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AN OTHER APPROACH FOR THE PROBLEM OF FINDING A COMMON FIXED POINT OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

TRUONG MINH TUYEN*

Department of Mathematics, Thainguyen University - Vietnam

ABSTRACT. The purpose of this paper is to give a Tikhonov regularization method and some regularization inertial proximal point algorithm for the problem of finding a common fixed point of a finite family of nonexpansive mappings in a uniformly convex and uniformly smooth Banach space E, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*.

KEYWORDS : Accretive operators; Uniformly smooth and uniformly convex Banach space; Sunny nonexpansive retraction; Weak sequential continuous mapping; Regularization.
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1. INTRODUCTION

Let E be a Banach space. We consider the following problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} F(T_i), \tag{1.1}$$

where $F(T_i)$ is the set of fixed points of nonexpansive mappings $T_i : C \longrightarrow C$ and C is a closed convex nonexpansive retract subset of a uniformly convex and uniformly smooth Banach space E.

It is well-known that, numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. For instance, if the nonexpansive mappings are projections onto some closed and convex sets, then the fixed point problem becomes the famous convex feasibility problem. Due to the practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be flourishing topic of interest in fixed point theory. This problem has been investigated by many researchers: see, for instance, Bauschke [7], O' Hara et al. [22], Jung [16], Chang et al. [10], Takahashi and Shimoji [27], Ceng et al. [9], Chidume et al. [11, 12], Plubtieng and Ungchittrakool [23], Kang et al. [17], N. Buong et al. [8] and others.

* *Corresponding author.*
Email address : tm.tuyentm@gmail.com(T. M. Tuyen).
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On the other hand, the problem of finding a fixed point of a nonexpansive mapping $T: E \longrightarrow E$ is equivalent to the problem of finding a zero of $m-$accretive $A = I - T$. One of the methods to solve the problem $0 \in A(x)$, where A is maximal monotone in a Hilbert space H is proximal point algorithm. This algorithm suggested by Rockafellar [24], starting from any initial guess $x_0 \in H$, this algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J_{c_n}^A(x_n + e_n), \tag{1.2}$$

where $J_r^A = (I + rA)^{-1}\ \forall r > 0$ is the resolvent of A on the space H. Rockafellar [24] proved the weak convergence of the algorithm (1.2) provided that the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$. However Güler's example [15] shows that in infinite dimensional Hilbert space, proximal point algorithm (1.2) has only weak convergence.

Ryazantseva [25] extended the proximal point algorithm (1.2) for the case that A is an $m-$accretive mapping in a properly Banach space E and proved the weak convergence of the sequence generated by (1.2) to a solution of the equation $0 \in A(x)$ which is assumed to be unique. Then, to obtain the strong convergence for algorithm (1.2), Ryazantseva [26] combined the proximal algorithm with the regularization, named regularization proximal algorithm, in the form

$$c_n(A(x_{n+1}) + \alpha_n x_{n+1}) + x_{n+1} = x_n,\ x_0 \in E. \tag{1.3}$$

Under some conditions on c_n and α_n, the strong convergence of $\{x_n\}$ of (1.3) is guaranteed only when the dual mapping j is weak sequential continuous and strong continuous, and the sequence $\{x_n\}$ is bounded.

Attouch and Alvarez [6] considered an extension of the proximal point algorithm (1.2) in the form

$$c_n A(u_{n+1}) + u_{n+1} - u_n = \gamma_n(u_n - u_{n-1}),\ u_0,\ u_1 \in H, \tag{1.4}$$

which is called an inertial proximal point algorithm, where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers. With this algorithm we also only obtained weak convergence of the sequence $\{x_n\}$ to a solution of problem $A(x) \ni 0$ in Hilbert spaces. Note that this algorithm was proposed by Alvarez in [2] in the context of convex minimization.

Then, Moudafi [19] applied this algorithm for variational inequalities, Moudafi and Elisabeth [20] studied this algorithm by using enlargement of a maximal monotone operator, and Moudafi and Oliny [21] considered convergence of a splitting inertial proximal method. The main result in these papers is also the weak convergence of the algorithm in Hilbert spaces.

In this paper, we introduced the algorithms in the forms

$$\sum_{i=1}^{N} A_i(x_n) + \alpha_n(x_n - y) = 0, \tag{1.5}$$

$$c_n\Big(\sum_{i=1}^{N} A_i(u_{n+1}) + \alpha_n(u_{n+1} - y)\Big) + u_{n+1} = Q_C(u_n + \gamma_n(u_n - u_{n-1})), \tag{1.6}$$

where $y,\ u_0,\ u_1 \in C$, and $Q_C: E \longrightarrow C$ is a sunny nonexpansive retraction from E onto C to solve the problem (1.1).

And also, we give some analogue regularization methods for the more general problems, such as: the problem of finding a common fixed point of a finite family of nonexpansive nonself - mapping on a closed and convex subset of E. Finally, the stability of the regularization algorithms are considered in this paper.

2. PRELIMINARIES

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the same symbol $\|.\|$. We write $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ and $\longrightarrow$ to denote the weak convergence, weak* convergence and strong convergence, respectively.

Definition 2.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \ge 0$ such that

$$\frac{\|x+y\|}{2} \le 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| :\ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\} \tag{2.1}$$

is called the modulus of convexity of the space E. The function $\delta_E(\varepsilon)$ defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0, 2]$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x + y\| + \|x - y\|\big) - 1 :\ \|x\| = 1,\ \|y\| = \tau\}, \tag{2.2}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive. In what follows, we denote

$$h_E(\tau) := \frac{\rho_E(\tau)}{\tau}. \tag{2.4}$$

The function $h_E(\tau)$ is nondecreasing. In addition, we have the following estimate

$$h_E(K\tau) \le LKh_E(\tau),\ \forall K > 1,\ \tau > 0, \tag{2.5}$$

where L is the Figiel's constant [3, 4, 13], $1 < L < 1.7$.

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.6}$$

is called the normalized duality mapping of E.

In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$ and, if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J be single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x,\ J(x_n) \overset{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

Definition 2.4. An operator $A :\ D(A) \subseteq E \rightrightarrows E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \ge 0,\ \forall u \in A(x),\ v \in A(y). \tag{2.7}$$

Definition 2.5. A mapping $T: C \longrightarrow E$ is said to be nonexpansive on a closed and convex subset C of Banach space E if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall x, y \in C. \tag{2.8}$$

It is clear that, if $T: C \longrightarrow E$ is a nonexpansive, then $I - T$ is accretive operator.

Definition 2.6. Let G be a nonempty closed and convex subset of E. A mapping $Q_G: E \longrightarrow G$ is said to be

i) a retraction onto G if $Q_G^2 = Q_G$;

ii) a nonexpansive retraction if it also satisfies the inequality

$$\|Q_G x - Q_G y\| \leq \|x - y\|, \ \forall x, y \in E; \tag{2.9}$$

iii) a sunny retraction if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{2.10}$$

A closed and convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Proposition 2.7. [14] *Let C be a nonempty closed convex subset of a smooth Banach E. A mapping $Q_C: E \longrightarrow C$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_C x, j(\xi - Q_C x)\rangle \leq 0, \ \forall x \in E, \ \forall \xi \in C. \tag{2.11}$$

Definition 2.8. Let $C_1, \ C_2$ be convex subsets of E. The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| = \sup_{u \in C_1} d(u, C_2)$$

is said to be semideviation of the set C_1 from the set C_2. The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \ \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2.

Lemma 2.9. [5] *If E is a uniformly smooth Banach space, C_1 and C_2 are closed and convex subsets of E such that the Hausdorff $\mathcal{H}(C_1, C_2) \leq \delta$, and Q_{C_1} and Q_{C_2} are the sunny nonexpansive retractions onto the subsets C_1 and C_2, respectively, then*

$$\|Q_{C_1}x - Q_{C_2}x\|^2 \leq 16R(2r + d)h_E(\frac{16L\delta}{R}), \tag{2.12}$$

where L is Figiel's constant, $r = \|x\|, \ d = \max\{d_1, d_2\}$, and $R = 2(2r + d) + \delta$. Here $d_i = dist(\theta, C_i)$, $i = 1, 2$, and θ is the origin of the space E.

3. MAIN RESULTS

First, we need the following lemmas in the proof of our results.

Lemma 3.1. [3] *Let E be a uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping T then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \geq L^{-1}R^2\delta_E\left(\frac{\|Ax - Ay\|}{4R}\right), \tag{3.1}$$

where $\|x\| \leq R, \ \|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 3.2 (demiclosedness principle)**.** [1] *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping, C a nonempty closed convex subset of E, and $T: C \longrightarrow E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I-T)x_n \longrightarrow y$ imply that $x \in C$ and $(I-T)x = y$.*

Lemma 3.3. [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \sigma_n, \ \forall n \ge 0,$$

where $\{\alpha_n\} \subset (0,1)$ for each $n \ge 0$ such that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$, or (c) $\limsup \dfrac{\sigma_n}{\alpha_n} \le 0$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 3.4. [18] *Let C be a closed convex subset of a strictly convex Banach space E and let $T: C \longrightarrow E$ be a nonexpansive mapping from C into E. Suppose that C is be sunny nonexpansive retract of E. If $F(T) \ne \emptyset$, then $F(T) = F(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C.*

Theorem 3.5. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i: C \longrightarrow C$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^{N} F(T_i) \ne \emptyset$. Then*

- i) *For each $\alpha_n > 0$ the equation (1.5) has unique solution x_n;*
- ii) *If the sequence of positive numbers $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $Q_S y$, where $Q_S: E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.*

Moreover, we have the following estimate

$$\|x_{n+1} - x_n\| \le \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R_0 \ \forall n \ge 0, \tag{3.2}$$

where $R_0 = 2\|y - Q_S y\|$.

Proof. i) For each $n \ge 0$, equation (1.5) defines unique sequence $\{x_n\} \subset E$, because for each n, the element x_n is unique fixed point of the contraction mapping $T: C \longrightarrow C$ defined by

$$T(x) = \frac{1}{N+\alpha_n} \sum_{i=1}^{N} T_i(x) + \frac{\alpha_n}{N+\alpha_n} y. \tag{3.3}$$

ii) From equation (1.5), we have

$$\langle \sum_{i=1}^{N} A_i(x_n), j(x_n - x^*)\rangle + \alpha_n \langle x_n - y, j(x_n - x^*)\rangle = 0, \ \forall x^* \in S. \tag{3.4}$$

By virtue of the property of $\sum_{i=1}^{N} A_i$ and j, we obtain

$$\langle \sum_{i=1}^{N} A_i(x_n), j(x_n - x^*)\rangle \ge 0, \ \forall x^* \in S.$$

Thus,

$$\langle x_n - y, j(x_n - x^*)\rangle \le 0, \ \forall x^* \in S. \tag{3.5}$$

From inequality (3.5), we get

$$\|x_n - x^*\|^2 \leq \langle y - x^*, j(x_n - x^*)\rangle \leq \|y - x^*\|.\|x_n - x^*\|, \ \forall x^* \in S. \tag{3.6}$$

Therefore

$$\|x_n - x^*\| \leq \|y - x^*\|, \ \forall n \geq 0, \ \forall x^* \in S, \tag{3.7}$$

that implies the boundedness of the sequence $\{x_n\}$. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which converges weakly to a limit point $\overline{x}$. Since C is closed and convex, it is also weakly closed. Therefore $\overline{x} \in C$.

We will show that $\overline{x} \in S$. Indeed, for each $i \in \{1, 2, ..., N\}$, $x^* \in S$ and $R > 0$ satisfy $R \geq \max\{\sup \|x_n\|, \|x^*\|\}$, we have

$$\begin{aligned}
\delta_E\Big(\frac{\|A_i(x_n)\|}{4R}\Big) &\leq \frac{L}{R^2}\langle A_i(x_n), j(x_n - x^*)\rangle \\
&\leq \frac{L}{R^2}\langle \sum_{k=1}^{N} A_k(x_n), j(x_n - x^*)\rangle \\
&\leq \frac{L\alpha_n}{R^2}\|x_n - y\|.\|x_n - x^*\| \\
&\leq \frac{L\alpha_n}{R^2}(R + \|y\|).\|y - x^*\| \longrightarrow 0, \ n \longrightarrow \infty.
\end{aligned}$$

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_n) \longrightarrow 0, \ n \longrightarrow \infty$. From Lemma 3.2, it implies that $A_i(\overline{x}) = 0$. Since $i \in \{1, 2, ..., N\}$ is an arbitrary element, we obtain $\overline{x} \in S$.

In inequality (3.6) replacing x_n by x_{n_k} and x^* by $\overline{x}$, using the weak continuity of j we obtain $x_{n_k} \longrightarrow \overline{x}$. From inequality (3.5), we get

$$\langle \overline{x} - y, j(\overline{x} - x^*)\rangle \leq 0, \ \forall x^* \in S. \tag{3.8}$$

Now, we show that the inequality (3.8) has unique solution. Suppose that $\overline{x}_1 \in S$ is also its solution. Then

$$\langle \overline{x}_1 - y, j(\overline{x}_1 - x^*)\rangle \leq 0, \ \forall x^* \in S. \tag{3.9}$$

In inequalities (3.8) and (3.9) replacing x^* by $\overline{x}_1$ and $\overline{x}$, respectively, we obtain

$$\begin{aligned}
&\langle \overline{x} - y, j(\overline{x} - \overline{x}_1)\rangle \leq 0, \\
&\langle y - \overline{x}_1, j(\overline{x} - \overline{x}_1)\rangle \leq 0.
\end{aligned}$$

Their combination gives $\|\overline{x} - \overline{x}_1\|^2 \leq 0$, thus $\overline{x} = \overline{x}_1 = Q_S y$ and the sequence $\{x_n\}$ converges weakly to $\overline{x} = Q_S y$, because $Q_S y$ satisfies the inequality (3.8). Finally, from the first inequality in (3.6), implies that $x_n \longrightarrow Q_S y$.

Now, we prove the inequality (3.2). In equation (1.5) replacing n by $n+1$ we have

$$\sum_{i=1}^{N} A_i(x_{n+1}) + \alpha_{n+1}(x_{n+1} - y) = 0. \tag{3.10}$$

From (3.10) and (1.5), we get

$$\langle \alpha_{n+1}x_{n+1} - \alpha_n x_n, j(x_{n+1} - x_n)\rangle \leq (\alpha_{n+1} - \alpha_n)\langle y, j(x_{n+1} - x_n)\rangle. \tag{3.11}$$

Therefore,

$$\begin{aligned}
\alpha_n\|x_{n+1} - x_n\|^2 &\leq (\alpha_{n+1} - \alpha_n)\langle y - x_{n+1}, j(x_{n+1} - x_n)\rangle \\
&\leq |\alpha_{n+1} - \alpha_n|.\|y - x_{n+1}\|.\|x_{n+1} - x_n\| \\
&\leq 2\|y - Q_S y\|.|\alpha_{n+1} - \alpha_n|.\|x_{n+1} - x_n\|.
\end{aligned}$$

Thus,

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R_0 \ \forall n \geq 0,$$

where $R_0 = 2\|y - Q_S y\|$. □

Theorem 3.6. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i : C \longrightarrow C,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the sequences $\{c_n\},\ \{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty} \alpha_n = +\infty$;

ii) $\gamma_n \geq 0,\ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by (1.6) converges strongly to $Q_S y$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, for each $n \geq 1$, equation (1.6) defines unique sequence $\{u_n\} \subset E$, because for each n, the element u_{n+1} is unique fixed point of the contraction mapping $f : C \longrightarrow C$ defined by

$$f(x) = \frac{c_n}{c_n(N + \alpha_n) + 1} \sum_{i=1}^N T_i(x) + \frac{c_n \alpha_n}{c_n(N + \alpha_n) + 1} y + \frac{1}{c_n(N + \alpha_n) + 1} z, \quad (3.12)$$

where $z = Q_C(u_n + \gamma_n(u_n - u_{n-1})) \in C$.

Now, we rewrite equations (1.5) and (1.6) in their equivalent forms

$$d_n \sum_{i=1}^N A_i(x_n) + x_n - y = \beta_n(x_n - y), \quad (3.13)$$

$$d_n \sum_{i=1}^N A_i(u_{n+1}) + u_{n+1} - y = \beta_n[Q_C(u_n + \gamma_n(u_n - u_{n-1})) - y], \quad (3.14)$$

where $\beta_n = \dfrac{1}{1 + c_n \alpha_n}$ and $d_n = c_n \beta_n$.

From (3.13), (3.14) and by virtue of the property of $\sum_{i=1}^N A_i$, we get

$$\begin{aligned}\|u_{u+1} - x_n\| &\leq \beta_n \|Q_C(u_n + \gamma_n(u_n - u_{n-1})) - x_n\| \\ &= \beta_n \|Q_C(u_n + \gamma_n(u_n - u_{n-1})) - Q_C(x_n)\| \\ &\leq \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\leq \|u_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\ &\leq \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\| + \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R,\end{aligned} \quad (3.15)$$

or equivalent to

$$\|u_{n+1} - x_{n+1}\| \leq (1 - b_n)\|u_n - x_n\| + \sigma_n,\ b_n = \frac{c_n \alpha_n}{1 + c_n \alpha_n}, \quad (3.16)$$

where $\sigma_n = \beta_n\gamma_n\|u_n - u_{n-1}\| + \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n}R$.
By the assumption, we have

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_n}+1)\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2}R\\ &\le \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_0}+1)\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2}R \longrightarrow 0.\end{aligned}$$

Furthermore, since $\sum_{n=0}^{\infty}\alpha_n = +\infty$, $\sum_{n=0}^{\infty} b_n = +\infty$.
By Lemma 3.3, we obtain $\|u_n - x_n\| \longrightarrow 0$. Since $x_n \longrightarrow Q_S y$, $u_n \longrightarrow Q_S y$. □

Corollary 3.7. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i:\ E \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the sequences $\{c_n\},\ \{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty}\alpha_n = +\infty$;

ii) $\gamma_n \ge 0,\ \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by

$$c_n(\sum_{i=1}^N A_i(u_{n+1}) + \alpha_n u_{n+1}) + u_{n+1} = u_n + \gamma_n(u_n - u_{n-1}),\ u_0,\ u_1 \in E$$

converges strongly to $Q_S\theta$, where $Q_S:\ E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. Applying Theorem 3.6 for $C = E$ and $y = \theta$, we obtain the proof of this corollary. □

Corollary 3.8. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $f_i:\ C \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(f_i) \neq \emptyset$. If the sequences $\{c_n\},\ \{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty}\alpha_n = +\infty$;

ii) $\gamma_n \ge 0,\ \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by

$$c_n(\sum_{i=1}^N B_i(u_{n+1}) + \alpha_n(u_{n+1} - y)) + u_{n+1} = Q_C(u_n + \gamma_n(u_n - u_{n-1})), \tag{3.17}$$

converges strongly to $Q_S y$, where $B_i = I - Q_C f_i,\ i = 1, 2, ..., N$, Q_C is a sunny nonexpansive retraction from E onto C, Q_S is a sunny nonexpansive retraction from E onto S, and $y,\ u_0,\ u_1 \in C$.

Proof. By Lemma 3.4, we have $S = \cap_{i=1}^N F(Q_C f_i)$. Applying Theorem 3.6 for $T_i = Q_C f_i,\ i = 1, 2, ..., N$ we obtain the proof of this corollary. □

Finally, we study stability of the algorithms (1.5) and (1.6) with respect to perturbations of both operators T_i and constraint set C satisfying the following conditions:

(P1) Instead of C, there is a sequence of closed convex sunny nonexpasive retract subsets $C_n \subset E,\ n = 1, 2, 3, ...$ such that the Hausdorff $\mathcal{H}(C_n, C) \le \delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers with the propertie

$$\delta_{n+1} \le \delta_n,\ \forall n \ge 1. \tag{3.18}$$

(P2) On the each set C_n, there is a nonexpansive self-mapping $T_i^n :\ C_n \longrightarrow C_n,\ i = 1, 2, ..., N$ satisfying the conditions: there exists the increasing positive for all $t > 0$ function $g(t)$ and $\xi(t)$ such that $g(0) \ge 0,\ \xi(0) = 0$ and $x \in C_k,\ y \in C_m,\ \|x - y\| \le \delta$, then

$$\|T_i^k x - T_i^m y\| \le g(\max\{\|x\|, \|y\|\})\xi(\delta). \tag{3.19}$$

In this paper, we establish the convergence and stability of the Tikhonov regularization method (1.5) and the regularization inertial proximal point algorithm (1.6) in the forms

$$\sum_{i=1}^{N} A_i^n(z_n) + \alpha_n(z_n - Q_{C_n} y) = 0, \tag{3.20}$$

$$c_n(\sum_{i=1}^{N} A_i^n(u_{n+1}) + \alpha_n(u_{n+1} - Q_{C_n} y)) + u_{n+1} = Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})), \tag{3.21}$$

respectively, where $u_0,\ u_1$ and y are elements in E, and $A_i^n = I - T_i^n,\ i = 1, 2, ..., N$, with respect to perturbations of the set C, and $Q_{C_n} :\ E \longrightarrow C_n$ is the sunny nonexpansive retraction of E onto C_n.

Theorem 3.9. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i :\ C \longrightarrow C,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \ne \emptyset$.*

i) *For each $\alpha_n > 0$ equation (3.20) has unique solution z_n;*

ii) *If the conditions (P1) and (P2) are fulfilled and the sequences of positive numbers $\{\alpha_n\},\ \{\delta_n\}$ satisfy*

$$\alpha_n \longrightarrow 0,\ \frac{\delta_n + \xi(\delta_n)}{\alpha_n} \longrightarrow 0,\ \text{as } n \longrightarrow \infty, \tag{3.22}$$

then $\{z_n\}$ converges strongly to $Q_S(Q_C y)$, where $Q_S :\ E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Moreover, if $\{\alpha_n\}$ is a decreasing sequence, then we have the following estimate

$$\begin{aligned} \|z_{n+1} - z_n\| \le 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} R \\ + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)},\ \forall n \ge 0, \end{aligned} \tag{3.23}$$

where $R,\ K,\ K_3,\ K_4$ are any constants.

Proof. i) For each $n \ge 0$, by an argument similar to the proof of Theorem 3.5, it follows that, the equation (3.20) has a unique solution z_n.
ii) Since the distance Hausdorff $\mathcal{H}(C_n, C) \le \delta_n$, therefore for each solution x_n of equation (1.5) (note that, in the case that the element y in (1.5) is replaced by $Q_C y$),

there exists an element $u_n \in C_n$ such that $\|x_n - u_n\| \leq \delta_n$.
From equations (1.5) and (3.20), we have

$$\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(u_n)) + \alpha_n(z_n - x_n) - \alpha_n(Q_{C_n}y - Q_C y) + \sum_{i=1}^{N}(A_i^n(u_n) - A_i(x_n)) = 0. \tag{3.24}$$

By virtue of the property of $\sum_{i=1}^{N} A_i^n$ and j, we get

$$\langle\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(u_n)), j(z_n - u_n)\rangle \geq 0,$$

that implies

$$\alpha_n\langle z_n - x_n, j(z_n - u_n)\rangle \leq \alpha_n\langle Q_{C_n}y - Q_C y, j(z_n - u_n)\rangle + \langle\sum_{i=1}^{N}(A_i(x_n) - A_i^n(u_n)), j(z_n - u_n)\rangle. \tag{3.25}$$

Thus,

$$\begin{aligned}\alpha_n\|z_n - u_n\| &\leq \alpha_n\|x_n - u_n\| + \alpha_n\|Q_{C_n}y - Q_C y\| + \sum_{i=1}^{N}\|A_i(x_n) - A_i^n(u_n)\| \\ &\leq \alpha_n\delta_n + \alpha_n\|Q_{C_n}y - Q_C y\| + \sum_{i=1}^{N}\|A_i(x_n) - A_i^n(u_n)\|.\end{aligned}$$

Since $\mathcal{H}(C_n, C) \leq \delta_n$, there exists constants $K_1 > 0$ and $K_2 > 1$ such that the inequalities

$$\|Q_{C_n}y - Q_C y\| \leq K_1\sqrt{h_E(K_2\delta_n)} \leq K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)}$$

hold.
Next, for each $i \in \{1, 2, ..., N\}$, we have

$$\begin{aligned}\|A_i(x_n) - A_i^n(u_n)\| &\leq \delta_n + g(\max\{\|x_n\|, \|u_n\|\})\xi(\delta_n) \\ &\leq \delta_n + g(M)\xi(\delta_n),\end{aligned}$$

where $M = \max\{\sup\|x_n\|, \sup\|u_n\|\} < +\infty$.
Consequently,

$$\alpha_n\|z_n - u_n\| \leq \alpha_n\delta_n + \alpha_n K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)} + N(\delta_n + g(M)\xi(\delta_n)). \tag{3.26}$$

Thus,

$$\begin{aligned}\|z_n - x_n\| &\leq \|z_n - u_n\| + \|x_n - u_n\| \\ &\leq 2\delta_n + K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)} + N\frac{\delta_n + g(M)\xi(\delta_n)}{\alpha_n}.\end{aligned} \tag{3.27}$$

Since $\alpha_n \longrightarrow 0$, $\dfrac{\delta_n + \xi(\delta_n)}{\alpha_n} \longrightarrow 0$, hence $\delta_n \longrightarrow 0$ and $h_E(\delta_n) \longrightarrow 0$. By inequality (3.27), we obtain $\|x_n - z_n\| \longrightarrow 0$. By Theorem 3.5, it implies that $x_n \longrightarrow Q_S(Q_C y)$, thus the sequence $\{z_n\}$ also converges strongly to $Q_S(Q_C y)$.

Finally, we prove the inequality (3.23). In equation (3.20) replacing n by $n+1$, we have

$$\sum_{i=1}^{N} A_i^{n+1}(z_{n+1}) + \alpha_n(z_{n+1} - Q_{C_{n+1}}y) = 0. \tag{3.28}$$

Since

$$\mathcal{H}(C_n, C_{n+1}) \le \mathcal{H}(C_n, C) + \mathcal{H}(C, C_{n+1}) \le 2\delta_n,$$

we assert that for any $z_{n+1} \in C_{n+1}$ there exists an element $v_n \in C_n$ such that $\|z_{n+1} - v_n\| \le 2\delta_n$.
From equations (3.20) and (3.28), we obtain

$$\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(v_n)) + \alpha_n(z_n - Q_{C_n}y) - \alpha_{n+1}(z_{n+1} - Q_{C_{n+1}}y) + \sum_{i=1}^{N}(A_i^n(v_n) - A_i^{n+1}(z_{n+1})) = 0.$$

By virtue of the property of $\sum_{i=1}^{N} A_i^n$ and j, we get

$$\begin{aligned} \alpha_n\|z_n - v_n\| \le \alpha_{n+1}\|v_n - z_{n+1}\| + |\alpha_n - \alpha_{n+1}|.\|v_n - Q_{C_n}y\| \\ + \alpha_{n+1}\|Q_{C_n}y - Q_{C_{n+1}}y\| + \sum_{i=1}^{N}\|A_i^n(v_n) - A_i^{n+1}(z_{n+1})\| \end{aligned} \tag{3.29}$$

Since $\mathcal{H}(C_n, C_{n+1}) \le 2\delta_n$, there exists constants $K_3 > 0$ and $K_4 > 1$ such that the inequalities

$$\|Q_{C_n}y - Q_{C_{n+1}}y\| \le K_3\sqrt{h_E(K_4\delta_n)} \le K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)} \tag{3.30}$$

hold.
Since $v_n \in C_n$, therefore

$$\|v_n - Q_{C_n}y\| \le \|v_n - y\| \le \sup\|z_n\| + \|y\| + 2\delta_1 := R. \tag{3.31}$$

Next, for each $i \in \{1, 2, ..., N\}$, we have

$$\begin{aligned} \|A_i^n(v_n) - A_i^{n+1}(z_{n+1})\| &\le 2\delta_n + \|T_i^n(v_n) - T_i^{n+1}(z_{n+1})\| \\ &\le 2\delta_n + g(\max\{\|v_n\|, \|z_{n+1}\|\})\xi(2\delta_n) \\ &\le 2\delta_n + g(M')\xi(2\delta_n), \end{aligned} \tag{3.32}$$

where $M' = \max\{\sup\|v_n\|, \sup\|z_n\|\} < +\infty$.
Combining (3.29), (3.30), (3.31) and (3.32), we obtain

$$\|z_n - v_n\| \le 2\delta_n + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)} + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n}, \tag{3.33}$$

where $K = \max\{2N, Ng(M')\}$.
Consequently,

$$\|z_{n+1} - z_n\| \le 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)}. \tag{3.34}$$

□

Next, we will prove the strong convergence and stability of regularization inertial proximal point algorithm (3.21) by the following theorem.

Theorem 3.10. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i : C \longrightarrow C,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{\alpha_n\},\ \{\delta_n\},\ \{\tilde{c}_n\}$ and $\{\gamma_n\}$ satisfy*

i) $\alpha_n \searrow 0,\ \dfrac{\alpha_n - \alpha_{n+1}}{\alpha_n^2} \longrightarrow 0,$ *as* $n \longrightarrow \infty,\ \sum_{n=1}^\infty \alpha_n = +\infty,$

ii) $\dfrac{\delta_n + \xi(2\delta_n)}{\alpha_n^2} \longrightarrow 0,\ \dfrac{\sqrt{h_E(\delta_n)}}{\alpha_n} \longrightarrow 0,$ *as* $n \longrightarrow \infty,$

iii) $0 < c_0 < c_n,\ \gamma_n \geq 0,\ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0,$ *as* $n \longrightarrow \infty,$

then the sequence $\{u_n\}$ defined by (3.21) converges strongly to $Q_S(Q_C y)$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, for each n by an argument similar to the proof of Theorem 3.6, it follows that, the equation (3.21) has unique solution $u_{n+1} \in C_n$.

Now, we rewrite equations (3.20) and (3.21) in their equivalent forms

$$d_n \sum_{i=1}^N A_i^n(z_n) + z_n - Q_{C_n} y = \beta_n(z_n - Q_{C_n} y), \tag{3.35}$$

$$d_n \sum_{i=1}^N A_i^n(u_{n+1}) + u_{n+1} - Q_{C_n} y = \beta_n[Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - Q_{C_n} y], \tag{3.36}$$

where $\beta_n = \dfrac{1}{1 + c_n\alpha_n}$ and $d_n = c_n\beta_n$.

From (3.35), (3.36) and by virtue of the property of $\sum_{i=1}^N A_i^n$, we have

$$\begin{aligned} \|u_{u+1} - z_n\| &\leq \beta_n \|Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - z_n\| \\ &= \beta_n \|Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - Q_{C_n}(z_n)\| \\ &\leq \beta_n \|u_n - z_n\| + \beta_n\gamma_n \|u_n - u_{n-1}\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \|u_{n+1} - z_{n+1}\| &\leq \|u_{n+1} - z_n\| + \|z_{n+1} - z_n\| \\ &\leq \beta_n \|u_n - z_n\| + \beta_n\gamma_n \|u_n - u_{n-1}\| + 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} \\ &+ R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)}, \end{aligned} \tag{3.37}$$

or equivalent to

$$\|u_{n+1} - z_{n+1}\| \leq (1 - b_n)\|u_n - z_n\| + \sigma_n, \tag{3.38}$$

where $b_n = \dfrac{c_n\alpha_n}{1 + \tilde{c}_n\alpha_n}$ and

$$\begin{aligned} \sigma_n &= \beta_n\gamma_n \|u_n - u_{n-1}\| + 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &+ K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)}. \end{aligned}$$

By the assumption, we obtain

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_n} + \alpha_n)\left[\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}R + 4\frac{\delta_n}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n^2}\right] \\ &+ (\frac{1}{c_n} + \alpha_n)K_3\sqrt{LK_4}\frac{\sqrt{h_E(\delta_n)}}{\alpha_n} \\ &\le \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_0} + \alpha_n)\left[\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}R + 4\frac{\delta_n}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n^2}\right] \\ &+ (\frac{1}{c_0} + \alpha_n)K_3\sqrt{LK_4}\frac{\sqrt{h_E(\delta_n)}}{\alpha_n} \longrightarrow 0,\ n \longrightarrow \infty.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n = +\infty$, $\sum_{n=0}^{\infty} b_n = +\infty$.
By Lemma 3.3, it implies that $\|u_n - z_n\| \longrightarrow 0$. Since $z_n \longrightarrow Q_S(Q_C y)$, $u_n \longrightarrow Q_S(Q_C y)$.

□

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