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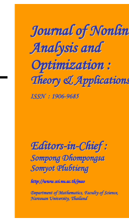

ON SET-VALUED MIXED VECTOR VARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

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ABSTRACT. In this paper we introduce the concept of generalized η-pseudomonotone mappings and generalized version of vector mixed variational-like inequalities in Banach spaces. Utilizing Ky Fan's Lemma and Nadler's Lemma, we derive the solvability for this class of vector mixed variational-like inequalities involving generalized η-pseudomonotone mappings. The results presented in this work are extensions and improvements of some earlier and recent results in the literature.

KEYWORDS : H-uniform continuity; Hausdorff metric; Complete continuity; Generalized η-pseudomonotone.
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1. INTRODUCTION

Vector variational inequality theory was initially introduced and studied by Giannessi [8] in the setting of finite dimensional Euclidean spaces. Ever since it has been widely studied and generalized in infinite dimensional spaces since it covers many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance, etc., as special cases. For details we refer [2,4,9-11,14-15,18,20-22] and references therein.

In recent past, a number of authors have studied generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, quasimonotonicity and semimonotonicity; see [1,3,5,7,11-13,19] and the references therein. Bai *et al.* [1] introduced η-α-pseudomonotonicity and established some existence results for variational-like inequalities in reflexive Banach spaces. Recently, Zeng and Yao [21] considered and studied the solvability for a class of generalized vector variational inequalities in reflexive Banach spaces. They proved the solvability for this class of generalized vector variational inequalities with monotonicity assumption. Also they removed the boundedness assumption of K and extended to the general case of a nonempty closed and convex subset K.

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Inspired and motivated by the work of Bai *et al.*[1], Usman *et al.*[18] and Zeng and Yao [21], in this work we introduce the concept of generalized η-pseudomonotone mappings. Further, a more general vector mixed variational-like inequality problem for set-valued mappings which is a extension of the corresponding vector variational-like inequalities in [2,14-15], is considered. Furthermore, utilizing the Ky Fan's Lemma and the Nadler's Lemma, we establish some solvability results for this class of generalized vector mixed variational-like inequality problem involving generalized η-pseudomonotone mappings. The results presented in this work extend and unify corresponding results of [1,7,10,18,21].

2. PRELIMINARIES

Throughout the paper unless otherwise stated, let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset of X and $P \subset Y$ be a nonempty subset of Y. $P \neq Y$ be a closed, convex and pointed cone. The partial order $\leq_P$ in Y, induced by the pointed cone P is defined by declaring $x \leq_P y$ if and only if $y - x \in P$ for all x, y in Y. An ordered Banach space is a pair (Y, P) with the partial order induced by P. The weak order $\not\leq_{\text{int}\, P}$ in an ordered Banach space (Y, P) with $\text{int}\, P \neq \emptyset$ is defined as $x \not\leq_{\text{int}\, P} y$ if and only if $y - x \notin \text{int}\, P$ for all x, y in Y, where $\text{int}\, P$ denotes the interior of P. Let $L(X, Y)$ be the space of all continuous linear mappings from X into Y. Let $P : K \longrightarrow 2^Y$ be a set-valued mapping such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $\text{int}\, P(x) \neq \emptyset$ and let $P_- = \bigcap_{x \in K} P(x)$.

Let $A : L(X, Y) \longrightarrow L(X, Y)$ be a mapping $\eta : X \times X \longrightarrow X$ and $f : K \times K \longrightarrow Y$ are the two bi-mappings and $V : K \longrightarrow 2^Y$ and $H : K \times Y \longrightarrow 2^{L(X,Y)}$ be set-valued mappings. In this paper we consider the following generalized vector mixed variational-like inequality problem (for short, GVMVLIP): Find $x \in K$, $z \in V(x)$ and $\xi \in H(x, z)$ such that

$$\langle A\xi, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\, P(x)} 0, \ \forall y \in K. \tag{2.1}$$

Some special cases of GVMVLIP (2.1)

(I) If $f \equiv 0$ and $A \equiv I$, the identity mapping of $L(X, Y)$, then GVMVLIP (2.1) reduces to the following generalized vector pre-variational inequality problem of finding $x \in K$ $z \in V(x)$ and $\xi \in H(x, z)$ such that

$$\langle \xi, \eta(y, x)\rangle \not\leq_{\text{int}\, P(x)} 0, \ \forall y \in K,$$

which was introduced and considered in real topological vector spaces by Chadli *et al.* [2] in 2004.

(II) If $V \equiv 0$, $H \equiv T : K \longrightarrow 2^{L(X,Y)}$ and $P(x) = P$, $\forall x \in K$, then GVMVLIP (2.1) reduces to the following generalized mixed vector variational-like inequality problem of finding $x \in K$ and $u \in T(x)$ such that

$$\langle Au, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\, P} 0, \ \forall y \in K,$$

which was introduced and studied by Usman *et al.* [18] in 2009.

(III) If $V \equiv 0$, $H \equiv T : K \longrightarrow 2^{L(X,Y)}$ and $A \equiv I$, the identity mapping of $L(X, Y)$, then GVMVLIP (2.1) reduces to the following generalized vector variational-type inequality problem of finding $x \in K$ such that for all $y \in K$, there exists $s_o \in T(x)$ such that

$$\langle s_0, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\, P(x)} 0, \ \forall y \in K,$$

which was introduced and considered in Hausdorff topological vector spaces by Lee *et al.* [14] in 2000.

(IV) If we take $T : K \longrightarrow L(X,Y)$ and $P(x) = P$, $\forall x \in K$ in (II), then it reduces to the following generalized weak vector variational-like inequality of finding $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle + f(y,x) \nleq_{\text{int}\, P} 0, \ \forall y \in K,$$

which was studied by Lee *et al.* [15] in 2008.

First, we recall the following concepts and results which are needed in the sequel.

Definition 2.1. A mapping $f : K \longrightarrow Y$ is said to be

(i) P_--*convex*, if $f(tx+(1-t)y) \leq_{P_-} tf(x)+(1-t)f(y), \ \forall x, y \in K, \ t \in [0,1]$;
(ii) P_--*concave*, if $-f$ is P_--convex.

Definition 2.2. [22] Let $P : K \longrightarrow 2^Y$ be a set-valued mapping such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $T : K \longrightarrow L(X,Y)$ and $\eta : K \times K \longrightarrow X$ be two mappings. T is said to be η-*pseudomonotone*, if for any $x, y \in K$

$$\langle T(x), \eta(y,x)\rangle \geq_{P_-} 0 \implies \langle T(y), \eta(x,y)\rangle \leq_{P_-} 0, \text{ where } P_- = \bigcap_{x\in K} P(x).$$

Remark that, if $\eta(y,x) = y-x, \ \forall x, y \in K$, then η-pseudomonotonicity of T reduces to pseudomonotonicity of T.

Lemma 2.3. [4] *Let (Y, P) be an ordered Banach space with a closed, convex and pointed cone P with int $P \neq \emptyset$. Then $\forall x, y, z \in Y$, we have*

(i) $z \nleq_{\text{int}\, P} x$ *and* $x \geq_P y \Rightarrow z \nleq_{\text{int}\, P} y$*;*
(ii) $z \ngeq_{\text{int}\, P} x$ *and* $x \leq_P y \Rightarrow z \ngeq_{\text{int}\, P} y$*.*

Definition 2.4. A mapping $g : X \longrightarrow Y$ is said to be *completely continuous* if and only if the weak convergence of x_n to x in X implies the strong convergence of $g(x_n)$ to $g(x)$ in Y.

Lemma 2.5. [6] *Let K be a subset of a topological vector space X and let $F : K \longrightarrow 2^X$ be a KKM mapping. If for each $x \in K$, $F(x)$ is closed and for at least one $x \in K$, $F(x)$ is compact, then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Lemma 2.6. [16]*Let X, Y and Z are real topological vector spaces, K be nonempty subset of X. Let $H : K \times Y \longrightarrow 2^Z$, $V : K \longrightarrow 2^Y$ be set-valued mapping. If both H, V are upper semicontinuous with compact values, then the set-valued mapping $T : K \longrightarrow 2^Z$ defined by*

$$T(x) = \bigcup_{z\in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values.

Lemma 2.7. [17]*Let $(X, \|.\|)$ be a normed vector space and H be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X, induced by a metric d in terms of $d(u,v) = \|u-v\|$, defined by*

$$H(U,V) = \max\{\sup_{u\in U}\inf_{v\in V}\|u-v\|, \sup_{v\in V}\inf_{u\in U}\|u-v\|\},$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq \|H(U, V)\|$.

Definition 2.8. A nonempty, compact set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ is called H-uniformly continuous if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in K$ with $\|x - y\| < \delta$, there holds $H(Tx, Ty) < \epsilon$, where H is the Hausdorff metric defined on $CB(L(X,Y))$.

3. EXISTENCE RESULTS FOR GVMVLIP (2.1)

Now we shall derive the solvability for the GVMVLIP (2.1) involving generalized η-pseudomonotone mappings under some quite mild conditions by using Ky Fan's Lemma [6] and Nadler's Lemma [17].

First, we give the concept of generalized η-pseudomonotone mappings.

Definition 3.1. Let $f : K \times K \longrightarrow Y$ and $\eta : X \times X \longrightarrow X$ are the two bi-mappings, let $A : L(X,Y) \longrightarrow L(X,Y)$ be the mapping, $V : K \longrightarrow 2^Y$ and $H : K \times Y \longrightarrow 2^{L(X,Y)}$ are the set-valued mappings. Then H, V are said to be *generalized η-pseudomonotone* mappings with respect to A, if for any $x \in K,\ z_1 \in V(x)$ and $\xi_1 \in H(x, z_1)$, we have

$$\langle A\xi_1, \eta(y,x)\rangle + f(y,x) \not\leq_{\text{int}\, P(x)} 0, \text{ implies that}$$

$$\langle A\xi_2, \eta(y,x)\rangle + f(y,x) - \alpha(x,y) \not\leq_{\text{int}\, P(x)} 0, \ \forall y \in K,\ z_2 \in V(y),\ \xi_2 \in H(y, z_2),$$

where $\alpha : X \times X \longrightarrow Y$ is a mapping such that $\lim_{t \longrightarrow 0^+} \frac{\alpha(x, ty+(1-t)x)}{t} = 0$.

Remark 3.2. (i) If $f, V \equiv 0$, $H \equiv T$: $K \longrightarrow L(X,Y)$ and $A \equiv I$, the identity mapping of $L(X,Y)$ and $\alpha(x,y) = \alpha(y - x)$, where $\alpha : X \longrightarrow \mathbb{R}$ with $\alpha(\lambda z) = \lambda^p \alpha(z)$ for $\lambda > 0,\ p > 1$ and if $P(x) = \mathbb{R}_+,\ \forall x \in K$, then Definition 3.1 reduces to

$$\langle Ty, \eta(y,x)\rangle \geq 0 \text{ implies } \langle Tx, \eta(y,x)\rangle \geq \alpha(y - x),\ \forall x, y \in K.$$

Then T is said to be relaxed η-α-pseudomonotone, introduced and studied by Bai *et al.* [1].

(ii) In the case (i), if we take $\eta(y,x) = y - x$, for all $x, y \in K$ and $\beta \equiv 0$, then it reduces to

$$\langle Ty, y - x\rangle \geq 0 \text{ implies } \langle Tx, y - x\rangle \geq 0,\ \forall x, y \in K.$$

Then T is said to be pseudomonotone; see for example [5, 11, 13]

Now we prove Minty's type Lemma for GVMVLIP (2.1) with the help of generalized η-pseudomonotone mappings.

Lemma 3.3. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X,Y) \longrightarrow L(X,Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty set-valued mapping. Suppose the following conditions hold:*

(i) *$f : K \times K \longrightarrow Y$ be a P_--convex in first argument with the condition $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$;*
(ii) *$\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is P_--convex for each $(\xi, y) \in L(X,Y) \times K$ is fixed;*
(iii) *$\langle A\xi, \eta(x,x)\rangle = 0,\ \forall(\xi, y) \in L(X,Y) \times K$;*

(iv) *Let $H : K \times Y \longrightarrow 2^{L(X,Y)}$, $V : K \longrightarrow 2^Y$ be two upper semicontinuous mappings with compact values such that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by*

$$T(x) = \bigcup_{z\in V(x)} H(x,z) = H(x, V(x))$$

is H-uniformly continuous.

Then following two problems are equivalent:

(A) *there exists $x_0 \in K,\ z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that*

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K. \tag{3.1}$$

(B) *there exists $x_0 \in K$ such that*

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z). \tag{3.2}$$

Proof. Suppose that there exists $x_0 \in K,\ z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K.$$

Since $H,\ V$ are generalized η-pseudomonotone with respect to A, we have

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

For any given $y \in K$, we know that $y_t = ty + (1-t)x_0 \in K$, for each $t \in (0, 1)$, we have $y_t \in K$ as K is convex. Hence for each $\xi_t \in T(y_t) = H(y_t, V(y_t))$

$$\langle A\xi_t, \eta(y_t, x_0)\rangle + f(y_t, x_0) - \alpha(x_0, y_t) \not\leq_{\text{int}\, P(x_0)} 0. \tag{3.3}$$

Since f is P_--convex in first argument, it follows that

$$f(y_t, x_0) \leq_{P_-} tf(y, x_0) + (1-t)f(x_0, x_0) = tf(y, x_0). \tag{3.4}$$

From assumptions (ii) and (iii) on η, we have

$$\begin{aligned}\langle A\xi_t, \eta(y_t, x_0)\rangle &= \langle A\xi_t, \eta(ty + (1-t)x_0, x_0)\rangle \\ &\leq_{P_-} t\langle A\xi_t, \eta(y, x_0)\rangle + (1-t)\langle A\xi_t, \eta(x_0, x_0)\rangle \\ &= t\langle A\xi_t, \eta(y, x_0)\rangle.\end{aligned} \tag{3.5}$$

It follows from inclusions (3.3)-(3.5) and Lemma 2.3 that for $t > 0$ and $p > 1$

$$t[\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0)] - \alpha(x_0, y_t) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall v_t \in T(y_t),\ t \in (0, 1).$$

$$\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0) - \frac{\alpha(x_0, y_t)}{t} \not\leq_{\text{int}\, P(x_0)} 0,\ \forall v_t \in T(y_t),\ t \in (0, 1). \tag{3.6}$$

We remark that according to Lemma 2.6, the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z\in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(x_0)$ are compact and from Lemma 2.7, it follows that for each fixed $\xi_t \in T(y_t)$, there exists an $\zeta_t \in T(x_0)$ such that

$$\|\xi_t - \zeta_t\| \leq H(T(y_t), T(x_0)).$$

Since $T(x_0)$ is compact, without loss of generality, we may assume that $\zeta_t \longrightarrow \xi_0 \in T(x_0)$ as $t \longrightarrow 0^+$. Since T is H-uniformly continuous and $\|y_t - x_0\| = t\|y - x_0\| \longrightarrow 0$ as $t \longrightarrow 0^+$ so $H(T(y_t), T(x_0)) \longrightarrow 0$ as $t \longrightarrow 0^+$. Thus one has

$$\begin{aligned}\|\xi_t - \xi_0\| &\leq \|\xi_t - \zeta_t\| + \|\zeta_t - \xi_0\| \\ &\leq \|H(T(y_t), T(x_0))\| + \|\zeta_t - \xi_0\| \longrightarrow 0.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\xi_t, \eta(y, x_0)\rangle - \langle A\xi_0, \eta(y, x_0)\rangle\| &= \|\langle A\xi_t - A\xi_0, \eta(y, x_0)\rangle\| \\ &\leq \|A\xi_t - A\xi_0\|\|\eta(y, x_0)\| \longrightarrow 0\end{aligned}$$

Also by inclusion (3.6), we deduce that

$$\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0) - \frac{\alpha(x_0, y_t)}{t} \in Y\backslash(-\text{int } P(x_0)).$$

Since $Y\backslash(-\text{int } P(x_0))$ is closed and letting $t \longrightarrow 0^+$, we have

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \in Y\backslash(-\text{int } P(x_0)),$$

and so

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int } P(x_0)} 0.$$

Next we claim that there holds

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \not\leq_{\text{int } P(x_0)} 0, \ \forall v \in K.$$

Indeed, let v be an arbitrary element in K and let $v_t = tv + (1-t)x_0 \in K$, for each $t \in (0, 1)$. Then one has $\|y_t - v_t\| = t\|y - v\| \longrightarrow 0$ as $t \longrightarrow 0^+$. Hence from H-uniform continuity of T it follows that $H(Ty_t, Tv_t) \longrightarrow 0$ as $t \longrightarrow 0^+$. Let $\{\xi_t\}_{t\in(0,1)}$ be any net choosen such that $\xi_t \longrightarrow \xi_0$ as $t \longrightarrow 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.7, it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$\|\xi_t - \gamma_t\| \leq H(Ty_t, Tv_t).$$

Consequently

$$\begin{aligned}\|\gamma_t - \xi_0\| &\leq \|\xi_t - \gamma_t\| + \|\xi_t - \xi_0\| \\ &\leq H(Ty_t, Tv_t) + \|\xi_t - \xi_0\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned}$$

Note that A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\gamma_t, \eta(v, x_0)\rangle - \langle A\xi_0, \eta(v, x_0)\rangle\| &= \|\langle A\gamma_t - A\xi_0, \eta(v, x_0)\rangle\| \\ &\leq \|A\gamma_t - A\xi_0\|\|\eta(v, x_0)\| \longrightarrow 0.\end{aligned}$$

Replacing y, y_t and ξ_t in inclusion (3.6) by v, v_t and γ_t, respectively, one deduces that

$$\langle A\gamma_t, \eta(v, x_0)\rangle + f(v, x_0) - \frac{\alpha(x_0, v_t)}{t} \not\leq_{\text{int } P(x_0)} 0, \ \forall t \in (0, 1),$$

which implies that

$$\langle A\gamma_t, \eta(v, x_0)\rangle + f(v, x_0) - \frac{\alpha(x_0, v_t)}{t} \in Y\backslash(-\text{int } P(x_0)).$$

Since $Y\backslash(-\text{int } P(x_0))$ is closed and letting $t \longrightarrow 0^+$, one has that

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \in Y\backslash(-\text{int } P(x_0)),$$

and hence

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \not\leq_{\text{int } P(x_0)} 0.$$

Thus, according to arbitrariness of v the assertion is valid.

Since, $\xi_0 \in T(x_0) = \bigcup_{z \in V(x_0)} H(x_0, z) = H(x_0, V(x_0))$, it follows that there exists $z_0 \in V(x_0)$ such that $\xi_0 \in H(x_0, z_0)$. Therefore, (3.1) holds. This completes the proof. □

Now, with the help of above Minty's type Lemma, we have following existence theorem for GVMVLIP (2.1).

Theorem 3.4. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X,Y) \longrightarrow L(X,Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$ and completely continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x,x)\rangle = 0$, for each $x \in K$ and $\xi \in L(X,Y)$;*
- (iii) *for each $(\xi, y) \in L(X,Y) \times K$ fixed, $\langle A\xi, \eta(.,y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y,.) : K \longrightarrow X$ is completely continuous;*
- (v) *for each fixed $y \in K,\ \alpha(.,y)$ is weakly lower semicontinuous.*

Suppose additionally that $H : K \times Y \longrightarrow 2^{L(X,Y)}$, $V : K \longrightarrow 2^Y$ be two upper semicontinuous mappings with compact values such that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K,\ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \not\leq_{\text{int}\, P(x^\star)} 0,\ \forall y \in K.$$

Proof. We divide the proof into four steps.

Step I. We claim that for every finite subset E of K, there exists $\bar{x} \in \mathrm{co}E$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0,\ \forall y \in \mathrm{co}E.$$

Indeed, let E be any finite subset of K and let us define a vector set-valued mapping $F : \mathrm{co}E \longrightarrow 2^{\mathrm{co}E}$ as follows:

$$F(y) = \{x \in \mathrm{co}E : \exists z \in V(x), \xi \in H(x,z) \text{ such that } \langle A\xi, \eta(y,x)\rangle + f(y,x) \not\leq_{\text{int}\, P(x)} 0\},$$

for all $y \in \mathrm{co}E$. From assumption (ii), one has $F(y) \neq \emptyset$ since $y \in F(y)$. The set $F(y)$ is also closed. Indeed, let $\{x_n\} \subseteq F(y)$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Hence for each n, there exists $z_n \in V(x_n)$ and $\xi_n \in H(x_n, z_n)$ such that

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n) \not\leq_{\text{int}\, P(x_n)} 0.$$

Since V is upper semicontinuous with compact values, $V(\mathrm{co}E)$ is compact. Therefore, without loss of generality one deduces that $z_n \longrightarrow z \in V(x)$ as $n \longrightarrow \infty$. On the other hand, since H is upper semicontinuous with compact values $H(\mathrm{co}E, V(\mathrm{co}E))$ is compact. It follows without loss of generality that $\xi_n \longrightarrow \xi \in H(x,z)$. Now, let $\{y_1, ..., y_n\} \subseteq \mathrm{co}E$ and let us verify $\mathrm{co}\{y_1, ..., y_n\} \subseteq \bigcup_{i=1}^{n} F(y_i)$. Let

$x \in \mathrm{co}\{y_1, ..., y_n\}$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0, i = 1, ..., n$ and $\sum_{i=1}^{n} t_i = 1$.
Utilizing assumptions (i)-(iii), we have

$$\begin{aligned} 0 &= \langle A\xi, \eta(x,x)\rangle + f(x,x) \\ &= \langle A\xi, \eta(\sum_{i=1}^{n} t_i y_i, x)\rangle + f(\sum_{i=1}^{n} t_i y_i, x) \\ &= \sum_{i=1}^{n} t_i[\langle A\xi, \eta(y_i,x)\rangle + f(y_i,x)] \end{aligned}$$

which hence implies that

$$\sum_{i=1}^{n} t_i[\langle A\xi, \eta(y_i,x)\rangle + f(y_i,x)] \not\leq_{\text{int}\, P(x)} 0.$$

Therefore there exists $i \in \{1, ..., n\}$ such that

$$\langle A\xi, \eta(y_i,x)\rangle + f(y_i,x) \not\leq_{\text{int}\, P(x)} 0.$$

Thus $x \in F(y_i) \subseteq \bigcup_{j=1}^{n} F(y_j)$. Consequently, from Lemma 2.5, we know that $\bigcap_{y\in \mathrm{co}E} F(y) \neq \emptyset$. Let $\bar{x} \in \bigcap_{y\in \mathrm{co}E} F(y)$. Then for each fixed $y \in coE$ there exists $\xi_y \in T\bar{x} = H(\bar{x}, V(\bar{x}))$ such that

$$\langle A\xi_y, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0.$$

Let $y_t = \bar{x} + t(y - \bar{x})$, $\forall t \in (0,1)$. Then, observe that

$$\begin{aligned} \langle A\xi_y, \eta(y_t,\bar{x})\rangle + f(y_t,\bar{x}) &= \langle A\xi, \eta(\bar{x} + t(y-\bar{x}),\bar{x})\rangle + f(\bar{x} + t(y-\bar{x}),\bar{x}) \\ &= t[\langle A\xi_y, \eta(y,\bar{x})\rangle + f(y,\bar{x})]. \end{aligned}$$

Hence

$$\langle A\xi_y, \eta(y_t,\bar{x})\rangle + f(y_t,\bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0.$$

Since H and V are generalized η-pseudomonotone with respect to A, we have

$$\langle A\xi_t, \eta(y_t,\bar{x})\rangle + f(y_t,\bar{x}) - \alpha(\bar{x}, y_t) \not\leq_{\text{int}\, P(\bar{x})} 0, \ \forall \xi_t \in Ty_t, \ t \in (0,1). \tag{3.7}$$

If it was false then for some $t_0 \in (0,1)$ and some $\xi_{t_0} \in Ty_{t_0}$

$$\langle A\xi_{t_0}, \eta(y_{t_0},\bar{x})\rangle + f(y_{t_0},\bar{x}) - \alpha(\bar{x}, y_{t_0}) \leq_{\text{int}\, P(\bar{x})} 0.$$

Consequently

$$\langle A\xi_y, \eta(y_{t_0},\bar{x})\rangle + f(y_{t_0},\bar{x}) = \langle A\xi_{t_0}, \eta(y_{t_0},\bar{x})\rangle + f(y_{t_0},\bar{x}) - \alpha(\bar{x}, y_{t_0}) \leq_{\text{int}\, P(\bar{x})} 0,$$

which hence implies that

$$\langle A\xi_y, \eta(y_{t_0},\bar{x})\rangle + f(y_{t_0},\bar{x}) \leq_{\text{int}\, P(\bar{x})} 0.$$

Which leads to a contradiction and hence (3.7) is valid. Now observe that

$$\begin{aligned} \langle A\xi_t, \eta(y_t,\bar{x})\rangle + f(y_t,\bar{x}) - \alpha(\bar{x}, y_t) &= \langle A\xi_t, \eta(ty+(1-t)\bar{x},\bar{x})\rangle + f(ty+(1-t)\bar{x},\bar{x}) - \alpha(\bar{x}, ty+(1-t)\bar{x}) \\ &= t[\langle A\xi_t, \eta(y,\bar{x})\rangle + f(y,\bar{x}) - \frac{\alpha(\bar{x}, ty+(1-t)\bar{x})}{t}]. \end{aligned}$$

Which together with (3.7) implies that

$$\langle A\xi_t, \eta(y,\bar{x})\rangle + f(y,\bar{x}) - \frac{\alpha(\bar{x}, y_t)}{t} \not\leq_{\text{int}\, P(\bar{x})} 0, \ \forall \xi_t \in Ty_t, \ t \in (0,1). \tag{3.8}$$

We remark that according to Lemma 2.6, the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(\bar{x})$ are compact and from Lemma 2.7, it follows that for each fixed $\xi_t \in T(y_t)$, there exists an $\zeta_t \in T(\bar{x})$ such that

$$\|\xi_t - \zeta_t\| \le H(T(y_t), T(\bar{x})).$$

Since $T(\bar{x})$ is compact, without loss of generality, we may assume that $\zeta_t \longrightarrow \bar{\xi} \in T(\bar{x})$ as $t \longrightarrow 0^+$. Since T is H-uniformly continuous and $\|y_t - \bar{x}\| = t\|y - \bar{x}\| \longrightarrow 0$ as $t \longrightarrow 0^+$, so $H(T(y_t), T(\bar{x})) \longrightarrow 0$ as $t \longrightarrow 0^+$. Thus one has

$$\begin{aligned}\|\xi_t - \bar{\xi}\| &\le \|\xi_t - \zeta_t\| + \|\zeta_t - \bar{\xi}\| \\ &\le H(T(y_t), T(\bar{x})) + \|\zeta_t - \bar{\xi}\| \longrightarrow 0.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\xi_t, \eta(y,\bar{x})\rangle - \langle A\bar{\xi}, \eta(y,\bar{x})\rangle\| &= \|\langle A\xi_t - A\bar{\xi}, \eta(y,\bar{x})\rangle\| \\ &\le \|A\xi_t - A\bar{\xi}\|\|\eta(y,\bar{x})\rangle\| \longrightarrow 0.\end{aligned}$$

Also by inclusion (3.8), we deduce that

$$\langle A\xi_t, \eta(y,\bar{x})\rangle + f(y,\bar{x}) - \frac{\alpha(\bar{x}, y_t)}{t} \in Y \backslash (-\text{int } P(\bar{x})).$$

Since $Y\backslash(-\text{int } P(\bar{x}))$ is closed and letting $t \longrightarrow 0^+$, we have that

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \in Y\backslash(-\text{int } P(\bar{x})),$$

and so

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\le_{\text{int } P(\bar{x})} 0.$$

Next we claim that there holds

$$\langle A\bar{\xi}, \eta(v,\bar{x})\rangle + f(v,\bar{x}) \not\le_{\text{int } P(\bar{x})} 0, \ \forall v \in \text{co}E.$$

Indeed, let v be an arbitrary element in $\text{co}E$ and set $v_t = tv + (1-t)\bar{x} \in K$, for each $t \in (0,1)$. Then one has $\|y_t - v_t\| = t\|y - v\| \longrightarrow 0$ as $t \longrightarrow 0^+$. Hence from H-uniform continuity of T it follows that $H(Ty_t, Tv_t) \longrightarrow 0$ as $t \longrightarrow 0^+$. Let $\{\xi_t\}_{t\in(0,1)}$ be any net choosen as above such that $\xi_t \longrightarrow \bar{\xi}$ as $t \longrightarrow 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.6, it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$\|\xi_t - \gamma_t\| \le H(T(y_t), T(v_t)).$$

Consequently

$$\begin{aligned}\|\gamma_t - \bar{\xi}\| &\le \|\xi_t - \gamma_t\| + \|\xi_t - \bar{\xi}\| \\ &\le H(T(y_t), T(v_t)) + \|\xi_t - \bar{\xi}\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\gamma_t, \eta(v,\bar{x})\rangle - \langle A\bar{\xi}, \eta(v,\bar{x})\rangle\| &= \|\langle A\gamma_t - A\bar{\xi}, \eta(v,\bar{x})\rangle\| \\ &\le \|A\gamma_t - A\bar{\xi}\|\|\eta(v,\bar{x})\rangle\| \longrightarrow 0.\end{aligned}$$

Replacing y, y_t and ξ_t in inclusion (3.8) by v, v_t and γ_t, respectively, one deduces that

$$\langle A\gamma_t, \eta(v,\bar{x})\rangle + f(v,\bar{x}) - \frac{\alpha(\bar{x}, v_t)}{t} \not\le_{\text{int } P(\bar{x})} 0, \ \forall t \in (0,1),$$

which implies that

$$\langle A\gamma_t, \eta(v,\bar{x})\rangle + f(v,\bar{x}) - \frac{\alpha(\bar{x}, v_t)}{t} \in Y\backslash(-\text{int } P(\bar{x})).$$

Since $Y\backslash(-\text{int } P(\bar{x}))$ is closed and letting $t \longrightarrow 0^+$, one has that

$$\langle A\bar{\xi}, \eta(v,\bar{x})\rangle + f(v,\bar{x}) \in Y\backslash(-\text{int } P(\bar{x})),$$

and hence

$$\langle A\bar{\xi}, \eta(v,\bar{x})\rangle + f(v,\bar{x}) \not\leq_{\text{int } P(\bar{x})} 0.$$

Thus, according to arbitrariness of v the assertion is valid.

Since, $\bar{\xi} \in T(\bar{x}) = \bigcup_{z\in V(\bar{x})} H(\bar{x}, z) = H(\bar{x}, V(\bar{x}))$, it follows that there exists $\bar{z} \in V(\bar{x})$ such that $\bar{\xi} \in H(\bar{x}, \bar{z})$. Therefore, the assertion of Step I is valid.

Step II. We claim that for every finite subset of E, there exists $\bar{x} \in \text{co}E$ such that

$$\langle A\xi, \eta(y,\bar{x})\rangle + f(y,\bar{x}) - \alpha(\bar{x}, y) \not\leq_{\text{int } P(\bar{x})} 0, \ \forall y \in \text{coE}, z \in V(y), \ \xi \in H(y,z).$$

Indeed, the assertion follows imediately from Step I and Lemma 2.7.

Step III. We claim that there exists $x^\star \in K$ such that

$$\langle A\xi, \eta(y,x^\star)\rangle + f(y,x^\star) - \alpha(x^\star, y) \not\leq_{\text{int } P(x^\star)} 0, \ \forall y \in K, \ z \in V(y), \ \xi \in H(y,z).$$

Indeed, since X is reflexive and K is nonempty, bounded, closed and convex subset of X, so K is compact with respect to the weak topology of X. Let $\mathcal{F}$ be the family of all finite subsets of K. For each $E \in \mathcal{F}$, consider the following set:

$$M_E = \{x \in K : \langle A\xi, \eta(y,x)\rangle + f(y,x) - \alpha(x,y) \not\leq_{\text{int } P(x)} 0, \ \forall y \in \text{coE}, \ z \in V(y), \ \xi \in H(y,z)\}.$$

From Step II, one has $M_E \neq \emptyset$ for each $E \in \mathcal{F}$. We shall prove that $\bigcap_{E\in\mathcal{F}} \overline{M_E}^w \neq \emptyset$, where $\overline{M_E}^w$ denotes the closure of E with respect to the weak topology of X. For this, it suffices to show that the family $\{\overline{M_E}^w\}_{E\in\mathcal{F}}$ has the finite intersection property. Let $E, F \in \mathcal{F}$ and set $G = E \cup F \in \mathcal{F}$. Then $M_G \subseteq M_E \cap M_F$ and it follows that $\overline{M_E}^w \cap \overline{M_F}^w \neq \emptyset$. This shows that the family $\{\overline{M_E}^w\}_{E\in\mathcal{F}}$ has the finite intersection property. Since K is compact with respect to weak topology of X, it follows that $\bigcap_{E\in\mathcal{F}} \overline{M_E}^w \neq \emptyset$. Let $x^\star \in \bigcap_{E\in\mathcal{F}} \overline{M_E}^w$ and for an arbitrary $y \in K$ fixed, consider $F = \{y, x^\star\}$. Since $x^\star \in \overline{M_F}^w$, there exists $\{x_n\} \subseteq \overline{M_F}^w$ such that $\{x_n\} \subseteq K$, $x_n \longrightarrow x^\star$ and for each n

$$\langle A\xi, \eta(v,x_n)\rangle + f(v,x_n) - \alpha(x_n, v) \not\leq_{\text{int } P(x_n)} 0, \ \forall v \in \text{coE}, z \in V(v), \ \xi \in H(v,z).$$

In particular, whenever $v = y$, one derive for each n

$$\langle A\xi, \eta(y,x_n)\rangle + f(y,x_n) - \alpha(x_n, y) \not\leq_{\text{int } P(x_n)} 0, \ \forall z \in V(y), \ \xi \in H(y,z).$$

$$\langle A\xi, \eta(y,x_n)\rangle + f(y,x_n) - \alpha(x_n, y) \notin -\text{int } P(x_n), \ \forall z \in V(y), \ \xi \in H(y,z).$$

Since for each fixed $y \in K$, $\eta(y,.)$ and $f(y,.)$ are completelty continuous and for each fixed $y \in K$, $\alpha(.,y)$ is lower semicontinuous, we conclude that for each $y \in K$, $z \in V(y)$ and $\xi \in H(y,z)$ fixed,

$$\langle A\xi, \eta(y,x_n)\rangle + f(y,x_n) - \alpha(x_n, y) \longrightarrow \langle A\xi, \eta(y,x^\star)\rangle + f(y,x^\star) - \alpha(x^\star, y) \text{ as } n \longrightarrow \infty.$$

Since $Y\backslash(-\text{int } P(x_n))$ is closed,

$$\langle A\xi, \eta(y,x^\star)\rangle + f(y,x^\star) - \alpha(x^\star, y) \in Y\backslash(-\text{int } P(x^\star)), \ \forall y \in K, \ z \in V(y), \ \xi \in H(y,z)$$

Thus, the assertion of Step III is proved.

Step IV. We claim that there exists $x^\star \in K, z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi, \eta(y, x^\star)\rangle + f(y, x^\star) \nleq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Indeed, the assertion follows immediately from Step III and Lemma 3.3. This completes the proof. □

If the boundedness of K is dropped off, then we have the following theorem under certain coercivity condition:

Theorem 3.5. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X, Y) \longrightarrow L(X, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and completely continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(X, Y)$;*
- (iii) *for each $(\xi, y) \in L(X, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow X$ is completely continuous;*
- (v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous;*
- (vi) *there exists some $r > 0$ such that* **(a)** *$H : K_r \times Y \longrightarrow 2^{L(X,Y)}$, $V : K_r \longrightarrow 2^Y$ are two upper semicontinuous with compact convex values where $K_r = \{x \in K : \|x\| \leq r\}$, and*

(**b**) $\langle A\xi, \eta(0, x)\rangle + f(0, x) \leq_{\text{int}\, P(x)} 0, \ \forall z \in V(x), \ \xi \in H(x, z)$ and $x \in K$ with $\|x\| = r$. (3.9)

Suppose additionally that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K, \ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \nleq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Proof. One can readly see that all conditions of Theorem 3.4 are fulfilled for a nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. Thus according to Theorem 3.4, there exist $x_r \in K_r, z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A\xi_r, \eta(v, x_r)\rangle + f(v, x_r) \nleq_{\text{int}\, P(x_r)} 0, \ \forall v \in K_r. \tag{3.10}$$

Putting $v = 0$ in the above inclusion, one has

$$\langle A\xi_r, \eta(0, x_r)\rangle + f(0, x_r) \nleq_{\text{int}\, P(x_r)} 0. \tag{3.11}$$

Combining (3.9) with (3.11), we know that $\|x_r\| < r$. For any $y \in K$, choose $t \in (0, 1)$ small enough such that $(1 - t)x_r + ty \in K_r$. Putting $v = (1 - t)x_r + ty$ in (3.10), one has

$$\langle A\xi_r, \eta((1 - t)x_r + ty, x_r)\rangle + f((1 - t)x_r + ty, x_r) \nleq_{\text{int}\, P(x_r)} 0.$$

Since the mappings $f(., x_r)$ and $\eta(., x_r)$ are affine, we have

$$\langle A\xi_r, \eta((1 - t)x_r + ty, x_r)\rangle + f((1 - t)x_r + ty, x_r) = t[\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r)].$$

Consequently, we have

$$\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r) \nleq_{\text{int}\, P(x_r)} 0, \ \forall y \in K.$$

This completes the proof. ☐

If $X = \mathbb{R}^n$, then complete continuity is equivalent to continuity. Also bounded and closed subset is equivalent to compact subset. By Theorem 3.4 and Theorem 3.5, we can obtain the following results:

Corollary 3.6. *Let K be a nonempty, compact and convex subset of a real reflexive Banach space $\mathbb{R}^n$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(\mathbb{R}^n, Y) \longrightarrow L(\mathbb{R}^n, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

(i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and continuous in second argument;*
(ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(\mathbb{R}^n, Y)$;*
(iii) *for each $(\xi, y) \in L(\mathbb{R}^n, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
(iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow \mathbb{R}^n$ is continuous;*
(v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous.*

Suppose additionally that $H : K \times Y \longrightarrow 2^{L(\mathbb{R}^n,Y)}$, $V : K \longrightarrow 2^Y$ be two upper semicontinuous mappings with compact values such that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K, \ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \nleq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Corollary 3.7. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space $\mathbb{R}^n$ with $0 \in K$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(\mathbb{R}^n, Y) \longrightarrow L(\mathbb{R}^n, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

(i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and continuous in second argument;*
(ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(\mathbb{R}^n, Y)$;*
(iii) *for each $(\xi, y) \in L(\mathbb{R}^n, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
(iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow \mathbb{R}^n$ is continuous;*
(v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous;*
(vi) *there exists some $r > 0$ such that* **(a)** *$H : K_r \times Y \longrightarrow 2^{L(\mathbb{R}^n,Y)}$, $V : K_r \longrightarrow 2^Y$ are two upper semicontinuous with compact convex values where $K_r = \{x \in K : \|x\| \leq r\}$, and*

$(\mathbf{b})$ $\langle A\xi, \eta(0, x)\rangle + f(0, x) \leq_{\text{int}\, P(x)} 0, \ \forall z \in V(x), \ \xi \in H(x, z)$ and $x \in K$ with $\|x\| = r$.

Suppose additionally that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K,\ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \not\leq_{\mathit{int}\, P(x^\star)} 0,\ \forall y \in K.$$

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