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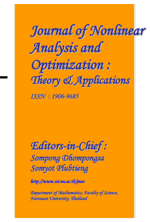

FIXED POINT THEOREMS IN MENGER SPACES USING THE (CLR_{ST}) PROPERTY AND APPLICATIONS

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ABSTRACT. In the present paper, we prove common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces employing the (CLR_{ST}) property. Some examples are furnished which demonstrate the validity of the hypotheses and degree of generality of our results. We extend our main result to four finite families of self mappings. As applications to our results, we obtain the corresponding common fixed point theorems in metric spaces. Our results improve and extend the results of Cho et al. [4] and Pathak et al. [21] besides several known results.

KEYWORDS : t-norm; Menger space; Weakly compatible mappings; (E.A) property; Common property (E.A); (CLR_S) property; (CLR_{ST}) property; Fixed point.
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1. INTRODUCTION

In 1942, Karl Menger [15] introduced the notion of a probabilistic metric space (shortly, PM-space). The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of the distances. In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [24]. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis especially due to its extensive applications in random differential as well as random integral equations.

In 1991, Mishra [17] extended the notion of compatibility (introduced by Jungck [8] in metric spaces) to PM-space. Cho et al. [4] studied the notion of compatible mappings of type (A) (introduced by Jungck et al. [9] in metric spaces) in Menger

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spaces which is equivalent to the concept of compatible mappings under some conditions. Further, Pathak et al. [21] improved and generalized the results of Cho et al. [4] by introducing the notion of weak compatible mappings of type (A) in Menger spaces. The fixed point theorems for contraction mappings in Menger spaces were obtained by many mathematicians (e.g. [16, 18, 20, 23]).

It is seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. However, the study of common fixed points of non-compatible mappings is also equally interesting which was initiated by Pant [19] in metric spaces. In 2002, Aamri and El Moutawakil [1] defined the (E.A) property for self mappings whose class contains the class of non-compatible as well as compatible mappings. Kubiaczyk and Sharma [11] studied the common fixed points of weakly compatible mappings satisfying the property (E.A) in PM-spaces and used it to prove results on common fixed points. In the recent years, there are a number of results via (E.A) property in PM-spaces (e.g. [5, 7, 12, 13]). Inspired by Liu et al. [14], Ali et al. [2] (also, see [3]) defined the common property (E.A) for the existence of a common fixed point in Menger spaces and generalized several known results in Menger spaces as well as metric spaces. It is observed that (E.A) property and common property (E.A) require the closedness of the subspaces for the existence of fixed point. Recently, Sintunavarat and Kumam [27] coined the idea of "common limit in the range property" which never requires the closedness of the subspaces for the existence of fixed point (also see [26, 28]).

The aim of this paper is to prove common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces employing the (CLR_{ST}) property. Illustrative examples are also furnished to support our results. We extend our main result to four finite families of mappings using the notion of pairwise commuting property of two finite families of mappings due to Imdad et al. [6]. As applications to our results, we obtain the corresponding common fixed point theorems in metric spaces.

2. PRELIMINARIES

Definition 2.1. [24] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We denote by $\Im$ the set of all distribution functions while H always denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \le 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. [24] A PM-space is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;
(ii) $F_{x,y}(0) = 0$;
(iii) $F_{x,y}(t) = F_{y,x}(t)$;
(iv) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3. [24] A mapping $\triangle : [0, 1]\times[0, 1] \to [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0, 1]$

(i) $\triangle(a, 1) = a$ for all $a \in [0, 1]$;
(ii) $\triangle(a, b) = \triangle(b, a)$;
(iii) $\triangle(a, b) \leq \triangle(c, d)$ for $a \leq c$, $b \leq d$;
(iv) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$;

Examples of t-norms are $\triangle(a, b) = \min\{a, b\}$, $\triangle(a, b) = ab$ and $\triangle(a, b) = \max\{a + b - 1, 0\}$.

Definition 2.4. [24] A Menger space is a triplet $(X, \mathcal{F}, \triangle)$ where $(X, \mathcal{F})$ is a PM-space and t-norm $\triangle$ is such that the inequality

$$F_{x,z}(t + s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X, d) can be realized as a Menger space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5. [17] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible if and only if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 2.6. [4] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible of type (A) if $F_{SAx_n,AAx_n}(t) \to 1$ and $F_{ASx_n,SSx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Remark 2.7. [4] If self mappings A and S are both continuous, then A and S are compatible if and only if they are compatible of type (A).

It is noted that Remark 2.7 is not true if self mappings A and S are not continuous on X. For examples, we refer to Jungck et al. [9].

Definition 2.8. [21] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be weak compatible of type (A) if

$$\lim_{n\to\infty} F_{ASx_n,SSx_n}(t) \geq \lim_{n\to\infty} F_{SAx_n,SSx_n}(t)$$

and

$$\lim_{n\to\infty} F_{SAx_n,AAx_n}(t) \geq \lim_{n\to\infty} F_{ASx_n,AAx_n}(t),$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Remark 2.9. [21] If self mappings A and S are both continuous. Then

(i) A and S are compatible of type (A) if and only if they are weak compatible of type (A).
(ii) A and S are compatible if and only if they are weak compatible of type (A).

It is noted that Remark 2.9 is not true if self mappings A and S are not continuous on X. For examples, we refer to Pathak et al. [21].

Definition 2.10. [10] Two self mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ (for $z \in X$), then $ASz = SAz$.

Remark 2.11. Two compatible self mappings are weakly compatible, but the converse is not true (see [25, Example 1]). Therefore the concept of weak compatibility is more general than that of compatibility.

Definition 2.12. [11] A pair (A, S) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ is said to satisfy the (E.A) property, if there exists a sequence $\{x_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Here, it can be pointed out that weak compatibility and the (E.A) property are independent to each other (see [22, Example 2.2]).

Remark 2.13. From Definition 2.5, it is inferred that two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, $\lim_{n\to\infty} F_{ASx_n, SAx_n}(t)$ is either less than 1 or nonexistent.

Therefore, from Definition 2.12, it is straight forward to notice that every pair of non-compatible self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ satisfies the (E.A) property but not conversely (see [5, Example 1]).

Definition 2.14. [2] Two pairs (A, S) and (B, T) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ are said to satisfy the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some z in X.

On the lines of Sintunavarat and Kumam [27], we define the (CLR_S) property (with respect to mapping S) in Menger space as follows:

Definition 2.15. A pair (A, S) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ is said to satisfy the (CLR_S) property with respect to mapping S if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$.

Now, we present examples of self mappings A and S satisfying the (CLR_S) property.

Example 2.16. Let $(X, \mathcal{F}, \triangle)$ be a Menger space with $X = [0, \infty)$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Define self mappings A and S on X by $A(x) = x + 2$ and $S(x) = 3x$ for all $x \in X$. Let a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 3 = S(1),$$

that is, $3 \in S(X)$, which shows that A and S satisfy the (CLR_S) property.

Example 2.17. The conclusion of Example 2.16 remains true if the self mappings A and S are defined on X by $A(x) = \frac{x}{2}$ and $S(x) = \frac{2x}{3}$ for all $x \in X$. Let a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0 = S(0),$$

that is, $0 \in S(X)$, hence A and S satisfy the (CLR_S) property.

Remark 2.18. From the Examples 2.16-2.17, it is evident that a pair (A, S) satisfying the (E.A) property along with closedness of the subspace $S(X)$ always enjoys the (CLR_S) property.

With a view to extend the (CLR_S) property to two pair of self mappings, we define the (CLR_{ST}) property (with respect to mappings S and T) as follows.

Definition 2.19. Two pairs (A, S) and (B, T) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ are said to satisfy the (CLR_{ST}) property (with respect to mappings S and T) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Lemma 2.20. [17] *Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $\triangle$ is a continuous t-norm. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3. RESULTS

In 1992, Cho et al. [4] proved the following fixed point theorem for compatible mappings of type (A) in Menger space.

Theorem 3.1. [4, Theorem 4.2] *Let $(X, \mathcal{F}, \triangle)$ be a complete Menger space with $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and A, B, S and T be mappings from X into itself such that*

(i) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*
(ii) *the pairs (A, S) and (B, T) are compatible of type (A),*
(iii) *one of A, B, S and T is continuous,*
(iv) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{Ax,By}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,By}(t), F_{Sx,By}(2t)F_{Ty,Ax}(t), \\ F_{Ty,Ax}(t), F_{Sx,By}(2t)F_{Ty,By}(t) \end{array} \right\}, \tag{3.1}$$

for all $x, y \in X$ and $t \geq 0$. Then A, B, S and T have a unique common fixed point in X.

Further, Pathak et al. [21] improved and generalized the results of Cho et al. [4] by using the notion of weak compatible mappings of type (A) which is more general than compatible mappings of type (A).

The attempted improvements in this paper are four fold.

(i) The condition on containment of ranges amongst the involved mappings are relaxed.
(ii) Continuity requirements of all the involved mappings are completely relaxed.
(iii) The mappings of compatible of type (A) or weak compatible of type (A) are replaced by weakly compatible mappings which are more general among all existing weak commutativity concepts.
(iv) The condition on completeness of the whole space is relaxed.

Before proving our main result, we begin with the following observation.

Lemma 3.2. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1. Suppose that*

(i) *the pair* (A, S) *satisfies the* (CLR_S) *property* $\Big($*or the pair* (B, T) *satisfies the* (CLR_T) *property*$\Big)$,

(ii) $A(X) \subset T(X)$ $\Big($*or* $B(X) \subset S(X)\Big)$,

(iii) $T(X)$ $\Big($*or* $S(X)\Big)$ *is a closed subset of* X,

(iv) $B(y_n)$ *converges for every sequence* $\{y_n\}$ *in* X *whenever* $T(y_n)$ *converges* $\Big($*or* $A(x_n)$ *converges for every sequence* $\{x_n\}$ *in* X *whenever* $S(x_n)$ *converges*$\Big)$.

Then the pairs (A, S) *and* (B, T) *share the* (CLR_{ST}) *property.*

Proof Suppose the pair (A, S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{3.2}$$

where $z \in S(X)$. As $A(X) \subset T(X)$ (wherein $T(X)$ is a closed subset of X), for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \tag{3.3}$$

where $z \in S(X) \cap T(X)$. Thus in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. Now we are required to show that $By_n \to z$ as $n \to \infty$. On using inequality (3.1) with $x = x_n$, $y = y_n$, we get

$$(F_{Ax_n,By_n}(kt))^2 \quad \geq \quad \min\left\{\begin{array}{c} (F_{Sx_n,Ty_n}(t))^2, F_{Sx_n,Ax_n}(t)F_{Ty_n,By_n}(t), \\ F_{Sx_n,By_n}(2t)F_{Ty_n,Ax_n}(t), F_{Ty_n,A_n}(t), \\ F_{S_n,By_n}(2t)F_{Ty_n,By_n}(t) \end{array}\right\}.$$

Let $By_n \to l(\neq z)$ for $t > 0$ as $n \to \infty$. Then, passing to limit as $n \to \infty$, we get

$$\begin{aligned} (F_{z,l}(kt))^2 \quad &\geq \quad \min\left\{\begin{array}{c} (F_{z,z}(t))^2, F_{z,z}(t)F_{z,l}(t), F_{z,l}(2t)F_{z,z}(t), \\ F_{z,z}(t), F_{z,l}(2t)F_{z,l}(t) \end{array}\right\} \\ &= \quad (F_{z,l}(t))^2. \end{aligned}$$

Owing to Lemma 2.20, we have $z = l$ which contradicts. Hence the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Remark 3.3. In general, the converse of Lemma 3.2 is not true. For a counter example, one can see Example 3.5.

Now we prove a common fixed point theorem for two pairs of self mappings in Menger space.

Theorem 3.4. *Let* A, B, S *and* T *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1. If the pairs* (A, S) *and* (B, T) *share the* (CLR_{ST}) *property, then* (A, S) *and* (B, T) *have a coincidence point each. Moreover,* A, B, S *and* T *have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof Since the pairs (A, S) and (B, T) share the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. We assert that $Au = Su$. On using inequality (3.1) with $x = u$, $y = y_n$, we get

$$(F_{Au,By_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Su,Ty_n}(t))^2, F_{Su,Au}(t)F_{Ty_n,By_n}(t), \\ F_{Su,By_n}(2t)F_{Ty_n,Au}(t), \\ F_{Ty_n,Au}(t), F_{Su,By_n}(2t)F_{Ty_n,By_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we obtain

$$\begin{aligned} (F_{Au,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,Au}(t)F_{z,z}(t), F_{z,z}(2t)F_{z,Au}(t), \\ F_{z,Au}(t), F_{z,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Au,z}(t))^2. \end{aligned}$$

On employing Lemma 2.20, we have $Au = z$. Therefore $Au = Su = z$ and hence u is a coincidence point of (A, S).

Also $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. We show that $Bv = Tv$. On using inequality (3.1) with $x = u$, $y = v$, we get

$$\begin{aligned} (F_{Au,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Su,Tv}(t))^2, F_{Su,Au}(t)F_{Tv,Bv}(t), F_{Su,Bv}(2t)F_{Tv,Au}(t), \\ F_{Tv,Au}(t), F_{Su,Bv}(2t)F_{Tv,Bv}(t) \end{array} \right\} \\ (F_{z,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,z}(t)F_{z,Bv}(t), F_{z,Bv}(2t)F_{z,z}(t), \\ F_{z,z}(t), F_{z,Bv}(2t)F_{z,Bv}(t) \end{array} \right\} \\ &= (F_{z,Bv}(t))^2. \end{aligned}$$

Appealing to Lemma 2.20, we have $z = Bv$. Therefore $Bv = Tv = z$ and hence v is a coincidence point of (B, T).

Since the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Putting $x = z$ and $y = v$ in inequality (3.1), we have

$$\begin{aligned} (F_{Az,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Sz,Tv}(t))^2, F_{Sz,Az}(t)F_{Tv,Bv}(t), F_{Sz,Bv}(2t)F_{Tv,Az}(t), \\ F_{Tv,Az}(t), F_{Sz,Bv}(2t)F_{Tv,Bv}(t) \end{array} \right\} \\ (F_{Az,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,z}(t), F_{Az,z}(2t)F_{z,Az}(t), \\ F_{z,Az}(t), F_{Az,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Az,z}(t))^2. \end{aligned}$$

In view of Lemma 2.20, we have $Az = z = Sz$ which shows that z is a common fixed point of the pair (A, S). Also the pair (B, T) is weakly compatible, therefore $Bz = BTv = TBv = Tz$. On using inequality (3.1) with $x = u$, $y = z$, we have

$$\begin{aligned} (F_{Au,Bz}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Su,Tz}(t))^2, F_{Su,Au}(t)F_{Tz,Bz}(t), F_{Su,Bz}(2t)F_{Tz,Au}(t), \\ F_{Tz,Au}(t), F_{Su,Bz}(2t)F_{Tz,Bz}(t) \end{array} \right\} \\ (F_{z,Bz}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,Bz}(t))^2, F_{z,z}(t)F_{Bz,Bz}(t), F_{z,Bz}(2t)F_{Bz,z}(t), \\ F_{Bz,z}(t), F_{z,Bz}(2t)F_{Bz,Bz}(t) \end{array} \right\} \\ &= (F_{z,Bz}(t))^2. \end{aligned}$$

Owing to Lemma 2.20, we have $Bz = z = Tz$ which shows that z is a common fixed point of the pair (B, T). Therefore z is a common fixed point of both the pairs (A, S) and (B, T). The uniqueness of common fixed point is an easy consequence of inequality (3.1).

The following example illustrates Theorem 3.4.

Example 3.5. Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $X = [1, 15)$, with t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Define the self mappings A, B, S and T by

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 14, & \text{if } x \in (1, 3]. \end{cases} \quad B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 5, & \text{if } x \in (1, 3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 6, & \text{if } x \in (1, 3]; \\ \frac{x+1}{4}, & \text{if } x \in (3, 15). \end{cases} \quad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 11, & \text{if } x \in (1, 3]; \\ x-2, & \text{if } x \in (3, 15). \end{cases}$$

Taking $\{x_n\} = \left\{3 + \frac{1}{n}\right\}$, $\{y_n\} = \{1\}$ or $\{x_n\} = \{1\}$, $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$, it is clear that both the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X).$$

Then $A(X) = \{1, 14\} \nsubseteq [1, 13) = T(X)$ and $B(X) = \{1, 5\} \nsubseteq [1, 4) \cup \{6\} = S(X)$. Thus, all the conditions of Theorem 3.4 are satisfied for some fixed $k \in (0, 1)$ and 1 is the unique common fixed point of the pairs (A, S) and (B, T). Here, it is worth noting that in this example $S(X)$ and $T(X)$ are not closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

Remark 3.6. Theorem 3.4 improves the results of Cho et al. [4, Theorem 4.2] and Pathak et al. [21, Theorem 4.2] .

Theorem 3.7. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1 and conditions (i)-(iv) of Lemma 3.2. Then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof In view of Lemma 3.2, the pairs (A, S) and (B, T) share the (CLR_{ST}) property, that is, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof can be completed on the lines of the proof of Theorem 3.4, therefore details are omitted.

Example 3.8. In the setting of Example 3.5, replace the self mappings A, B, S and T by the following besides retaining the rest:

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 10, & \text{if } x \in (1, 3]. \end{cases} \quad B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 4, & \text{if } x \in (1, 3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 4, & \text{if } x \in (1, 3]; \\ \frac{x+1}{4}, & \text{if } x \in (3, 15). \end{cases} \quad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 10+x, & \text{if } x \in (1, 3]; \\ x-2, & \text{if } x \in (3, 15). \end{cases}$$

Clearly, both the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X).$$

Notice that $A(X) = \{1, 10\} \subset [1, 13] = T(X)$ and $B(X) = \{1, 4\} \subset [1, 4] = S(X)$. Also the remaining conditions of Theorem 3.7 can be easily verified for some fixed $k \in (0, 1)$ while 1 is the unique common fixed point of the pairs (A, S) and (B, T). Here, it is worth noting that Theorem 3.4 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

By choosing A, B, S and T suitably, we can deduce corollaries for a pair as well as for a triode of self mappings. The details of two possible corollaries for a triode of mappings are not included. As a sample, we obtain the following natural result for a pair of self mappings with an independent proof.

Theorem 3.9. *Let A and S be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Suppose that*

(i) *the pair (A, S) satisfies the (CLR_S) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{Ax,Ay}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Sx,Sy}(t))^2, F_{Sx,Ax}(t)F_{Sy,Ay}(t), F_{Sx,Ay}(2t)F_{Sy,Ax}(t), \\ F_{Sy,Ax}(t), F_{Sx,Ay}(2t)F_{Sy,Ay}(t) \end{array} \right\}, \tag{3.4}$$

for all $x, y \in X$ and $t > 0$. Then (A, S) has a coincidence point. Moreover if the pair (A, S) is weakly compatible then it has a unique common fixed point in X.

Proof Since the pair (A, S) satisfies the (CLR_S) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. We assert that $Au = Su$. On using inequality (3.4) with $x = u$, $y = x_n$, we get

$$(F_{Au,Ax_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Su,Sx_n}(t))^2, F_{Su,Au}(t)F_{Sx_n,Ax_n}(t), F_{Su,Ax_n}(2t)F_{Sx_n,Au}(t), \\ F_{Sx_n,Au}(t), F_{Su,Ax_n}(2t)F_{Sx_n,Ax_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} (F_{Au,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,Au}(t)F_{z,z}(t), F_{z,z}(2t)F_{z,Au}(t), \\ F_{z,Au}(t), F_{z,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Au,z}(t))^2. \end{aligned}$$

On employing Lemma 2.20, we have $Au = z$. Therefore $Au = Su = z$ and hence u is a coincidence point of (A, S). Since the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Putting $x = u$, $y = x_n$ in inequality (3.4), we get

$$(F_{Az,Ax_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Sz,Sx_n}(t))^2, F_{Sz,Az}(t)F_{Sx_n,Ax_n}(t), F_{Sz,Ax_n}(2t)F_{Sx_n,Az}(t), \\ F_{Sx_n,Az}(t), F_{Sz,Ax_n}(2t)F_{Sx_n,Ax_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} (F_{Az,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,z}(t), F_{Az,z}(2t)F_{z,Az}(t), \\ F_{z,Az}(t), F_{Az,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Az,z}(t))^2. \end{aligned}$$

In view of Lemma 2.20, we have $Az = z = Sz$. Therefore z is a common fixed point of the pair (A, S). The uniqueness of common fixed point is an easy consequence of inequality (3.4).

Example 3.10. Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $X = [2, 15)$, with t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Now we define the self mappings A and S by

$$A(x) = \begin{cases} 2, & \text{if } x \in \{2\} \cup (3, 15); \\ 9, & \text{if } x \in (2, 3]. \end{cases} \quad S(x) = \begin{cases} 2, & \text{if } x = 2; \\ 10, & \text{if } x \in (2, 3]; \\ \frac{x+1}{2}, & \text{if } x \in (3, 15). \end{cases}$$

Taking $\{x_n\} = \left\{3 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ or $\{x_n\} = \{2\}$, it is clear that the pair (A, S) satisfies the (CLR_S) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2 \in S(X).$$

It is noted that $A(X) = \{2, 9\} \nsubseteq [2, 8) \cup \{10\} = S(X)$. Thus, all the conditions of Theorem 3.9 are satisfied and 2 is a unique common fixed point of the pair (A, S). Also, all the involved mappings are even discontinuous at their unique common fixed point 2.

4. APPLICATION

The following definition is essentially contained in Imdad et al. [6].

Definition 4.1. [6] Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

- (i) $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
- (ii) $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
- (iii) $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

Now, we utilize Definition 4.1 (which is indeed a natural extension of commutativity condition to two finite families) to prove a common fixed point theorem for four finite families of weakly compatible mappings in Menger space (as an application of Theorem 3.4).

Theorem 4.2. *Let $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm with $A = A_1A_2\ldots A_m, B = B_1B_2\ldots B_n, S = S_1S_2\ldots S_p$ and $T = T_1T_2\ldots T_q$ satisfying inequality (3.1) of Theorem 3.1 such that the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a point of coincidence each.*

Then $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ have a unique common fixed point provided the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_h\})$ commute pairwise wherein $i \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, p\}, r \in \{1, 2, \ldots, n\}$ and $h \in \{1, 2, \ldots, q\}$.

Proof Owing to pairwise commuting property, we can prove that $AS = SA$ as

$$\begin{aligned} AS &= (A_1A_2\ldots A_m)(S_1S_2\ldots S_p) = (A_1A_2\ldots A_{m-1})(A_mS_1S_2\ldots S_p) \\ &= (A_1A_2\ldots A_{m-1})(S_1S_2\ldots S_pA_m) = (A_1A_2\ldots A_{m-2})(A_{m-1}S_1S_2\ldots S_pA_m) \\ &= (A_1A_2\ldots A_{m-2})(S_1S_2\ldots S_pA_{m-1}A_m) = \ldots = A_1(S_1S_2\ldots S_pA_2\ldots A_{m-1}A_m) \\ &= (S_1S_2\ldots S_p)(A_1A_2\ldots A_m) = SA. \end{aligned}$$

Similarly, we can also easily prove that $BT = TB$ so that the pairs (A, S) and (B, T) are weakly compatible. Now using Theorem 3.4, we conclude that A, B, S and T have a unique common fixed point z in X.

Now, we prove that w remains the fixed point of all the component mappings. To do this, consider

$$\begin{aligned} A(A_iw) &= ((A_1A_2\ldots A_m)A_i)\,w = (A_1A_2\ldots A_{m-1})(A_mA_i)w \\ &= (A_1A_2\ldots A_{m-1})(A_iA_m)w = (A_1A_2\ldots A_{m-2})(A_{m-1}A_iA_m)w \\ &= (A_1A_2\ldots A_{m-2})(A_iA_{m-1}A_m)w = \ldots = A_1(A_iA_2\ldots A_m)w \\ &= (A_1A_i)(A_2\ldots A_m)w \\ &= (A_iA_1)(A_2\ldots A_m)w = A_i(A_1A_2\ldots A_m)w = A_iw. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
A(S_k w) &= S_k(Aw) = S_k w, S(S_k w) = S_k(Sw) = S_k w, S(A_i w) = A_i(Sw) = A_i w,\\
B(B_r w) &= B_r(Bw) = B_r w, B(T_h w) = T_h(Bw) = T_h w, T(T_h w) = T_h(Tw) = T_h w,\\
T(B_r w) &= B_r(Tw) = B_r w,
\end{aligned}$$

which shows that (for all i, r, k and h) $A_i w$ and $S_k w$ are other fixed point of the pair (A, S) whereas $B_r w$ and $T_h w$ are other fixed points of the pair (B, T).

Now appealing to the uniqueness of common fixed points of mappings A, B, S and T, we get

$$w = A_i w = S_k w = B_r w = T_h w,$$

for all $i \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, p\}, r \in \{1, 2, \ldots, n\}, h \in \{1, 2, \ldots, q\}$, which shows that w is the unique common fixed point of $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$.

By setting $A_1 = A_2 = \ldots = A_m = A, B_1 = B_2 = \ldots = B_n = B, S_1 = S_2 = \ldots = S_p = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Theorem 4.2, we deduce the following:

Corollary 4.3. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Suppose that*

(i) *the pairs (A^m, S^p) and (B^n, T^q) share the (CLR_{S^p,T^q}) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{A^m x, B^n y}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{S^p x, T^q y}(t))^2, F_{S^p x, A^m x}(t) F_{T^q y, B^n y}(t), \\ F_{S^p x, B^n y}(2t) F_{T^q y, A^m x}(t), \\ F_{T^q y, A^m x}(t), F_{S^p x, B^n y}(2t) F_{T^q y, B^n y}(t) \end{array} \right\}, \quad (4.1)$$

for all $x, y \in X, t > 0$, m, n, p and q are fixed positive integers. Then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 4.4. The results similar to Theorem 4.2 and Corollary 4.3 can be obtained in respect of Theorem 3.9.

Remark 4.5. Theorem 4.2 and Corollary 4.3 extend the results of Cho et al. [4] and Pathak et al. [21] to four finite families of self mappings.

5. CORRESPONDING RESULTS IN METRIC SPACES

In this section, as a sample, we utilize Theorem 3.4 to derive corresponding common fixed point theorem in metric space.

Theorem 5.1. *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that*

(i) *the pairs (A, S) and (B, T) share the (CLR_{ST}) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(d(Ax, By))^2 \leq k \max \left\{ \begin{array}{c} (d(Sx, Ty))^2, d(Sx, Ax) d(Ty, By), \\ \frac{1}{2} d(Sx, By) d(Ty, Ax), d(Ty, Ax), \\ \frac{1}{2} d(Sx, By) d(Ty, By) \end{array} \right\}, \quad (5.1)$$

for all $x, y \in X$. Then the pairs (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof Define $F_{x,y}(t) = H(t - d(x, y))$ and $\triangle(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$. Then metric space (X, d) can be realized as a Menger space $(X, \mathcal{F}, \triangle)$. It is straightforward to notice that Theorem 5.1 satisfies all the conditions of Theorem 3.4. Also inequality (5.1) of Theorem 5.1 implies inequality (3.1) of Theorem 3.1. For any

$x, y \in X$ and $t > 0$, $F_{Ax,By}(kt) = 1$ if $kt > d(Ax, By)$ which confirms the verification of inequality (3.1) of Theorem 3.1. Otherwise, if $kt \leq d(Ax, By)$, then

$$t \leq \max \left\{ \begin{array}{c} (d(Sx,Ty))^2, d(Sx,Ax)d(Ty,By), \frac{1}{2}d(Sx,By)d(Ty,Ax), \\ d(Ty,Ax), \frac{1}{2}d(Sx,By)d(Ty,By) \end{array} \right\},$$

which shows that inequality (3.1) of Theorem 3.1 is satisfied. Thus, all the conditions of Theorem 3.4 are satisfied so that conclusions follow immediately from Theorem 3.4.

Remark 5.2. The results similar to Theorem 5.1 can also be outlined in respects of Theorem 3.7, Theorem 3.9, Theorem 4.2 and Corollary 4.3.

Remark 5.3. Theorem 5.1 improves the results of Cho et al. [4, Theorem 4.3] and Pathak et al. [21, Theorem 4.3].

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