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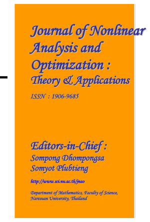

POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

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ABSTRACT. In this paper, we study the existence of positive solution to boundary value problem for fractional differential equation with a one-dimensional p-Laplacian operator

$$\begin{cases} D_{0+}^{\sigma}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and f is a lower semi-continuous function. By using Krasnoselskii's fixed point theorems in a cone, the existence of one positive solution and multiple positive solutions for nonlinear singular boundary value problems is obtained.

KEYWORDS : Cone; Multi point boundary value problem; Fixed point theorem; Riemann-Liouville fractional derivative.
AMS Subject Classification: 34B07 34D05 34L20.

1. INTRODUCTION

The purpose of this paper is to study the existence of positive solutions for the following m-point boundary value problem for fractional differential equation with p-Laplacian

$$\begin{cases} D_{0+}^{\sigma}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.1}$$

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where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $m > 2$ ($m \in \mathbb{N}$), $a, b, c, d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0, +\infty)$ ($i = 1, 2, \ldots, m-2$), $g \in C((0,1); [0, +\infty))$ and $0 < \int_0^1 g(r)dr < \infty$, and f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [5, 8, 9] and the references therein. In [12], Liu, and Jia investigated the existence of multiple solutions for problem:

$$\begin{cases} {}^cD_{0+}^{\sigma}(p(t)u'(t)) + q(t)f(t, u(t)) = 0, \quad t > 0, \ \ 0 < \sigma < 1, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases}$$

where ${}^cD_{0+}^{\sigma}$ is the standard Caputo derivative of order σ. Some existence results were given for the problem (1.1) with $\sigma = 2$ by Yanga et al. [24] and Zhao et al. [25].

The solution of differential equations of fractional order is much involved. Some analytical methods are presented, such as the popular Laplace transform method [20, 21], the Fourier transform method [15], the iteration method [22] and Green function method [14, 23]. Numerical schemes for solving fractional differential equations are introduced, for example, in [3, 4, 17]. Recently, a great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [18], homotopy perturbation method [19], homotopy analysis method [2], differential transform method [16] and variational method [6] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations.
The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [1, 11, 20, 22] and the references therein.

In this paper, We show that the problem (1.1) has positive solutions by using Krasnoselskii's fixed point theorems in a cone.

The paper has been organized as follows. In Sect. 2, we give some preliminary facts and provide basic properties which are needed later. We also state the Krasnoselakii's fixed point theorem. In Sect. 3, we establish the existence of at least one or multiple positive solutions result for problem (1.1). In Section 4 we give an example as application.

2. PRELIMINARIES

In this section, we present some notation and preliminary lemmas that will be used in the proofs of the main results.

We work in the space $C([0, 1])$ with respect to the norm $||u|| = \max_{0\leq t\leq 1} |u(t)|$. For convenience, we make the following assumptions:

(H1) $f \in C([0, +\infty); [0, +\infty))$;

(H1*) f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$, i.e. $\exists I \subset [0, +\infty)$; $\forall x_n \in I$, $x_n \to x_0$ ($n \to \infty$), one has $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$.

Moreover, f has only a finite number of discontinuity points in each compact subinterval of $[0,+\infty)$.

(H2) $g \in C((0,1);[0,+\infty))$ and $0 < \int_0^1 g(r)dr < +\infty$. Moreover, $g(t)$ does not vanish identically on any subinterval of $[0,1]$;

(H3) $a,b,c,d \geq 0$, $\rho = ac+bc+ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)$ ($i = 1,2,\ldots,m-2$), $\rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$ and $\Delta < 0$, where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}$$

and

$$\psi(t) = b + at, \quad \varphi(t) = c + d - ct, \quad t \in [0,1], \tag{2.1}$$

are linearly independent solutions of the equation $x''(t) = 0$, $t \in [0,1]$. Obviously, ψ is non-decreasing on $[0,1]$ and φ is non-increasing on $[0,1]$.

Definition 2.1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.4. *([7]). The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0,1)$.*

Lemma 2.5. *([7]). Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1,\ldots,n$, there $n-1 < \alpha \leq n$.

Lemma 2.6. *([7]). Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1,\ldots,n$, there $n-1 < \alpha \leq n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

Let $y(t) = \phi_p(u''(t))$, then the problem

$$\begin{cases} D_{0+}^{\sigma}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, & t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \end{cases}$$

is turned into problem

$$\begin{cases} D_{0+}^{\sigma}y(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.2}$$

Lemma 2.7. *If (H1) and (H2) hold, then the boundary value problem (2.2) has a unique solution*

$$y(t) = -\int_0^1 H(t,s)g(s)f(u(s))ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le s \le t \le 1, \\ \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

proof. According to Lemma 2.6, we can obtain that

$$y(t) = I_{0+}^{\sigma}(g(t)f(u(t))) - c_1t^{\sigma-1} - c_2t^{\sigma-2} = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - c_1t^{\sigma-1} - c_2t^{\sigma-2}.$$

By the boundary conditions of (2.2), there are $c_2 = 0$ and $c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1(1-s)^{\sigma-1}g(s)f(u(s))ds$.

Thus, the unique solution of problem (2.2) is

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - \frac{t^{\sigma-1}}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds \\ &= -\int_0^t \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds - \int_t^1 \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds \\ &= -\int_0^1 H(t,s)g(s)f(u(s))ds. \end{aligned}$$

□

Lemma 2.8. *If (H3) holds, then for $y \in C[0,1]$, the boundary value problem*

$$\begin{cases} u''(t) = \phi_q(y(t))), & t \in (0,1), \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_iu(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_iu(\xi_i), \end{cases} \tag{2.5}$$

has a unique solution

$$u(t) = -\Big[\int_0^1 G(t,s)\phi_q(y(s))ds + A(\phi_q(y(s)))\psi(t) + B(\phi_q(y(s)))\varphi(t)\Big], \tag{2.6}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}\varphi(t)\psi(s), & 0\le s\le t\le 1,\\ \varphi(s)\psi(t), & 0\le t\le s\le 1,\end{cases} \tag{2.7}$$

$$A(\phi_q(y(s)))=\frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds & \rho-\sum_{i=1}^{m-2}a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds & -\sum_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix} \tag{2.8}$$

$$B(\phi_q(y(s)))=\frac{1}{\Delta}\begin{vmatrix}-\sum_{i=1}^{m-2}a_i\psi(\xi_i) & \sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds\\ \rho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds\end{vmatrix} \tag{2.9}$$

proof. The proof is similar to that of Lemma 5.5.1 in [13], so we omit it here. □
we assume that $\theta\in(0,\frac{1}{2})$. Furthermore, for convenience, we set

$$\Lambda_1=\min\Big\{\frac{\varphi(1-\theta)}{\varphi(0)},\frac{\psi(\theta)}{\psi(1)}\Big\},\qquad \Gamma=\min\{\Lambda_1,\frac{\Lambda_2}{\Lambda_3}\},$$
$$\Lambda_2=\min\{\min_{\theta\le t\le 1-\theta}\varphi(t),\min_{\theta\le t\le 1-\theta}\psi(t),1\},\qquad \Lambda_3=\max\{1,||\varphi||,||\psi||\}.$$

Lemma 2.9. *Let $\rho,\Delta\neq 0$ and $\theta\in(0,\frac{1}{2})$, then we have the following results:*

$$0\le G(t,s)\le G(s,s),\quad \textit{for}\ \ t,s\in[0,1], \tag{2.10}$$

and

$$G(t,s)\ge\Lambda_1 G(s,s),\quad \textit{for}\ \ t\in\Big[\theta,1-\theta\Big]\ \ \textit{and}\ \ s\in[0,1]. \tag{2.11}$$

proof. The inequality (2.10) is obvious. In following, we are going to verify the inequality (2.11). Indeed, when $t\in[\theta,1-\theta]$, $s\in[0,1]$, we have

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \begin{cases}\frac{\varphi(t)}{\varphi(s)}, & 0\le s\le t\le 1-\theta,\\ \frac{\psi(t)}{\psi(s)}, & \theta\le t\le s\le 1,\end{cases}\\ &\ge \begin{cases}\frac{\varphi(1-\theta)}{\varphi(0)}, & 0\le s\le t\le 1-\theta,\\ \frac{\psi(\theta)}{\psi(1)}, & \theta\le t\le s\le 1,\end{cases}\\ &\ge \Lambda_1.\end{aligned}$$

This completes the proof. □

Proposition 2.10. *For $t,s\in[0,1]$, we have*

$$0\le H(t,s)\le H(s,s)\le\frac{1}{\Gamma(\sigma)}\big(\frac{1}{4}\big)^{\sigma-1}.$$

Proposition 2.11. *Let $\theta\in(0,\frac{1}{2})$, then there exists a positive function $\varrho\in C(0,1)$ such that*

$$\min_{\theta\le t\le 1-\theta}H(t,s)\ge\varrho(s)H(s,s),\qquad s\in(0,1).$$

proof. For $\theta\in(0,\frac{1}{2})$, we define

$$g_1(t,s)\ \ =\ \ t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1},\qquad 0\le s\le t\le 1,$$

$$g_2(t,s) \quad = \quad t^{\sigma-1}(1-s)^{\sigma-1} \qquad\qquad 0 \le t \le s \le 1.$$

Then

$$\begin{aligned} \frac{d}{dt}g_1(t,s) &= (\sigma-1)\Big(t^{\sigma-2}(1-s)^{\sigma-1} - (t-s)^{\sigma-2}\Big) \\ &= (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-\frac{s}{t})^{\sigma-2}\Big) \\ &\le (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-s)^{\sigma-2}\Big). \end{aligned}$$

which implies that $g_1(\cdot,s)$ is nonincreasing for all $s \in (0,1]$. Also, we have $g_2(\cdot,s)$ is nondecreasing for all $s \in (0,1)$. Then, we have

$$\begin{aligned} \min_{\theta \le t \le 1-\theta} H(t,s) &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\theta], \\ \min\{\frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, \frac{g_2(\theta,s)}{\Gamma(\sigma)}\}, & s \in [\theta, 1-\theta], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [1-\theta, 1). \end{cases} \\ &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [\mu, 1). \end{cases} \\ &= \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{\theta^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)} & s \in [\mu, 1), \end{cases} \end{aligned}$$

where $\theta < \mu < 1-\theta$ is solution of equation

$$(1-\theta)^{\sigma-1}(1-\mu)^{\sigma-1} - (1-\theta-\mu)^{\sigma-1} = \theta^{\sigma-1}(1-\mu)^{\sigma-1}.$$

It follows from the monotonicity of g_1 and g_2 that

$$\max_{0 \le t \le 1} H(t,s) = H(s,s) = \frac{s^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, \quad s \in (0,1).$$

Therefore, we set

$$\varrho(s) = \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{s^{\sigma-1}(1-s)^{\sigma-1}}, & s \in (0,\mu], \\ \left(\frac{\theta}{s}\right)^{\sigma-1} & s \in [\mu, 1). \end{cases}$$

Thus, we complete the proof. □

From Lemmas 2.7 and 2.8, we know that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \qquad (2.12)$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Lemma 2.12. *Let (H1), (H2) and (H3) hold. Then the solution u of the problem (1.1) satisfies*

(i) $u(t) \ge 0$, *for* $t \in [0,1]$,

and

(ii) $\min_{\theta \le t \le 1-\theta} u(t) \ge \Gamma||u||$.

proof. (i) By Lemma 2.9, Proposition 2.10, (2.3), (2.6)-(2.9) and the property of function ϕ_q it is obvious that we have

$$G(t,s)\geq 0,\quad \phi_q(W(s))\geq 0,\quad A(\phi_q(W(s)))\geq 0,\quad B(\phi_q(W(s)))\geq 0,$$

so we get $u(t)\geq 0$.

(ii) From Lemma 2.9 and (2.12), for $t\in[\theta,1-\theta]$, we have

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)\\
&\geq \Lambda_1\int_0^1 G(s,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)\\
&\geq \Lambda_1\int_0^1 G(s,s)\phi_q(W(s))ds + \frac{\Lambda_2}{\Lambda_3}\cdot\Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]\\
&\geq \Gamma\Big[\int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]\Big]\\
&\geq \Gamma||u||.
\end{aligned}$$

Therefore, we get $\min_{\theta\leq t\leq 1-\theta} u(t)\geq \Gamma||u||$. □

Then, choose a cone K is $C^1([0,1])$, by

$$K=\{u\in C[0,1]\mid u(t)\geq 0, \min_{\theta\leq t\leq 1-\theta} u(t)\geq \Gamma\|u\|\}.$$

Define an operator T by

$$(Tu)(t)=\int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \quad (2.13)$$

where $W(s)=\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

It is clear that the existence of a positive solution for the system (1.1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 2.13. *Suppose that the conditions (H1), (H2) and (H3) hold, then $T(K)\subseteq K$ and $T: K\to K$ is completely continuous.*

proof. For any $u\in K$, by (2.13), we obtain $(Tu)(t)\geq 0$ and, for $t\in[0,1]$,

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)\\
&\leq \int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))].
\end{aligned}$$

Thus, $||Tu||\leq \int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]$.

On the other hand, for $t\in[\theta,1-\theta]$, we have

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)\\
&\geq \Lambda_1\int_0^1 G(s,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)
\end{aligned}$$

$$\begin{aligned}
&\geq \Lambda_1 \int_0^1 G(s,s)\phi_q(W(s))ds + \frac{\Lambda_2}{\Lambda_3}\cdot\Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))] \\
&\geq \Gamma\Big[\int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]\Big] \\
&\geq \Gamma||Tu||.
\end{aligned}$$

Therefore, we get $TK \subseteq K$

By conventional arguments and Ascoli-Arzela theorem, one can prove $T: K \to K$ is completely continuous, so we omit it here. □

Our approach is based on the following Guo-Krasnoselskii fixed point theorem of cone expansion-compression type [10].

Theorem 2.14. *Let E be a Banach space and $K \subseteqq E$ a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T: K\bigcap(\overline{\Omega}_2\backslash\Omega_1) \to K$ be a completely continuous operator. In addition suppose either*
(A) $\|Tu\| \leq \|u\|, \forall u \in K\cap\partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in K\cap\partial\Omega_2$ or
(B) $\|Tu\| \geq \|u\|, \forall u \in K\cap\partial\Omega_1$ and $\|Tu\| \leq \|u\|, \forall u \in K\cap\partial\Omega_2$
holds. Then T has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. MAIN RESULTS

We define $\Omega_l = \{u \in K : ||u|| < l\}$, $\partial\Omega_l = \{u \in K : ||u|| = l\}$, where $l > 0$.

If $u \in \partial\Omega_l$, for $t \in [\theta, 1-\theta]$, we have $\Gamma l \leq u \leq l$.

For convenience, we introduce the following notations. Let

$$\begin{aligned}
f_l &= \inf\Big\{\frac{f(u)}{\phi_p(l)}\Big| u \in [\Gamma l, l]\Big\}, \qquad f^l = \sup\Big\{\frac{f(u)}{\phi_p(l)}\Big| u \in [0, l]\Big\}, \\
f_\varrho &= \liminf_{u\to\varrho}\frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty), \\
f^\varrho &= \limsup_{u\to\varrho}\frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty), \\
\eta &= \min_{\theta\leq s\leq 1-\theta}\varrho(s), \\
\frac{1}{\omega} &= \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\Big[\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big], \\
\frac{1}{M} &= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\Big].
\end{aligned}$$

We always assume that (H1) hold in the following theorems.

Theorem 3.1. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions*

(H4) $f^r \leq \phi_p(\omega)$,

and

(H5) $f_R \geq \phi_p(M)$,

hold. Then the problem (1.1) has at least one positive solution $u \in K$ such that

$$0 < r \leq ||u|| \leq R.$$

proof. Case 1. We shall prove that the result holds when (H1) is satisfied. Without loss of generality, we suppose that $r < \Gamma R$ for $r < R$.

By (H4), Proposition 2.10, (2.8) and (2.9), for $u \in \Omega_r$, we have

$$\begin{aligned} A(\phi_q(W)) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}, \\ &= \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\widetilde{A}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} B(\phi_q(W)) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds\end{vmatrix} \\ &= \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\widetilde{B}. \end{aligned} \tag{3.2}$$

Therefore, by (H4), Lemma 2.9, (2.13), (3.1) and (3.2), for $t\in[0,1]$ and $u\in\Omega_r$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\leq \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big) \\ &\quad +\Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\widetilde{A}\psi(t) + \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\widetilde{B}\varphi(t) \\ &\leq \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\omega r\Big[\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)+\Lambda_3\widetilde{A}+\Lambda_3\widetilde{B}\Big] \\ &= r = ||u||. \end{aligned}$$

This implies that $||Tu||\leq||u||$ for $u\in\Omega_r$.

on the other hand, by (H5), (2.13), Proposition 2.11, (2.8) and (2.9), for $u\in\Omega_R$, we have

$$\begin{aligned} A(\phi_q(W)) &\geq \frac{\big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}, \\ &= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{A}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} B(\phi_q(W)) &\geq \frac{\big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds\end{vmatrix} \\ &= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{B}. \end{aligned} \tag{3.4}$$

Therefore, by (H5), Lemma 2.9, (2.13), (3.3) and (3.4), for $t\in[0,1]$ and $u\in\Omega_R$, we have

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)$$

$$\begin{aligned}
&\geq \left(\frac{\eta}{\Gamma(\sigma)}\right)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)+\Lambda_2\widehat{A}+\Lambda_2\widehat{B}\Big]\\
&= R=||u||.
\end{aligned}$$

This implies that $||Tu|| \geq ||u||$ for $u \in \Omega_R$.

Therefore, by Theorem 2.14, it follows that T has a fixed-point u in $K\cap(\overline{\Omega_R}\setminus\Omega_r)$. This means that the problem (1.1) has at least one positive solution $u \in K$ such that $0 < r \leq ||u|| \leq R$.

Case 2. When (H1*) holds, by applying the linear approaching method on the domain of discontinuous points of f we can establish sequence $\{f_j\}_{j=1}^{\infty}$ satisfying the following two conditions

(i) $f_j \in C[0,\infty)$ and $0 \leq f_j \leq f_{j+1}$ on $[0,\infty)$,
and
(ii) $\lim_{j\to\infty} f_j = f$, $j = 1, 2, \dots$, is pointwisely convergent on $[0,\infty)$.

By virtue of proof of Case 1, we know that when $f = f_j$, the problem (1.1) has a positive solution $u_j(t)$ where

$$\begin{aligned}
u_j(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds\\
&+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2}a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix}\\
&+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}a_i\psi(\xi_i) & \sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds\\ \rho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds\end{vmatrix}\\
&= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds+\psi(t)A_j+\varphi(t)B_j,
\end{aligned}$$

for all $t \in [0,1]$ and $r \leq ||u_j|| \leq R$, r, R are independent of j.

By uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$,$\varphi(t)$ and $\psi(t)$ on $[0,1]$, for any $\epsilon > 0$ (small enough), there exists $\delta > 0$ such that for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, one has $|G(t_1,s) - G(t_2,s)| < \epsilon$, $|\varphi(t_1) - \varphi(t_2)| < \epsilon$ and $|\psi(t_1) - \psi(t_2)| < \epsilon$. Thus, for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, one has

$$\begin{aligned}
|u_j(t_1)-u_j(t_2)| &\leq \int_0^1 |G(t_1,s)-G(t_2,s)|\cdot\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds\\
&\quad +A_j|\psi(t_1)-\psi(t_2)|+B_j|\varphi(t_1)-\varphi(t_2)|\\
&\leq \left(\frac{1}{\Gamma(\sigma)}\right)^{q-1}\left(\frac{1}{4}\right)^{(\sigma-1)(q-1)}\cdot\max_{||u_j||\leq R} f_j(u_j)\cdot\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)\cdot\epsilon+A_j\cdot\epsilon+B_j\cdot\epsilon.
\end{aligned}$$

So we get that $\{u_j\}_{j=1}^{\infty}$ are equicontinuous on $[0,1]$. Thus, by the Arzela-Asoli theorem, we know that there exists a convergent subsequence of $\{u_j\}_{j=1}^{\infty}$. For convenience, we denote this convergent subsequence with $\{u_j\}_{j=1}^{\infty}$. Without loss of generality, we suppose $\lim_{j\to\infty} u_j(t) = u(t)$, $\forall t \in [0,1]$, and $r \leq ||u|| \leq R$. By the Fatou's Lemma and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{j\to\infty} u_j(t)\\
&\geq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau\Big)ds
\end{aligned}$$

$$+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=0.5}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}$$

$$+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=0.5}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds \end{vmatrix}$$

i.e.

$$u(t) \geq \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \tag{3.5}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$. On the other hand, by the conditions (i) and (ii), we have

$$\begin{aligned} u_j(t) \;\; \leq \;\; & \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau\Big)ds \\ & +\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} \\ & +\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds \end{vmatrix}, \end{aligned}$$

By the lower semi-continuity of f, taking limits in above inequality as $j \to \infty$, we have

$$\begin{aligned} u(t) \;\; \leq \;\; & \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds \\ & +\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} \\ & +\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds \end{vmatrix}, \end{aligned}$$

i.e

$$u(t) \leq \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \tag{3.6}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

By (3.5) and (3.6), we have

$$u(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t),$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Therefore $u(t)$ is a positive solution of the problem (1.1). This completes the proof of Theorem 3.1. □

Similarly, we can obtain the following conclusion.

Theorem 3.2. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions*

(H4)* $f^r < \phi_p(\omega)$,
and
(H5)* $f_R > \phi_p(M)$,
hold. Then the problem (1.1) has at least one positive solution $u \in K$ *such that*

$$0 < r < ||u|| < R.$$

Theorem 3.3. *Assume that one of the following two conditions*
(H6) $f^0 \leq \phi_p(\omega)$, $\quad f_\infty \geq \phi_p(\frac{M}{\Gamma})$,
and
(H7) $f_0 \geq \phi_p(\frac{M}{\Gamma})$, $\quad f^\infty \leq \phi_p(\omega)$
is satisfied. Then the problem (1.1) has at least one positive solution.

proof. We need to do is to prove that the results of Theorem 3.3 hold when f is nonnegative and continuous on $[0, \infty)$. And by the similar proof process of Theorem 3.1 we can prove the results of Theorem 3.3 when f is nonnegative and lower semi-continuous on $[0, \infty)$.

We show that (H6) implies (H4) and (H5). Suppose that (H6) holds, then there exist r and R with $0 < r < \Gamma R$, such that

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad 0 < u \leq r$$

and

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\Gamma}), \quad u \geq \Gamma R.$$

Hence, we obtain

$$f(u) \leq \phi_p(\omega)\phi_p(u) \leq \phi_p(\omega)\phi_p(r) = \phi_p(r\omega), \quad 0 < u \leq r$$

and

$$f(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(\Gamma R) = \phi_p(MR), \quad u \geq \Gamma R.$$

Thus, (H4) and (H5) holds.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution.

Now suppose that (H7) holds, then there exist $0 < r < R$ with $Mr < \omega R$ such that

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\Gamma}), \quad 0 < u \leq r. \tag{3.7}$$

and

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad u \geq R. \tag{3.8}$$

By (3.7), it follows that

$$f(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(\Gamma r) = \phi_p(Mr), \quad \Gamma r \leq u \leq r.$$

So, the condition (H5) holds for r.

For (3.8), we consider two cases.

(i) If $f(u)$ is bounded, there exists a constant $D > 0$ such that $f(u) \leq D$, for $0 \leq u < \infty$. By (3.8), there exists a constant $\lambda \geq R$ with $Mr < \omega R \leq \lambda\omega$ satisfying $\phi_p(\lambda) \geq \max\{\phi_p(R), \frac{D}{\phi_p(\omega)}\}$ such that $f(u) \leq D \leq \phi_p(\lambda\omega)$ for $0 \leq u \leq \lambda$. This means that the condition (H4) holds for λ.

(ii) If $f(u)$ is unbounded, there exist $\lambda_1 \geq R$ with $Mr < \omega R \leq \lambda_1\omega$ such that $f(u) \leq f(\lambda_1)$ for $0 \leq u \leq \lambda_1$. This yields $f(u) \leq f(\lambda_1) \leq \phi_p(\lambda_1\omega)$ for $0 \leq u \leq \lambda_1$. Thus, condition (H4) holds for λ_1.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution. Theorem 3.3 is proved. □

Remark 3.4. It is obvious that Theorem 3.3 holds if f satisfies conditions $f^0 = 0$, $f_\infty = +\infty$ or $f_0 = +\infty$, $f^\infty = 0$.

In this section, we give some conclusions about the existence of multiple positive solutions. We always suppose that (H1*), (H2) and (H3) hold in the following theorems.

Theorem 3.5. *Assume that one of the following two conditions*

(H8) $f^r < \phi_p(\omega)$,

and

(H9) $f_0 \geq \phi_p(\frac{M}{\Gamma})$, $\qquad f_\infty \geq \phi_p(\frac{M}{\Gamma})$

are satisfied. Then the problem (1.1) has at least two positive solutions such that

$$0 < ||u_1|| < r < ||u_2||.$$

proof. By the proof of Theorem 3.3, we can take $0 < r_1 < r < \Gamma r_2$ such that $f(u) \geq \phi_p(r_1 M)$ for $\Gamma r_1 \leq u \leq r_1$ and $f(u) \geq \phi_p(r_2 M)$ for $\Gamma r_2 \leq u \leq r_2$. Therefore, by Theorems 3.2 and 3.3, it follows that problem (1.1) has at least two positive solutions such that $0 < ||u_1|| < r < ||u_2||$. ■

Theorem 3.6. *Assume that one of the following two conditions*

(H10) $f_R > \phi_p(M)$,

and

(H11) $f^0 \leq \phi_p(\omega)$, $\qquad f^\infty \leq \phi_p(\omega)$,

are satisfied. Then the problem (1.1) has at least two positive solutions such that

$$0 < ||u_1|| < R < ||u_2||.$$

Theorem 3.7. *Assume (H6) (or (H7)) holds, and there exist constants $r_1, r_2 > 0$ with $r_1 M < r_2\omega$ (or $r_1 < \Gamma r_2$) such that (H8) holds for $r = r_2$ (or $r = r_1$) and (H10) holds for $R = r_1$ (or $R = r_2$). Then the problem (1.1) has at least three positive solutions such that*

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < ||u_3||.$$

The proofs of Theorems 3.6 and 3.7 are similar to that of Theorem 3.5, so we omit it here.

Theorem 3.8. *Let $n = 2k + 1$, $k \in \mathbb{N}$. Assume (H6) (or (H7)) holds. If there exist constants $r_1, r_2, \ldots, r_{n-1} > 0$ with $r_{2i} < \Gamma r_{2i+1}$, for $1 \leq i \leq k-1$ and $r_{2i-1}M < r_{2i}\omega$ for $1 \leq i \leq k$ (or with $r_{2i-1} < \Gamma r_{2i}$, for $1 \leq i \leq k$ and $r_{2i}M < r_{2i+1}\omega$ for $1 \leq i \leq k-1$)*

such that (H10) (or (H8)) holds for r_{2i-1}, $1 \leq i \leq k$ and (H8) (or (H10)) holds for r_{2i}, $1 \leq i \leq k$. Then the problem (1.1) has at least n positive solutions $u_1, \ldots, u_n$ such that

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < \cdots < ||u_{n-1}|| < r_{n-1} < ||u_n||.$$

4. APPLICATION

Example 4.1. Consider the following singular boundary value problems with a p-Laplacian operator

$$\begin{cases} D_{0+}^{\frac{3}{2}}(\phi_p(u''(t))) - t^{-\frac{1}{2}} f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ u(0) - u'(0) = \frac{1}{2} u(\frac{1}{2}), \\ u(1) + u'(1) = \frac{1}{2} u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $p = \frac{3}{2}$,

$$f(u) = \begin{cases} e^{-u}, & 0 \leq u \leq 10, \\ (n+1)e^{-u}, & n < u \leq n+1, \quad n = 10, 11, \ldots, 20, \\ e^{\sqrt{u}}, & u > 21. \end{cases}$$

We note that

$$a = b = c = d = 1, \quad \rho = 3, \quad q = 3, \quad m = 3, \quad \xi_1 = \frac{1}{2}, \quad \sigma = \frac{3}{2},$$
$$a_1 = b_1 = \frac{1}{2}, \quad f_0 = +\infty, \quad f_\infty = +\infty, \quad \Delta = -\frac{9}{2}, \quad g(t) = t^{-\frac{1}{2}}.$$

Let $\theta = \frac{1}{3}$, then

$$\Lambda_1 = \frac{2}{3}, \quad \Lambda_2 = 1, \quad \Lambda_3 = 2, \quad \Gamma = \frac{1}{2},$$
$$\omega = \frac{9\pi}{131}, \quad M = \frac{729\pi}{944(3 - 2\sqrt{2})\eta^2},$$

where $\eta = \min_{\frac{1}{3} \leq s \leq \frac{2}{3}} \varrho(s)$.

By calculating, we can let $\mu = \frac{2\sqrt{2}-1}{2\sqrt{2}}$. So, $f_\infty > \phi_p(\frac{M}{\Gamma})$ and $f_0 > \phi_p(\frac{M}{\Gamma})$. We choose $r = 10$, then

$$f^r = \sup\left\{ \frac{f(u)}{\phi_p(r)} \middle| u \in [0, r] \right\} = 0.316227 < 0.464462 = \phi_p(\omega).$$

Thus, (H8) and (H9) hold. Obviously, (H1*), (H2) and (H3) hold. By Theorem 3.5, the problem (4.1) has at least two positive solutions $u_1, u_2 \in K$ such that $0 < ||u_1|| < 4 < ||u_2||$.

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References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta.

[2] J. Cang, Y. Tan, H. Xu, S. J. Liao, Series solutions of non-linear Riccati differential equations with fractional order, Chaos Solitons Fractals 40(1)(2009) 1-9.

[3] K. Diethelm, N. Ford, A. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynam. 29(2002) 3-22.

[4] K. Diethelm, N. Ford, A. Freed, Detailed error analysis for a fractional Adams Method, Numer. Algorithms. 36(2004) 31-52.

[5] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal. 33(1998) 181-186.

[6] J. Feng, Z. Yong, Existence of solutions for a class of fractional boundary value problems via critical point theory, Comput. Math. Appl. 62(2011) 1181-1199.

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier. Amsterdam. (2006).

[8] A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: Methods, results and problems I, Appl. Anal. 78(2001) 153-192.

[9] A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: Methods, results and problems II, Appl. Anal. 81 (2002) 435-493.

[10] M.A. Krasnoselskii, Positive solutions of operator equations, Noordhoff. Groningen. Netherlands. 1964.

[11] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. TMA 69(8)(2008) 2677-2682.

[12] X. Liu, M. Jia, Multiple solutions of nonlocal boundary value problems for fractional differential equations on the half-line, Elect. J. Qual. Differen. Equat. 56(2011) 1-14.

[13] R.Y. Ma, Nonlocal Problems for the Nonlinear Ordinary Differential Equation, Science Press. Beijing. 2004 (in Chinese).

[14] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, Fract. Calc. Appl. Anal. 4(2001) 153-192.

[15] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley. New York. 1993.

[16] S. Momani, Z. Odibat, A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula, J. Comput. Appl. Math. 220(1-2)(2008) 85-95.

[17] Z. Odibat, S. Momani, An algorithm for the numerical solution of differential equations of fractional order, J. Appl. Math. Inform. 26(1-2)(2008) 15-27.

[18] Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Modelling. 32(12)(2008) 28-39.

[19] Z.Odibat, S. Momani, Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos Solitons Fractals 36(1)(2008) 167-174.

[20] I. Podlubny, Fractional Differential Equations, Academic Press, New York. (1999).

[21] I. Podlubny, The Laplace Transform Method for Linear Differential Equations of Fractional Order, Slovac Academy of Science. Slovak Republic. (1994).

[22] G. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach. Amsterdam. (1993).

[23] W. Schneider, W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 134-144.

[24] J. Yang, Z. Wei, Existence of positive solutions for fourth-order m-point boundary value problems with a one-dimensional p-Laplacian operator, Nonlinear Anal. 71(2009) 2985-2996.

[25] J. Zhaoa, L. Wang, W. Ge, Necessary and sufficient conditions for the existence of positive solutions of fourth order multi-point boundary value problems, Nonlinear Anal. 72(2010) 822-835.