_i, \theta_i)_{1 \leq i \leq n}$, be n complete fuzzy metric spaces with $M_i(x, x_i, t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x, x_i \in X_i$ and let $\{A_i\}_{i=1}^{i=n}$ be n -mappings such that $A_i : X_i \rightarrow X_{i+1}$ for all $i = 1, \dots, n-1$ and $A_n : X_n \rightarrow X_1$, satisfying the inequalities

$$\phi_1 \left(\begin{array}{c} M_1(A_n A_{n-1} \dots A_2 x_2, A_n A_{n-1} \dots A_2 A_1 x_1, kt), \\ M_2(x_2, A_1 A_n A_{n-1} \dots A_2 x_2, t), \\ M_1(x_1, A_n A_{n-1} \dots A_2 x_2, t), M_1(x_1, A_n A_{n-1} \dots A_2 A_1 x_1, t) \end{array} \right) \geq 0 \quad (2.1)$$

for all $x_1 \in X_1, x_2 \in X_2$ and $t > 0$, in general, we have

$$\phi_i \left(\begin{array}{c} M_i(A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, A_{i-1} \dots A_1 A_n \dots A_i x_i, kt), \\ M_{i+1}(x_{i+1}, A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, t), \\ M_i(x_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, t), \\ M_i(x_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i x_i, t), \end{array} \right) \geq 0 \quad (2.i)$$

for all $x_i \in X_i, x_{i+1} \in X_{i+1}, t > 0$ and $i = 2, \dots, n-1$ and

$$\phi_n \left(\begin{array}{c} M_n(A_{n-1} A_{n-2} \dots A_1 x_1, A_{n-1} A_{n-2} \dots A_1 A_n x_n, kt), \\ M_1(x_1, A_n A_{n-1} A_{n-2} \dots A_1 x_1, t), \\ M_n(x_n, A_{n-1} A_{n-2} \dots A_1 x_1, t), \\ M_n(x_n, A_{n-1} A_{n-2} \dots A_1 A_n x_n, t) \end{array} \right) \geq 0 \quad (2.n)$$

for all $x_1 \in X_1, x_n \in X_n$ and $t > 0$, where $\phi_i \in \Psi, i = 1, 2, \dots, n$ and $0 < k < 1$. Then $A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i$ has a unique fixed point $p_i \in X_i$ for $i = 1, \dots, n$. Further, $A_i p_i = p_{i+1}$ for $i = 1, \dots, n-1$ and $A_n p_n = p_1$.

Proof. Let $\{x_r^{(1)}\}, \{x_r^{(2)}\}, \dots, \{x_r^{(i)}\}, \dots, \{x_r^{(n)}\}, r \in \mathbb{N}$ be sequences in $X_1, X_2, \dots, X_i, \dots, X_n$ respectively. Now let $x_0^{(1)}$ be an arbitrary point in X_1 , we define the sequences $\{x_r^{(i)}\}_{r \in \mathbb{N}}$ for $i = 1, \dots, n$ by

$$x_r^{(1)} = (A_n A_{n-1} \dots A_1)^r x_0^{(1)},$$

$$x_r^{(i)} = A_{i-1} A_{i-2} \dots A_1 (A_n A_{n-1} \dots A_1)^r x_0^{(1)} \text{ for } i = 2, \dots, n.$$

For $n = 1, 2, \dots$, we assume that $x_r^{(1)} \neq x_{r+1}^{(1)}$. Applying the inequality (2.1) for $x_2 = A_1 (A_n A_{n-1} \dots A_1)^{r-1} x_0^{(1)}$, $x_1 = (A_n A_{n-1} \dots A_1)^r x_0^{(1)}$ we get

$$\begin{aligned} & \phi_1 \left(\begin{array}{c} M_1 \left((A_n A_{n-1} \dots A_1)^r x_0^{(1)}, (A_n A_{n-1} \dots A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_2 \left(A_1 (A_n A_{n-1} \dots A_1)^{r-1} x_0^{(1)}, A_1 (A_n A_{n-1} \dots A_1)^r x_0^{(1)}, t \right), \\ M_1 \left((A_n A_{n-1} \dots A_1)^r x_0^{(1)}, (A_n A_{n-1} \dots A_1)^r x_0^{(1)}, t \right), \\ M_1 \left((A_n A_{n-1} \dots A_1)^r x_0^{(1)}, (A_n A_{n-1} \dots A_1)^{r+1} x_0^{(1)}, t \right) \end{array} \right) \\ &= \phi_1 \left(M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right), M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right), 1, M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, t \right) \right) \geq 0 \end{aligned}$$

From the implicit relation we have

$$M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right) \geq M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right) \quad (3.1)$$

Applying the inequality (2.i) for $x_{i+1} = A_i \dots A_1 (A_n \dots A_1)^{r-1} x_0^{(1)}$ and $x_i = A_{i-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}$, we obtain

$$\begin{aligned} & \phi_i \left(\begin{array}{c} M_i \left(A_{i-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{i-1} \dots A_1 (A_n \dots A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_{i+1} \left(x_{i+1} = A_i \dots A_1 (A_n \dots A_1)^{r-1} x_0^{(1)}, A_i \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, t \right), \\ M_i \left(A_{i-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{i-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, t \right), \\ M_i \left(A_{i-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{i-1} \dots A_1 A_{i-1} \dots A_1 (A_n \dots A_1)^{r+1} x_0^{(1)}, t \right), \end{array} \right) \\ &= \phi_i \left(\begin{array}{c} M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, kt \right), M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t \right), \\ 1, M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, t \right) \end{array} \right) \geq 0 \end{aligned}$$

and so

$$M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, kt \right) \geq M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t \right) \quad (3.i)$$

for $i = 2, \dots, n-1$ and $r = 1, 2, \dots$. Now applying the inequality (2.n) for $x_n = A_{n-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}$ and $x_1 = (A_n A_{n-1} \dots A_1)^r x_0^{(1)}$ we have

$$\begin{aligned} & \phi_n \left(\begin{array}{c} M_n \left(A_{n-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{n-1} \dots A_1 (A_n \dots A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_1 \left((A_n \dots A_1)^r x_0^{(1)}, (A_n \dots A_1)^{r+1} x_0^{(1)}, t \right), \\ M_n \left(x_n = A_{n-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{n-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, t \right), \\ M_n \left(A_{n-1} \dots A_1 (A_n \dots A_1)^r x_0^{(1)}, A_{n-1} A_{n-2} \dots A_1 (A_n A_{n-1} \dots A_1)^{r+1} x_0^{(1)}, t \right) \end{array} \right) \\ &= \phi_n \left(\begin{array}{c} M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, kt \right), M_1 \left(x_{r-1}^{(1)}, x_r^{(1)}, t \right), \\ 1, M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, t \right) \end{array} \right) \geq 0 \end{aligned}$$

and so

$$M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, kt \right) \geq M_1 \left(x_{r-1}^{(1)}, x_r^{(1)}, t \right) \quad (3.n)$$

It now follows from (3.1), (3.i) and (3.n) that for large enough n we obtain

$$\begin{aligned} M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right) &\geq M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right) \\ M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, t \right) &\geq M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, \frac{t}{k} \right) \\ &\geq \dots \end{aligned}$$

$$\begin{aligned}
&\geq M_n \left(x_{r+i-n}^{(n)}, x_{r+i-n+1}^{(n)}, \frac{t}{k^{n-i}} \right) \\
&\geq M_1 \left(x_{r+i-n-1}^{(1)}, x_{r+i-n}^{(1)}, \frac{t}{k^{n-i+1}} \right) \\
&\geq \dots \\
&\quad M_1 \left(x_{r+i-2n-1}^{(1)}, x_{r+i-2n}^{(1)}, \frac{t}{k^{2n-i+1}} \right) \\
&\geq \dots \\
&\geq M_1 \left(x_{r+i-mn-1}^{(1)}, x_{r+i-mn}^{(1)}, \frac{t}{k^{mn-i+1}} \right) \\
&\geq \min \left\{ M_1 \left(x_1^{(1)}, x_2^{(1)}, \frac{t}{k^{mn}} \right), \dots, M_n \left(x_1^{(n)}, x_2^{(n)}, \frac{t}{k^{mn}} \right) \right\}
\end{aligned}$$

Since $0 < k < 1$, it follows from lemma 1.10 that $\{x_r^{(i)}\}$ is a Cauchy sequences in X_i with a limit p_i in X_i for $i = 1, 2, \dots, n$.

To prove that p_i is a fixed point of $A_{i-1} \dots A_1 A_n \dots A_i p_i$ for $i = 2, \dots, n-1$, suppose that $A_{i-1} \dots A_1 A_n \dots A_i p_i \neq p_i$. Using the inequality (2.i) for $x_i = p_i$ and $x_{i+1} = x_r^{(i+1)}$ we obtain

$$\phi_i \left(\begin{array}{c} M_i \left(x_r^{(i)}, A_{i-1} \dots A_1 A_n \dots A_i p_i, kt \right) \\ , M_{i+1} \left(x_r^{(i+1)}, x_{r+1}^{(i+1)}, t \right), M_i \left(p_i, x_r^{(i)}, t \right) \\ M_i \left(p_i, A_{i-1} \dots A_1 A_n \dots A_i p_i, t \right) \end{array} \right) \geq 0$$

Letting $r \rightarrow \infty$ we have

$$\phi_i \left(\begin{array}{c} M_i \left(p_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i p_i, kt \right), 1, 1, \\ d_i \left(p_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i p_i, t \right) \end{array} \right) \geq 0.$$

It follows from (iii) that $p_i = A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i p_i$ in X_i for $i = 2, \dots, n-1$ and $p_i = A_{i-1} p_{i-1} = \dots = A_{i-1} A_{i-2} \dots A_2 A_1 p_1$.

For the case $i = 1$, we use (2.1) for $x_1 = p_1$ and $x_2 = A_1 (A_n A_{n-1} \dots A_1)^{r-1} x_0^{(1)} = x_r^{(2)}$ giving

$$\phi_1 \left(\begin{array}{c} M_1 \left(x_r^{(1)}, A_n A_{n-1} \dots A_1 p_1, kt \right), \\ M_2 \left(x_r^{(2)}, x_{r+1}^{(2)}, t \right), M_1 \left(p_1, x_r^{(1)}, t \right) \\ M_1 \left(p_1, A_n A_{n-1} \dots A_1 p_1, t \right) \end{array} \right) \geq 0$$

letting $r \rightarrow \infty$ we have

$$\phi_1 \left(\begin{array}{c} M_1 \left(p_1, A_n A_{n-1} \dots A_1 p_1, kt \right), 1, 1, \\ M_1 \left(p_1, A_n A_{n-1} \dots A_1 p_1, t \right) \end{array} \right) \geq 0.$$

It follows from (iii) that $A_n A_{n-1} \dots A_2 A_1 p_1 = p_1$ in X_1 .

Finally, if $i = n$, using the inequality (2.n) for $x_n = p_n$ and $x_1 = x_r^{(1)}$ we get

$$\phi_n \left(\begin{array}{c} M_n \left(x_{r+1}^{(n)}, A_{n-1} A_{n-2} \dots A_1 A_n p_n, kt \right), \\ M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, t \right), M_n \left(p_n, x_{r+1}^{(n)}, t \right) \\ M_n \left(p_n, A_{n-1} A_{n-2} \dots A_1 A_n p_n, t \right) \end{array} \right) \geq 0.$$

Letting $r \rightarrow \infty$ we have

$$\phi_n \left(\begin{array}{c} M_n \left(p_n, A_{n-1} A_{n-2} \dots A_1 A_n p_n, kt \right), 1, 1, \\ M_n \left(p_n, A_{n-1} A_{n-2} \dots A_1 A_n p_n, t \right) \end{array} \right) \geq 0.$$

and by (iii), $p_n = A_{n-1}A_{n-2}..A_1A_n p_n$ in X_n and $p_n = A_{n-1}p_{n-1} = \dots = A_{n-1}A_{n-2}..A_2A_1p_1$.

To prove the uniqueness, suppose that $A_{i-1}..A_1A_n..A_i$ has a second fixed point $z_1 \neq p_1$ in X_1 . Using the inequality (2.1) for $x_{i+1} = A_i z_i$ and $x_i = p_i$ we get

$$\phi_i \left(\begin{array}{c} M_i(A_{i-1}..A_1A_n..A_i z_i, A_{i-1}..A_1A_n..A_i p_i, kt), \\ M_{i+1}(A_i z_i, A_{i-1}..A_1A_n..A_i z_i, t) \\ M_i(p_i, A_{i-1}..A_1A_n..A_i z_i, t), M_1(p_i, A_{i-1}..A_1A_n..A_i p_i, t) \end{array} \right) \geq 0$$

and so

$$\phi_i(M_i(z_i, p_i, kt), 1, M_i(p_i, z_i, t), 1) \geq 0$$

which implies that $z_i = p_i$, proving the uniqueness of p_i in X_i for $i = 2, \dots, n-1$.

The uniqueness of p_1 in X_1 and p_n in X_n follow similarly.

Finally, we note that

$$A_i p_i = A_i A_{i-1}..A_1A_n..A_{i+1}(A_i p_i),$$

hence, p_i is a fixed point of $A_i..A_1A_n..A_{i+1}$. Since the fixed point is unique, it follows that $A_i p_i = p_{i+1}$ for all $i = 1, \dots, n-1$. It follows similarly that $A_n p_n = p_1$. This complete the proof of the theorem. $\square$

Example 2.2. Let (M_i, X_i, θ_i) for $i = 1, \dots, n$ be n fuzzy metric spaces, where $M_i(x_i, y_i, t) = \frac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i-1 \leq x_i \leq i\}$ for $i = 1, \dots, n$. Define $A_i : X_i \rightarrow X_{i+1}$ for $i = 1, \dots, n-1$ and $A_n : X_n \rightarrow X_1$ by

$$\begin{aligned} A_1 x_1 &= \begin{cases} \frac{5}{4} & \text{if } 0 \leq x_1 < \frac{1}{2}, \\ \frac{3}{2} & \text{if } \frac{1}{2} \leq x_1 \leq 1 \end{cases}, \\ A_i x_i &= \begin{cases} i + \frac{1}{4} & \text{if } i-1 \leq x_i < i - \frac{3}{4}, \\ i + \frac{1}{2} & \text{if } i - \frac{3}{4} \leq x_i \leq i \end{cases} \quad \text{for all } i = 2, \dots, n-1, \\ A_n x_n &= \begin{cases} \frac{3}{4} & \text{if } n-1 \leq x_n < n - \frac{3}{4}, \\ 1 & \text{if } n - \frac{3}{4} \leq x_n \leq n \end{cases} \end{aligned}$$

Let $\phi_1 = \phi_2 = \dots = \phi_n = \phi$ and $\phi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}$. Note that there exists p_i in X_i such that $(A_{i-1}A_{i-2}..A_1A_n..A_i)p_i = p_i$ for $i = 1, \dots, n$. For example If we put $i = n$, we get $(A_{n-1}A_{n-2}..A_1A_n)p_n = p_n$ if $p_n = n - \frac{1}{2}$ because

$$\begin{aligned} A_{n-1}A_{n-2}..A_1A_n\left(n - \frac{1}{2}\right) &= A_{n-1}A_{n-2}..A_1(1), \\ &= A_{n-1}A_{n-2}..A_2\left(\frac{3}{2}\right) \\ &\vdots \\ &= A_{n-1}A_{n-2}..A_{i+1}\left(i + \frac{1}{2}\right) \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&= A_{n-1}A_{n-2}\left(n - \frac{5}{2}\right) \\
&= A_{n-1}\left(n - \frac{3}{2}\right) \\
&= n - \frac{1}{2} \quad \text{car } n - \frac{7}{4} \leq n - \frac{3}{2} \leq n - 1.
\end{aligned}$$

Note that for all $i = 1, \dots, n-1$ and $i - \frac{3}{4} \leq x_i < i$; $(i+1) - \frac{3}{4} \leq A_i x_i < i+1$ and $\frac{1}{2} \leq A_n x_n \leq 1$ with $n - \frac{3}{4} \leq x_n < n$, there exists $p_i = i - \frac{1}{2}$ such that $(A_{i-1} \dots A_1 A_n \dots A_i)\left(i - \frac{1}{2}\right) = i - \frac{1}{2}$ for $i = 1, \dots, n-1$.

The inequalities (1.i) for all $i = 1, \dots, n$ are satisfied since the value of the left hand side of each inequality is 1. In fact $M_i(A_{i-1} \dots A_1 A_n \dots A_{i+1} x_{i+1}, A_{i-1} \dots A_1 A_n \dots A_i x_i, t) = 1$ for $i = 1, \dots, n-1$ because

(1) If $i-1 \leq x_i < i - \frac{3}{4}$ we have

$$\begin{aligned}
A_{i-1} \dots A_1 A_n \dots A_i x_i &= A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} \left(i + \frac{1}{4}\right) \\
&= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+2} \left((i+1) + \frac{1}{2}\right) \\
&\vdots \\
&= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \left((n-2) + \frac{1}{2}\right) \\
&= A_{i-1} A_{i-2} \dots A_1 A_n \left(n - \frac{1}{2}\right) \\
&= A_{i-1} A_{i-2} \dots A_2 A_1 (1), \\
&= A_{i-1} A_{i-2} \dots A_2 \left(1 + \frac{1}{2}\right) \\
&\vdots \\
&= A_{i-1} \left((i-2) + \frac{1}{2}\right) \\
&= (i-1) + \frac{1}{2} = i - \frac{1}{2}.
\end{aligned}$$

(2) If $i - \frac{3}{4} \leq x_i \leq i$, we get

$$\begin{aligned}
A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i x_i &= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} \left(i + \frac{1}{2}\right) \\
&\vdots \\
&= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \left((n-2) + \frac{1}{2}\right) \\
&= A_{i-1} A_{i-2} \dots A_1 A_n \left(n - \frac{1}{2}\right), \\
&= A_{i-1} A_{i-2} \dots A_2 A_1 (1) =
\end{aligned}$$

$$\begin{aligned} & \vdots \\ A_{i-1} \left((i-2) + \frac{1}{2} \right) &= (i-1) + \frac{1}{2} = i - \frac{1}{2}. \end{aligned}$$

(3) If $i \leq x_{i+1} < i + \frac{1}{4}$ we get

$$\begin{aligned} A_{i-1} \dots A_1 A_n \dots A_{i+1} x_{i+1} &= A_{i-1} \dots A_1 A_n \dots A_{i+2} \left((i+1) + \frac{1}{4} \right) \\ &\vdots \\ &= A_{i-1} \dots A_1 A_n \left(n - \frac{1}{2} \right) \\ &= A_{i-1} \dots A_2 A_1 (1) = \\ &\vdots \\ &= A_{i-1} \left((i-2) + \frac{1}{2} \right) \\ &= (i-1) + \frac{1}{2} = i - \frac{1}{2}. \end{aligned}$$

(4) If $i + \frac{1}{4} \leq x_{i+1} \leq i + 1$ we obtain

$$\begin{aligned} A_{i-1} \dots A_1 A_n \dots A_{i+1} x_{i+1} &= A_{i-1} \dots A_1 A_n \dots A_{i+2} \left((i+1) + \frac{1}{2} \right) \\ &\vdots \\ &= A_{i-1} \dots A_1 A_n \left(n - \frac{1}{2} \right), \\ &= A_{i-1} \dots A_2 A_1 (1) = \\ &\vdots \\ &= A_{i-1} \left((i-2) + \frac{1}{2} \right) \\ &= (i-1) + \frac{1}{2} = i - \frac{1}{2}. \end{aligned}$$

Thus, all the conditions of theorem 2.1 are satisfied.

If we take $n = 5$ in theorem 2.1, we get the following corollary.

Corollary 2.3. Let (X_i, M_i, θ_i) , $i = 1, \dots, 5$ be 5 complete fuzzy metric spaces, $A_i : X_i \rightarrow X_{i+1}$, $i = 1, 2, 3, 4$ and $A_5 : X_5 \rightarrow X_1$ be 5 mappings satisfying

$$\phi_1 \left(\begin{array}{c} M_1 (A_5 A_4 A_3 A_2 x_2, A_5 A_4 A_3 A_2 A_1 x_1, kt), \\ M_1 (x_1, A_5 A_4 A_3 A_2 x_2, t), \\ M_1 (x_1, A_5 A_4 A_3 A_2 A_1 x_1, t), \\ M_2 (x_2, A_1 A_5 A_4 A_3 A_2 x_2, t) \end{array} \right) \geq 0 \quad (4.1)$$

for all $x_1 \in X_1$ and $x_2 \in X_2$,

$$\phi_2 \left(\begin{array}{c} M_2 (A_1 A_5 A_4 A_3 x_3, A_1 A_5 A_4 A_3 A_2 x_2, kt), \\ M_3 (x_3, A_2 A_1 A_5 A_4 A_3 x_3, t), \\ M_2 (x_2, A_1 A_5 A_4 A_3 A_2 x_2, t) \\ M_2 (x_2, A_1 A_5 A_4 A_3 x_3, t) \end{array} \right) \geq 0 \quad (4.2)$$

for all $x_2 \in X_2$ and $x_3 \in X_3$

$$\phi_3 \left(\begin{array}{c} M_3 (A_2 A_1 A_5 A_4 x_4, A_2 A_1 A_5 A_4 A_3 x_3, kt), \\ M_4 (x_4, A_3 A_2 A_1 A_5 A_4 x_4, t), \\ M_3 (x_3, A_2 A_1 A_5 A_4 A_3 x_3, t), \\ M_3 (x_3, A_2 A_1 A_5 A_4 x_4, t) \end{array} \right) \geq 0 \quad (4.3)$$

for all $x_3 \in X_3$ and $x_4 \in X_4$

$$\phi_4 \left(\begin{array}{c} M_4 (A_3 A_2 A_1 A_5 x_5, A_3 A_2 A_1 A_5 A_4 x_4, kt), \\ M_5 (x_5, A_4 A_3 A_2 A_1 A_5 x_5, t), \\ M_4 (x_4, A_3 A_2 A_1 A_5 A_4 x_4, t), \\ M_4 (x_4, A_3 A_2 A_1 A_5 x_5, t) \end{array} \right) \geq 0 \quad (4.4)$$

for all $x_4 \in X_4$ and $x_5 \in X_5$

$$\phi_5 \left(\begin{array}{c} M_5 (A_4 A_3 A_2 A_1 x_1, A_4 A_3 A_2 A_1 A_5 x_5, kt), \\ M_1 (x_1, A_5 A_4 A_3 A_2 A_1 x_1, t), \\ M_5 (x_5, A_4 A_3 A_2 A_1 A_5 x_5, t), \\ M_5 (x_5, A_4 A_3 A_2 A_1 x_1, t) \end{array} \right) \geq 0 \quad (4.5)$$

for all $x_1 \in X_1$, $x_5 \in X_5$ and for all $t > 0$, where $0 < k < 1$. Then

(a₁) $A_5 A_4 A_3 A_2 A_1$ has a unique fixed point $w_1 \in X_1$,

(a₂) $A_1 A_5 A_4 A_3 A_2$ has a unique fixed point $w_2 \in X_2$,

(a₃) $A_2 A_1 A_5 A_4 A_3$ has a unique fixed point $w_3 \in X_3$,

(a₄) $A_3 A_2 A_1 A_5 A_4$ has a unique fixed point $w_4 \in X_4$,

(a₅) $A_4 A_3 A_2 A_1 A_5$ has a unique fixed point $w_5 \in X_5$,

Further, $A_1 w_1 = w_2$, $A_2 w_2 = w_3$, $A_3 w_3 = w_4$, $A_4 w_4 = w_5$ and $A_5 w_5 = w_1$.

If we take $n = 2$ in theorem 2.1, we obtain theorem 2.9 of Rao et al. [13] and a fuzzy version of theorem 3 of [1].

The following example illustrates our corollary 2.3.

Example 2.4. Let (M_i, X_i, θ_i) for $i = 1, \dots, 5$ be 5 fuzzy metric spaces where $M_i(x_i, y_i, t) = \frac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i - 1 \leq x_i \leq i\}$ for $i = 1, \dots, 5$. Define $A_i : X_i \rightarrow X_{i+1}$ for $i = 1, \dots, 4$ and $A_5 : X_5 \rightarrow X_1$ by

$$\begin{aligned} A_1 x_1 &= \begin{cases} 1 & \text{if } x_1 \in [0, \frac{3}{4}[\\ \frac{3}{2} & \text{if } x_1 \in [\frac{3}{4}, 1] \end{cases}, \quad A_2 x_2 = \begin{cases} \frac{5}{2} & \text{if } x_2 \in [1, \frac{3}{2}[\\ 3 & \text{if } x_2 \in [\frac{3}{2}, 2] \end{cases} \\ A_3 x_3 &= \begin{cases} \frac{13}{4} & \text{if } x_3 \in [2, \frac{5}{2}[\\ \frac{7}{2} & \text{if } x_3 \in [\frac{5}{2}, 3] \end{cases}, \quad A_4 x_4 = \begin{cases} \frac{17}{4} & \text{if } x_4 \in [3, \frac{7}{2}[\\ \frac{9}{2} & \text{if } x_4 \in [\frac{7}{2}, 4] \end{cases} \\ A_5 x_5 &= \begin{cases} \frac{3}{4} & \text{if } x_5 \in [4, \frac{9}{2}[\\ 1 & \text{if } x_5 \in [\frac{9}{2}, 5] \end{cases} \end{aligned}$$

Let $\phi_1(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$

Further, the inequalities (4.1), (4.2), (4.3), (4.4), (4.5) are satisfied since the left hand side of each inequality is 1 and

$$A_5 A_4 A_3 A_2 A_1(1) = 1$$

$$\begin{aligned}
A_1 A_5 A_4 A_3 A_2 \left(\frac{3}{2} \right) &= \frac{3}{2} \\
A_2 A_1 A_5 A_4 A_3 \left(\frac{5}{2} \right) &= \frac{5}{2} \\
A_3 A_2 A_1 A_5 A_4 \left(\frac{7}{2} \right) &= \frac{7}{2} \\
A_4 A_3 A_2 A_1 A_5 \left(\frac{9}{2} \right) &= \frac{9}{2}.
\end{aligned}$$

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