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A GENERAL ITERATIVE ALGORITHM FOR MONOTONE OPERATORS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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ABSTRACT. Let $VI(A, H)$ be the set of all solutions of the following variational inequality problem:

$$\text{find } \ u \in H \ \text{ such that } \langle v - u, Au \rangle \geq 0, \quad \text{for all } v \in H.$$

Where H is a Hilbert space, $A : H \to H$ is a Lipschitz continuous and monotone operator. Assume that $F : H \to H$ is a Lipschitz continuous and strongly monotone operator. Let $f : H \to H$ be a Lipschitz continuous mapping. In this paper, we consider a demiclosed, demicontractive mapping T on H such that $Fix(T) \cap VI(A, H) \neq \varnothing$.

For finding an element x^* which solves the following variational inequality problem: find an $x^* \in Fix(T) \cap VI(A, H)$ such that

$$\langle v - x^*, \mu F x^* - \gamma f x^* \rangle \geq 0, \qquad \text{for all } v \in Fix(T) \cap VI(A, H),$$

when μ and γ are positive real numbers which satisfy appropriate conditions, we introduce a new general iterative algorithm and obtain strong convergence results.

KEYWORDS : Demicontractive mapping; Viscosity method; Monotone operator; Variational inequality, Fixed point.
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1. INTRODUCTION

Many problems arising in engineering sciences and structural analysis, are reduced to variational inequalities and fixed point problems, and iterative algorithms to solve these problems have been proposed.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Recall that a mapping $F : H \to H$ is called $\eta-$strongly monotone operator if there is a positive real number η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \qquad \text{for all } x, y \in H.$$

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Assume that $f : H \to H$ is a α-contraction: that is, there is a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$. Let T be a nonexpansive mapping on H. i.e. $|T(x) - T(y)\| \leq \|x - y\|$ for $x, y \in H$. We use $Fix(T)$ to denote the set of all fixed points of T.

The viscosity approximation method of selecting a particular fixed point of given nonexpansive mapping was proposed by Moudafi [9]. Particularly, he introduced the following process: Let $x_1 \in H$ be arbitrary and

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T(x_n) \qquad n \geq 0, \tag{1.1}$$

where f is a contraction with the coefficient $\alpha \in [0, 1)$, T is a nonexpansive mapping on H and $\{\varepsilon_n\}$ is a sequence in $(0, 1)$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \Sigma_{n=0}^{\infty} \varepsilon_n = \infty, \text{ and } \lim_{n\to\infty} (\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}) = 0.$$

It is showed that the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution $x^* \in Fix(T)$ of the variational inequality:

$$\langle (f - I)x^*, x - x^* \rangle \leq 0, \qquad \text{for all } x \in Fix(T).$$

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min\{\frac{1}{2}\langle Bx, x\rangle \ - \langle x, b\rangle : x \in C\}, \tag{1.2}$$

where C is the set of all fixed points of a nonexpansive mapping T on H and b is a given point in H, B is a strongly positive bounded linear map on H : That is, there is a constant $\gamma \geq 0$ with the following property

$$\langle Bx, x\rangle \geq \gamma\|x\|^2, \qquad \text{for all } x \in H. \tag{1.3}$$

In [17], Xu proved that the sequence $\{x_n\}$ generated by the recursive relation

$$x_{n+1} = \alpha_n b + (1 - \alpha_n B)Tx_n, \qquad n \geq 0, \tag{1.4}$$

converges strongly to the unique solution of the the quadratic minimization problem (1.2) under suitable hypotheses on $\{\alpha_n\}$. In 2006, Marino and Xu combined the iterative method (1.4) with the viscosity approximation method (1.1) and consider the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)Tx_n, \qquad n \geq 0. \tag{1.5}$$

They showed that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution $\tilde{x}$ of the variational inequality

$$\langle (\gamma f - B)\tilde{x}, x - \tilde{x}\rangle \leq 0 \qquad \text{for all } x \in C, \tag{1.6}$$

which is an optimal condition for the minimization problem

$$\min\{\frac{1}{2}\langle Bx, x\rangle \ - h(x) : x \in C\},$$

where h is a potential function for γf.

In 2009, Mainge [6] generalized the moudafi's scheme (1.1), and proved strong convergence results for quasi-nonexpansive mapping in Hilbert spaces.

In 2010, Tian defined the following iterative scheme

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu F)T(x_n) \tag{1.7}$$

where $f : H \to H$ is a contraction and $F : H \to H$ is a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0$, $\eta > 0$. He offered some strong convergence results for the case that T is a nonexpansive mapping on H.

In [14], Tian extended the algorithm (1.7) and acquired a more general result: suppose that T is a nonexpansive mapping on H, f is a $L-$Lipschitzian continuous operator with $L > 0$ and $F : H \to H$ is a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/\kappa^2$, $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$, and the sequence $\{\alpha_n\}$ satisfies the following conditions,

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \Sigma_{n=0}^{\infty}\alpha_n = \infty \quad \text{and} \quad \Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty,$$

then, the sequence $\{x_n\}$ defined by the recursive relation

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1 - \alpha_n\mu F)T(x_n), \qquad \text{for all } n \geq 0, \tag{1.8}$$

converges strongly to the unique solution $x^* \in Fix(T)$ of the following variational inequality:

$$\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0, \qquad \text{for all } x \in Fix(T). \tag{1.9}$$

Currently, Tian and Jin [15] considered the following iterative algorithm. Let $x_0 = x$ be an arbitrary element in H,

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1 - \alpha_n\mu F)T_w(x_n), \qquad \text{for all } n \geq 0, \tag{1.10}$$

where $w \in (0, \frac{1}{2})$, $T_w := (1 - w)I + wT$, T is a quasi-nonexpansive mapping on H and the sequence $\{\alpha_n\}$ satisfies the following two conditions:

(i) $\lim \alpha_n = 0$.
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$.

They obtained strong convergence results over the class of quasi-nonexpansive mappings in Hilbert spaces.

Before introducing our work in this paper, we need to offer a few background on the Korpelevich extragradient method. Note that in this paper, we denote by $VI(A, C)$ the set of solutions of the following variational inequality problem:

$$\text{find } u \in C \text{ such that} \quad \langle v - u, Au\rangle \geq 0, \qquad \text{for all } v \in C, \tag{1.11}$$

where C is a nonempty closed convex set in H and $A : H \to H$ is a monotone mapping on C : that is,

$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \text{for all } x, y \in C.$$

It should be noted that $VI(A, C)$ is closed and convex (see [4] and [1]).

In 1976, Korpelevich [3] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda Ax_n), \\ x_{n+1} &= P_C(x_n - \lambda Ay_n), \end{aligned} \tag{1.12}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{\theta})$, C is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and $\theta-$Lipschitzian continuous mapping of C into $\mathbb{R}^n$. Korpelevich proved that if $VI(A, C)$ is nonempty, then both sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.12), converge strongly to a point $z \in VI(A, C)$.

The following iterative algorithm which is based on Korpelevich's extragradient method [3] and Mann's iteration [7] was introduced by Nadezhkina and Takahashi

[10], when T is a nonexpansive and A is monotone and $\theta-$Lipschitz continuous:

$$\begin{aligned} &x_0 \in H,\\ &y_n = P_C(x_n - \lambda_n A x_n),\\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - \lambda_n A y_n) \quad n \geq 0, \end{aligned} \tag{1.13}$$

where $\lambda_n \subset [a, b]$ for some $a, b \in (0, \frac{1}{\theta})$ and $\alpha_n \subset [c, d]$ for some $c, d \in (0, 1)$. They proved that both sequences $\{x_n\}$ and $\{y_n\}$ given by (1.13) converge weakly to the same point in $Fix(T) \cap VI(A, C)$.

The next algorithm in this direction was introduced by Zeng and Yao [20]. They proved the following Theorem:

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, $\theta-$Lipschitz continuous mapping and $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \cap VI(A, C) \neq \varnothing$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by*

$$\begin{aligned} &x_0 \in H,\\ &y_n = P_C(x_n - \lambda_n A x_n),\\ &x_{n+1} = \alpha_n x_0 + (1-\alpha_n) T P_C(x_n - \lambda_n A y_n) \quad n \geq 0, \end{aligned} .$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions

(H1) $\{\alpha_n\} \subset [0, 1), \quad \sum_{n\geq 0} \alpha_n = \infty, \quad \alpha_n \to 0;$
(H2) $\{\theta\lambda_n\} \subset [a, b] \quad (where\ 0 < a \leq b < 1).$

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{Fix(T)\cap VI(A,C)}x_0$ provided that $\|x_{n+1} - x_n\| \to 0$.

Suppose that $T : H \to H$ is a demicontractive mapping. In [4], for finding a solution of the following variational inequality problem:

$$\text{find } x^* \in Fix(T) \cap VI(A, H) \text{ such that}$$
$$\langle v - x^*, Fx^* \rangle \geq 0, \text{ for all } v \in Fix(T) \cap VI(A, H), \tag{1.14}$$

Mainge suggested a new iterative algorithm. Particularly, he proved the following excellent criterion:

Theorem 1.2. ([4]) *Assume that $A : H \to H$ is monotone on C and $\theta-$Lipschitz continuous on H. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, C) \neq \varnothing$. Let $F : H \to H$ be a $L-$Lipschitzian, $\eta-strongly$ monotone operator with $L > 0, \eta > 0$, and assume that the following conditions hold:*

(H1) $w \in (0, \frac{1-\beta}{2}];$
(H2) $\{\alpha_n\} \subset [0, 1),\ \alpha_n \to 0;$
(H3) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1);$
(H4) $\sum_{n\geq 0} \alpha_n = \infty.$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H,\\ &y_n = P_C(x_n - \lambda_n A x_n),\\ &t_n = P_C(x_n - \lambda_n A y_n),\\ &x_{n+1} = [(1-w)I + wT]v_n, \qquad v_n := t_n - \alpha_n F(t_n), \end{aligned} \tag{1.15}$$

converge strongly to x^*, the unique solution of (1.14).

In this paper, motivated by the above-mentioned works, we consider a demicontractive mapping T on H such that $Fix(T) \cap VI(A,H) \neq \varnothing$. For finding an element x^* which solves the following variational inequality problem:

$$\begin{aligned} &\text{find } x^* \in Fix(T) \cap VI(A,H) \text{ such that} \\ &\langle v - x^*, \mu F x^* - \gamma f x^* \rangle \geq 0, \text{ for all } v \in Fix(T) \cap VI(A,H), \end{aligned} \tag{1.16}$$

we introduce the following iterative algorithm

$$\begin{cases} x_0 \in H, \\ y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n \gamma f(t_n) + (1 - \alpha_n \mu F)[(1-w)I + wT](t_n), \end{cases}$$

and prove that under appropriate assumptions, the sequences $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to the same point x^* which is the unique solution of (1.16)

We note that the class of demicontractive mappings includes important operators such as quasi-nonexpansive mappings and the strictly pseudocontractive mappings with fixed points. Hence, our algorithm, which deals with demicontractive mappings and is based on the extragradient, visosity and Hybrid steepest descent method, enables us to obtain more extended results.

2. PRELIMINARIES

Throughout this paper, we denote $x_n \to x$ (respectively, $x_n \rightharpoonup x$) the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Let C be a closed convex subset of a Hilbert space H. Let us recall that a mapping $T : H \to H$ is called

(i) quasi-nonexpansive if $\|Tx - q\| \leq \|x - q\|$ for all $(x,q) \in H \times Fix(T)$.
(ii) demicontractive if there is $\beta \in [0,1)$ such that $\|Tx - q\|^2 \leq \|x - q\|^2 + \beta\|x - Tx\|^2$ for all $(x,q) \in H \times Fix(T)$.

Also recall that $T : H \to H$ is demiclosed at the origin if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, we have $x \in Fix(T)$.

Lemma 2.1. ([4]) *Let $T : H \to H$ be a $\beta-$demicontractive mapping and let $T_w := (1-w)I + wT$. Then T_w is a quasi-nonexpansive mapping on H if $w \in [0, 1-\beta]$. Besides we have*

$$\|T_w x - q\|^2 \leq \|x - q\|^2 - w(1 - \beta - w)\|x - Tx\|^2.$$

Recall that the projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

The following lemma characterizes the projection P_C.

Lemma 2.2. ([12]) *Let C be a closed convex subset of a real Hilbert space H, $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if*

$$\langle x - y, y - z \rangle \geq 0, \qquad \textit{for all } z \in C. \tag{2.1}$$

Lemma 2.3. *Let H be a real Hilbert space. Then, the following simple well-known result holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \qquad x, y \in H.$$

Lemma 2.4. ([17]) *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n \qquad n \geq 1$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *Let $f : H \to H$ be a L-Lipschitzian continuous operator with the coefficient $L > 0$, $F : H \to H$ be a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0, \eta > 0$. Then, for $0 < \gamma \leq \frac{\mu\eta}{L}$,*

$$\langle x-y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \geq (\mu\eta - \gamma L)\|x-y\|^2.$$

That is, $\mu F - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma L$.

Lemma 2.6. *Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta - strongly$ monotone operator with $\kappa > 0, \eta > 0$, and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that $0 < \mu < 2\eta/\kappa^2$, $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$. Then we have*

$$\|(1-\alpha_n\mu F)x - (1-\alpha_n\mu F)y\| \leq (1-\alpha_n\tau)\|x-y\| \quad \text{for all } x, y \in H.$$

Lemma 2.7. ([6]) *Let $\{\Gamma_n\}$ be a sequence of nonnegative real numbers which is not decreasing at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, let $\{\tau(n)\}_{n\geq 0}$ be a sequence of integers defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq 0}$ is a nondecreasing sequence, $\lim_{n\to\infty}\tau(n) = \infty$, and for all $n \geq 0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ and $\Gamma_n < \Gamma_{\tau(n)+1}$.

The following lemma helps us to prove the main result of this paper in the next section.

Lemma 2.8. ([4, Lemma 4.2]) *Let A be a $\theta-$Lipschitzian continuous and monotone mapping on a real Hilbert space H. Assume that $VI(A,C) \neq \varnothing$. Let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in H such that*

$$y_n = P_C(x_n - \lambda_n Ax_n), \qquad t_n = P_C(x_n - \lambda_n Ay_n).$$

Then, we have the following inequalities

$$\|y_n - t_n\| \leq \theta\lambda_n\|x_n - y_n\| \qquad \text{and} \qquad \|t_n - u\|^2 \leq \|x_n - u\|^2 - (1-\theta^2\lambda_n^2)\|x_n - y_n\|^2,$$

where u is any element in $VI(A,C)$.

Let C be a closed convex subset of a real Hilbert space H and let $A : C \to H$ be a monotone mapping. Let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0 \text{ for all } u \in C\}$, and define $B : H \to 2^H$ by

$$Bv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \varnothing & \text{otherwise}, \end{cases} \tag{2.2}$$

then, B is maximal monotone: that is, the graph of B defined by $G(B) = \{(x,y) \in H \times H; \quad y \in B(x)\}$ is not contained in the graph of any other monotone mapping. Furthermore, we have the well-known result that $0 \in Bv$ if and only if $v \in VI(A,C)$ (see, for instance, [20] and [11]). Thus,

$$\langle u-v, -w\rangle \geq 0, \qquad \text{for all } (v,w) \in G(B) \Rightarrow u \in VI(A,C). \tag{2.3}$$

The following lemma gives us sufficient conditions ensuring that the weak cluster points of the sequence $\{t_n\}$, defined in the Lemma, belong to $VI(A, C)$. The lemma was proven implicitly in [20] and in [4], but, we bring the proof for the sake of completeness.

Lemma 2.9. *Let C be a closed convex subset of a real Hilbert space H and let $A : C \to H$ be a $\theta-$Lipschitzian continuous and monotone mapping. Let $\{\lambda_n\} \subset [\delta, \infty)$ (for some $\delta > 0$) and let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in C such that*

$$y_n = P_C(x_n - \lambda_n A x_n), \qquad t_n = P_C(x_n - \lambda_n A y_n).$$

Assume that

(i) *$\{t_{n_k}\}$ converges weakly to some u in C;*
(ii) *$\|x_{n_k} - y_{n_k}\| \to 0$ and $\|t_{n_k} - y_{n_k}\| \to 0$.*

Then, u belongs to the set $VI(A, C)$.

Proof. Assume that B is the mapping defined as in the (2.2). By the above comments, since A is monotone and Lipschitz continuous on C, it follows that B is maximal monotone. Hence, we can use the property (2.3). Suppose that $\{t_{n_k}\}$ converges weakly to some u in C. We show that $\langle u - v, -w\rangle \geq 0$ for all $(v, w) \in G(B)$. For this purpose, let (v, w) be an arbitrary element of $G(B)$. It follows from the definition that $w \in Av + N_C v$. Hence, $w - Av \in N_C v$ and $\langle v - z, w - Av\rangle \geq 0$ for all $z \in C$. Since $\{t_n\} \subset C$, we deduce that

$$\langle v - t_{n_k}, w\rangle \geq \langle v - t_{n_k}, Av\rangle. \tag{2.4}$$

Using (2.1) we have $\langle x_{n_k} - \lambda_{n_k} A y_{n_k} - t_{n_k}, t_{n_k} - v\rangle \geq 0$. Hence, we have

$$\begin{aligned}\langle v - t_{n_k}, w\rangle &\geq \langle v - t_{n_k}, Av\rangle - \tfrac{1}{\lambda_{n_k}}\langle t_{n_k} - v, x_{n_k} - \lambda_{n_k} A y_{n_k} - t_{n_k}\rangle \\ &= \langle v - t_{n_k}, Av - At_{n_k}\rangle + \langle v - t_{n_k}, At_{n_k} - Ay_{n_k}\rangle \\ &\quad - \langle v - t_{n_k}, \tfrac{t_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle,\end{aligned}$$

Now, since A is monotone, we deduce that

$$\langle v - t_{n_k}, w\rangle \geq \langle v - t_{n_k}, At_{n_k} - Ay_{n_k}\rangle - \langle v - t_{n_k}, \frac{t_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle. \tag{2.5}$$

Using (ii) and the fact that A is Lipschitz continuous it follows that $\|At_{n_k} - Ay_{n_k}\| \to 0$. On the other hand, since $\{t_{n_k}\}$ converges weakly to u, it follows that $\{t_{n_k}\}$ is bounded. Let $k \to \infty$ in (2.5) and note that $\lambda_{n_k} \geq \delta > 0$, hence, we deduce that $\langle v - u, w\rangle \geq 0$. Now (2.3) implies that $u \in VI(A, C)$. □

3. MAIN RESULTS

In this section, we assume that H is a real Hilbert space. Let F be a κ-Lipschitzian continuous and $\eta-$ strongly monotone operator with $\kappa > 0$, $\eta > 0$, let A be a monotone and $\theta-$Lipschizian operator on H, let T be a $\beta-$demicontractive mapping on H, and let $f : H \to H$ be a L-Lipschitzian continuous operator. Assume that $Fix(T) \neq \varnothing$. Suppose that $w \in (0, 1 - \beta)$. We note that $Fix(T) = Fix(T_w)$. It follows from the Lemma 2.1 that $Fix(T)$ is closed and convex.

Lemma 3.1. *Let A be a $\theta-$Lipschitzian continuous and monotone mapping on H. Assume that $VI(A, H) \neq \varnothing$. Let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in H such that*

$$y_n = x_n - \lambda_n A x_n, \qquad t_n = x_n - \lambda_n A y_n.$$

Let x^* be the solution of the variational inequality (1.16). Assume that $T : H \to H$ is demiclosed on H and $Fix(T) \neq \varnothing$. Suppose that $\{t_n\}$ is a bounded sequence in

H and $\|Tt_n - t_n\| \to 0$. Suppose that $\|y_n - t_n\| \to 0$ and $\|x_n - y_n\| \to 0$. Then, we have

$$\liminf_{n\to\infty} (\mu F - \gamma f)x^*, t_n - x^*\rangle \geq 0.$$

Proof. Since $\{t_n\}$ is bounded, we can take a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that

$$\liminf_{n\to\infty} \langle(\mu F - \gamma f)x^*, t_n - x^*\rangle = \lim_{j\to\infty} \langle(\mu F - \gamma f)x^*, t_{n_j} - x^*\rangle,$$

and that $t_{n_j} \rightharpoonup \tilde{t}$. Since T is demiclosed on H, we have $\tilde{t} \in Fix(T)$. Now, since $\|y_n - t_n\| \to 0$ and $\|x_n - y_n\| \to 0$ it follows from the Lemma 2.9 that $\tilde{t}$ belongs to $VI(A, H)$. Thus, $\tilde{t} \in Fix(T) \cap VI(A, H)$. Since x^* is the solution of the variational inequality (1.16), we obtain that

$$\liminf_{n\to\infty} (\mu F - \gamma f)x^*, t_n - x^*\rangle = \langle(\mu F - \gamma f)x^*, \tilde{t} - x^*\rangle \geq 0.$$

□

Now, we are ready to prove the main result of this paper:

Theorem 3.1. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, H) \neq \varnothing$. Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta-strongly$ monotone operator with $\kappa > 0, \eta > 0$, and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

- (H1) $0 < \mu < 2\eta/\kappa^2, \quad 0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L;$
- (H2) $w \in (0, 1 - \beta);$
- (H3) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0;$
- (H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1);$
- (H5) $\sum_{n\geq 0} \alpha_n = \infty.$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n \gamma f(t_n) + (1 - \alpha_n \mu F)[(1 - w)I + wT](t_n), \end{aligned} \tag{3.1}$$

converge strongly to x^*, the unique solution of (1.16).

Proof. First, we show that $\{x_n\}$ is bounded.

Indeed, if $p \in Fix(T) \cap VI(A, H)$, by Lemmas 2.1, 2.6 and 2.8, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \gamma f(t_n) + (1 - \alpha_n \mu F)T_w(t_n) - p\| \\ &= \|\alpha_n \gamma (f(t_n) - f(p)) + \alpha_n(\gamma f(p) - \mu F p) \\ &\quad + (1 - \alpha_n \mu F)T_w(t_n) - (1 - \alpha_n \mu F)p\| \\ &\leq \alpha_n \gamma L\|t_n - p\| + \alpha_n\|\gamma f(p) - \mu F p\| + (1 - \alpha_n \tau)\|t_n - p\| \\ &\leq (1 - \alpha_n(\tau - \gamma L))\|t_n - p\| + \alpha_n\|\gamma f(p) - \mu F p\| \\ &\leq (1 - \alpha_n(\tau - \gamma L))\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu F p\| \end{aligned} \tag{3.2}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu F p\|}{\tau - \gamma L}\}, \qquad \text{for all } n \geq 0.$$

Thus, $\{x_n\}$ is bounded. Lemma 2.8 implies the boundedness of the sequences $\{t_n\}$ and $\{f(t_n)\}$. Also from the definition, we deduce the boundedness of the sequence $\{y_n\}$.

Since $x^* \in Fix(T) \cap VI(A, H)$, we can use the Lemmas 2.3 and 2.1 to deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n \mu F)T_w(t_n) - (1 - \alpha_n \mu F)x^* - \alpha_n(\mu F x^* - \gamma f(t_n))\|^2 \\
&= \|((1 - \alpha_n \mu F)T_w(t_n) - (1 - \alpha_n \mu F)x^*) \\
&\quad - \alpha_n(\mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n))\|^2 \\
&\leq \|((1 - \alpha_n \mu F)T_w(t_n) - (1 - \alpha_n \mu F)x^*)\|^2 \\
&\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \tau)^2 \|(T_w(t_n) - x^*)\|^2 \\
&\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \tau)^2 \|t_n - x^*\|^2 - (1 - \alpha_n \tau)^2 (w(1 - \beta - w)\|t_n - Tt_n\|^2) \\
&\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\quad - 2\alpha_n \langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \tau)^2 (\|x_n - x^*\|^2 - (1 - \theta^2 \lambda_n^2))\|x_n - y_n\|^2) \\
&\quad - (1 - \alpha_n \tau)^2 (w(1 - \beta - w)\|t_n - Tt_n\|^2) \\
&\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle - 2\alpha_n \langle \gamma f(x^*) \\
&\quad - \gamma f(t_n), x_{n+1} - x^* \rangle.
\end{aligned} \tag{3.3}$$

Let $\Gamma_n := \|x_n - x^*\|^2$. Now, we consider two cases to prove that $x_n \to x^*$.

Case 1. There is n_0 such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$. It follows that $\lim_{n\to\infty} \Gamma_n$ exists and hence $\lim_{n\to\infty} \Gamma_n - \Gamma_{n+1} = 0$.

The inequality (3.3) implies that

$$\begin{aligned}
&(1 - \alpha_n \tau)^2 (w(1 - \beta - w)\|t_n - Tt_n\|^2 + (1 - \theta^2 \lambda_n^2))\|x_n - y_n\|^2) \\
&\qquad \leq (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\qquad\quad - 2\alpha_n \langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle \\
&\qquad \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_n \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\qquad\quad - 2\alpha_n \langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle.
\end{aligned} \tag{3.4}$$

Since $\{x_n\}$ and $\{f(t_n)\}$ are bounded and $\alpha_n \to 0$ and $\lim_{n\to\infty} \Gamma_n - \Gamma_{n+1} = 0$, from the inequality (3.4) it follows that $\|t_n - Tt_n\| \to 0$ and $\|x_n - y_n\| \to 0$. By considering the Lemma 2.8 and the assumption $H(4)$, we also obtain that $\|y_n - t_n\| \to 0$.

Notice that, since $\|t_n - Tt_n\| \to 0$ and T is demiclosed on H, every weak cluster point of $\{t_n\}$ belongs to $Fix(T)$. On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - t_n\| &= \|\alpha_n \gamma f(t_n) + (1 - \alpha_n \mu F)T_w(t_n) - t_n\| \\
&= \|\alpha_n(\gamma f(t_n) - \mu F T_w(t_n)) + w(Tt_n - t_n)\| \\
&\leq \alpha_n \|\gamma f(t_n) - \mu F(T_w t_n)\| + w\|Tt_n - t_n\| \to 0.
\end{aligned} \tag{3.5}$$

Hence we deduce that

$$\|x_{n+1}-x_n\| \le \|x_{n+1}-t_n\| + \|t_n-y_n\| + \|y_n-x_n\| \to 0.$$

From (3.5) we have

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), t_n - x^*\rangle = \liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), x_{n+1} - x^*\rangle.$$

It can be checked that all of conditions of the Lemma 3.1 are satisfied, thus we may deduce that

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), t_n - x^*\rangle \ge 0.$$

Hence it follows that

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), x_{n+1} - x^*\rangle \ge 0. \tag{3.6}$$

Now, (3.3) implies that for all $n > n_0$:

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n\tau)^2(\|x_n-x^*\|^2 - (1-\theta^2\lambda_n^2))\|x_n-y_n\|^2)\\ &\quad -(1-\alpha_n\tau)^2 w(1-\beta-w)\|t_n-Tt_n\|^2\\ &\quad -2\alpha_n\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle\\ &\quad -2\alpha_n\langle \gamma f(x^*)-\gamma f(t_n), x_{n+1}-x^*\rangle\\ &\le (1-\alpha_n\tau)^2\|x_n-x^*\|^2 + 2\alpha_n\gamma L\|t_n-x^*\|\|x_{n+1}-x^*\|\\ &\quad -2\alpha_n\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle\\ &\le (1-\alpha_n\tau)^2\|x_n-x^*\|^2 + 2\alpha_n\gamma L\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\quad -2\alpha_n\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle\\ &\le (1-2\alpha_n\tau+(\alpha_n\tau)^2))\|x_n-x^*\|^2 + 2\alpha_n\gamma L\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\quad -2\alpha_n\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle\\ &\le (1-2\alpha_n\tau+(\alpha_n\tau)^2))\|x_n-x^*\|^2 + 2\alpha_n\gamma L\|x_n-x^*\|^2\\ &\quad -2\alpha_n\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle\\ &\le (1-2\alpha_n(\tau-\gamma L))\|x_n-x^*\|^2 + 2\alpha_n(\tau-\gamma L)(\tfrac{\alpha_n\tau^2\|x_n-x^*\|^2}{2(\tau-\gamma L)}\\ &\quad -\tfrac{\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle}{\tau-\gamma L})\end{aligned}$$

Recall that by the the assumption $0 < \gamma < \tau/L$. Since $\{x_n\}$, $\{f(x_n)\}$ and $\{f(x_n)\}$ are bounded and $\alpha_n \to 0$, we deduce from (3.6) that

$$\limsup_{n\to\infty}(\frac{\alpha_n\tau^2\|x_n-x^*\|^2}{2(\tau-\gamma L)} - \frac{\langle \mu Fx^*-\gamma f(x^*), x_{n+1}-x^*\rangle}{\tau-\gamma L})) \le 0,$$

Now, we can apply the Lemma 2.4 to conclude $x_n \to x^*$. Also, since $\|y_n - x_n\| \to 0$ and $\|t_n - y_n\| \to 0$, we have $y_n \to x^*$ and $t_n \to x^*$.

Case 2. Assume that there is a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}_{n\ge 0}$ such that $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ for all $j > 0$. In this case, it follows from Lemma 2.7 that there is a subsequence $\{\Gamma_{\tau(n)}\}_{n\ge 0}$ of $\{\Gamma_n\}_{n\ge 0}$ such that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ and $\{\tau(n)\}$ is defined as in Lemma 2.7. In this case, first, we show that $\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0$. It follows from inequality (3.3) that

$$\begin{aligned}&(1-\alpha_n\tau)^2(w(1-\beta-w)\|t_{\tau(n)}-Tt_{\tau(n)}\|^2 + (1-\theta^2\lambda_n^2)\|x_{\tau(n)}-y_{\tau(n)}\|^2)\\ &\qquad\le (1-\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)}-x^*\|^2 - \|x_{\tau(n)+1}-x^*\|^2\\ &\qquad\quad - 2\alpha_{\tau(n)}\langle \mu Fx^*-\gamma f(x^*), x_{\tau(n)+1}-x^*\rangle\\ &\qquad\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*)-\gamma f(t_{\tau(n)}), x_{\tau(n)+1}-x^*\rangle\\ &\qquad\le \|x_{\tau(n)}-x^*\|^2 - \|x_{\tau(n)+1}-x^*\|^2\\ &\qquad\quad - 2\alpha_{\tau(n)}\langle \mu Fx^*-\gamma f(x^*), x_{\tau(n)+1}-x^*\rangle\\ &\qquad\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*)-\gamma f(t_{\tau(n)}), x_{\tau(n)+1}-x^*\rangle.\end{aligned} \tag{3.7}$$

Since $\{x_n\}$, and $\{t_n\}$, $\{f(t_n)\}$ are bounded and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, it follows from the above inequality that

$$\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0 \quad \text{and} \quad \|x_{\tau(n)} - y_{\tau(n)}\| \to 0. \tag{3.8}$$

Now, we can apply the Lemma 2.8 to conclude that $\|y_{\tau(n)} - t_{\tau(n)}\| \to 0$. Also we have

$$\begin{aligned}\|x_{\tau(n)+1} - t_{\tau(n)}\| &= \|\alpha_{\tau(n)}\gamma f(t_{\tau(n)}) + (1-\alpha_{\tau(n)}\mu F)T_w(t_{\tau(n)}) - t_{\tau(n)}\| \\ &= \|\alpha_{\tau(n)}(\gamma f(t_{\tau(n)}) - \mu F T_w(t_{\tau(n)})) + w(Tt_{\tau(n)} - t_{\tau(n)})\| \\ &\le \alpha_{\tau(n)}\|\gamma f(t_{\tau(n)}) - \mu F(T_w t_{\tau(n)})\| + w\|Tt_{\tau(n)} - t_{\tau(n)}\| \to 0.\end{aligned}$$

Hence, we deduce that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \le \|x_{\tau(n)+1} - t_{\tau(n)}\| + \|t_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - x_{\tau(n)}\| \to 0.) \tag{3.9}$$

Since $\|x_{\tau(n)+1} - t_{\tau(n)}\| \to 0$, it follows that

$$\liminf_{n\to\infty}\langle \mu F x^* - \gamma f(x^*), t_{\tau(n)} - x^*\rangle = \liminf_{n\to\infty}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle. \tag{3.10}$$

On the other hand, as $\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0$ and T is demiclosed on H, we can use Lemma 3.1 to deduce that

$$\liminf_{n\to\infty}\langle \mu F x^* - \gamma f(x^*), t_{\tau(n)} - x^*\rangle \ge 0. \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$\liminf_{n\to\infty}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \ge 0. \tag{3.12}$$

Now, we use the inequality (3.3) to conclude that

$$\begin{aligned}\|x_{\tau(n)+1} - x^*\|^2 &\le (1-\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - x^*\|^2 \\ &\quad - (1-\alpha_{\tau(n)}\tau)^2(1-\theta^2\lambda^2_{\tau(n)})\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1-\alpha_{\tau(n)}\tau)^2 w(1-\beta-w)\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\ &\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x^*\rangle \\ &\le (1 - 2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2 \\ &\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)} - x^*\rangle \\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle \qquad (3.13) \\ &\le (1 - 2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2 \\ &\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad + 2\alpha_{\tau(n)}\gamma L\|t_{\tau(n)} - x^*\|\|x_{\tau(n)} - x^*\| \\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle \\ &\le (1 - 2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2 \\ &\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad + 2\alpha_{\tau(n)}\gamma L\|x_{\tau(n)} - x^*\|^2 \\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle.\end{aligned}$$

Recall $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ for all $n \geq 0$. From the inequality (3.13) we have

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\leq 2\alpha_{\tau(n)}(-\tau\|x_{\tau(n)} - x^*\|^2 + \gamma L\|x_{\tau(n)} - x^*\|^2 \\ &\quad + \frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad + \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|). \end{aligned} \tag{3.14}$$

Since $0 < \alpha_n < 1$, inequality (3.14) implies that

$$\begin{aligned} (\tau - \gamma L)\|x_{\tau(n)} - x^*\|^2 \leq (\frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ + \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|). \end{aligned} \tag{3.15}$$

Since $\{x_n\}$ and $\{f(t_n)\}$ are bounded, it follows from (3.9) and (3.12) that

$$\begin{aligned} \limsup_{n\to\infty}(\frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ + \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|) \leq 0. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we deduce that

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Since $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$, it follows that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x^*\|^2 = 0.$$

On the other hand, from Lemma 2.7 we have $\Gamma_n < \Gamma_{\tau(n)+1}$ for all $n \geq 0$, and therefore, $x_n \to x^*$. Since $\|y_n - x_n\| \to 0$ and $\|t_n - y_n\| \to 0$, we have $y_n \to x^*$ and $t_n \to x^*$. □

If we take $f \equiv 0$ in the above Theorem, we have the following Theorem:

Theorem 3.2. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, H) \neq \varnothing$. Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta - strongly$ monotone operator with $\kappa > 0, \eta > 0$, and assume that the following conditions hold:*

- (H1) $0 < \mu < 2\eta/\kappa^2$;
- (H2) $w \in (0, 1 - \beta)$;
- (H3) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0$;
- (H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
- (H5) $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = (1 - \alpha_n \mu F)[(1 - w)I + wT](t_n), \end{aligned}$$

converge strongly to x^* which is the unique solution of the following variational inequality problem

find $x^* \in Fix(T) \cap VI(A, H)$ such that

$$\langle v - x^*, \mu F x^*\rangle \geq 0, \ \textit{for all } v \in Fix(T) \cap VI(A, H).$$

Corollary 3.2. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A,H) \neq \varnothing$. Let $u \in H$ be arbitrary chosen and assume that the following conditions hold:*

(H1) $w \in (0, 1-\beta)$;
(H2) $\{\alpha_n\} \subset (0,1),\ \alpha_n \to 0$;
(H3) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H4) $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned}
&x_0 \in H,\\
&y_n = x_n - \lambda_n A x_n,\\
&t_n = x_n - \lambda_n A y_n,\\
&x_{n+1} = \alpha_n u + (1-\alpha_n)[(1-w)I + wT](t_n),
\end{aligned}$$

converge strongly to x^*, which satisfies $x^* = P_{Fix(T)\cap VI(A,H)}(u)$.

Proof. In Theorem 3.1 set $F := I - u$. Note that F is 1-Lipschizian and 1-strongly monotone operator and the result follows. □

We can also drive the following corollary from the Theorem 3.1:

Corollary 3.3. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A,C) \neq \varnothing$. Let $u \in H$ be arbitrary chosen and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

(H1) $0 < \mu < 2, \quad 0 < \gamma < \mu(1-\frac{\mu}{2})/L = \tau/L$;
(H2) $w \in (0, 1-\beta)$;
(H3) $\{\alpha_n\} \subset (0,1),\ \alpha_n \to 0$;
(H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H5) $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned}
&x_0 \in H,\\
&y_n = x_n - \lambda_n A x_n,\\
&t_n = x_n - \lambda_n A y_n,\\
&x_{n+1} = \alpha_n \gamma f(t_n) + \alpha_n \mu u + (1-\alpha_n\mu)[(1-w)I + wT](t_n),
\end{aligned}$$

converge strongly to x^*, which satisfies $x^* = P_{Fix(T)\cap VI(A,H)}(u + \frac{\gamma}{\mu} f(x^*))$.

In the next Corollary, we show that Theorem 3.1 can be applied to approximating common zeroes of monotone operators:

Corollary 3.4. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Let $D : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}(0) \cap D^{-1}(0) \neq \varnothing$. Let J_r^D be the resolvent of D for each $r > 0$. Let $u \in H$ be arbitrary chosen and let $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

(H1) $0 < \mu < 2, \quad 0 < \gamma < \mu(1-\frac{\mu}{2})/L = \tau/L$;
(H2) $w \in (0,1)$;
(H3) $\{\alpha_n\} \subset (0,1),\ \alpha_n \to 0$;
(H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H5) $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{t_n\}$ *generated by following algorithm*

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n \gamma f(t_n) + \alpha_n \mu u + (1-\alpha_n\mu)[(1-w)I + wJ_r^D](t_n), \end{aligned} \tag{3.17}$$

converge strongly to x^*, *which satisfies* $x^* = P_{A^{-1}(0)\cap D^{-1}(0)}(u + \frac{\gamma}{\mu} f(x^*))$.

Proof. Recall that J_r^D is a nonexpansive mapping (hence demiclosed and $0-$demicontractive). On the other hand, we have $A^{-1}(0) = VI(A, H)$ and $Fix(J_r^D) = D^{-1}(0)$. So we can apply Corollary 3.3 to conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by the iterative method (3.17) converge strongly to x^*, which satisfies $x^* = P_{A^{-1}(0)\cap D^{-1}(0)}(u + \frac{\gamma}{\mu} f(x^*))$. □

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